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# ANGULAR DERIVATIVES AND COMPACT COMPOSITION OPERATORS ON THE HARDY AND BERGMAN SPACES

BARBARA D. MacCLUER AND JOEL H. SHAPIRO

**1. Introduction.** Let $U$ denote the open unit disc of the complex plane, and $\varphi$ a holomorphic function taking $U$ into itself. In this paper we study the linear composition operator $C_\varphi$ defined by $C_\varphi f = f \circ \varphi$ for $f$ holomorphic on $U$. Our goal is to determine, in terms of geometric properties of $\varphi$, when $C_\varphi$ is a compact operator on the Hardy and Bergman spaces of $U$. For Bergman spaces we solve the problem completely in terms of the angular derivative of $\varphi$, and for a slightly restricted class of $\varphi$ (which includes the univalent ones) we obtain the same solution for the Hardy spaces $H^p$ $(0 < p < \infty)$. We are able to use these results to provide interesting new examples and to give unified explanations of some previously discovered phenomena.

The boundedness of $C_\varphi$ on $H^p$ is a consequence of Littlewood's Subordination Theorem, and it is in the $H^p$ setting that the study of such operators has attracted the most attention. In 1968 Nordgren initiated the study of spectra of composition operators on $H^p$ **[24]**. His work was carried on by Caughran and Schwartz **[8]**, Kamowitz **[16]**, C. Cowen **[12]**, and in the context of the unit ball of $\mathbf{C}^N$, by B. MacCluer **[18]**, **[19]**. However in this paper we pursue the line of investigation initiated by Ryff **[28]** and H. J. Schwartz **[29]**, and elaborated by Shapiro and Taylor **[30]**.

In **[30]** Shapiro and Taylor focused particular attention on the connection between compactness of $C_\varphi$ and existence of the angular derivative of $\varphi$. We say the *angular derivative* of $\varphi$ exists at a point $\zeta \in \partial U$ if there exists $\omega \in \partial U$ such that the difference quotient

$$\frac{\varphi(z) - \omega}{z - \zeta}$$

has a (finite) limit as $z$ tends non-tangentially to $\zeta$ through $U$. This concept will be discussed more fully in Section 2.3. Right now we simply remark that its existence at $\zeta$ guarantees several things: (i) that $\varphi$ itself has a non-tangential limit of modulus 1 at $\zeta$ (obvious), (ii) that the complex derivative $\varphi'(z)$ also has a non-tangential limit at $\zeta$, and (iii) that $\varphi$ is in a

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certain sense "conformal" at $\zeta$, so that it cannot push the unit disc too sharply into itself. Shapiro and Taylor proved that if $C_\varphi$ is to be compact on $H^p$ then $\varphi$ cannot have an angular derivative at even a single point of $\partial U$ ( [**30**], Theorem 2.1, and Theorem 5.3 below). They also gave a useful, but very special result in the converse direction; and then asked if non-existence of the angular derivative at every point of $\partial U$ might also be sufficient for compactness ( [**30**], page 481).

We are going to answer this last question affirmatively for weighted Bergman spaces (Theorems 3.5 and 5.3) and negatively, but "almost positively" for the Hardy spaces $H^p$ with $0 < p < \infty$ (Theorems 3.6 and 3.10). Our line of attack comprises three major elements. First there is the Theorem of Julia and Caratheodory which we use to make quantitative sense out of the non-existence of the angular derivative. Then there is the equivalence between certain Hardy, Bergman, and Dirichlet spaces. This allows us to treat these spaces in a unified way, and to use known facts about composition operators on the smaller spaces as a means for ascertaining their behavior on the larger ones. Finally there is the characterization, recently exploited in higher dimensions by MacCluer [**20**], of compact composition operators in terms of certain Carleson measures.

The plan of the rest of the paper is as follows. The next section is preparatory: its goal is to keep the rest of the paper reasonably self contained. Here we assemble, mostly without proof, some widely scattered facts about $H^p$ spaces, composition operators, and angular derivatives. We also include a brief discussion of the intuition behind the compactness problem. In the third section we state our main results (Theorems 3.5 and 3.10) and derive their major consequences. Our positive result about compact composition operators on $H^p$ (Theorem 3.10) involves the behavior of $C_\varphi$ on certain weighted Dirichlet spaces. We introduce these spaces, as well as their Bergman counterparts in Section 3.

The relationship between composition operators and Carleson measures is developed in Section 4. In the following section our main results are proved in the setting of weighted Dirichlet spaces: the versions given in Section 3 being derived as corollaries.

In the final section we shift our attention from the unit disc to the unit ball $B$ of $\mathbf{C}^N$ for $N > 1$. Here the situation is complicated by the fact that composition operators need not be bounded on the Hardy or Bergman spaces of $B$, in sharp contrast with the case $N = 1$. However we are able to show that a certain reasonable class of holomorphic maps $\varphi: B \to B$ does induce bounded composition operators, and we characterize the compacts among these in terms of an "angular derivative". We give a class of examples which show that in general however, non-existence of this angular derivative does not imply compactness, even on the weighted Bergman spaces of the ball.

**2. Background.** Here we collect some known facts about $H^p$ spaces, composition operators, and angular derivatives. We also discuss some of the intuition behind the compactness problem for composition operators.

2.1. *$H^p$ spaces.* A good general reference for this material is [**13**], Chapters 1-3. For $0 < p < \infty$ we denote by $H^p$ the space of functions $f$ holomorphic in $U$ for which

$$\|f\|_p^p = \sup_{0 \leqq r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

We denote by $H^\infty$ the space of bounded holomorphic functions on $U$, taken in the supremum norm:

$$\|f\|_\infty = \sup_{|z|<1} |f(z)|.$$

These are all complete linear topological spaces in their natural metrics, and of course they are Banach spaces when $p \geqq 1$ ( [**13**], page 37).

If $f$ belongs to $H^p(p < \infty)$ then for each $z$ in $U$,

$$|f(z)| \leqq 2^{1/p}\|f\|_p(1 - |z|)^{-1/p}.$$

It is this inequality which yields the completeness of $H^p$: it shows that the unit ball of $H^p$ is a normal family, and that the functionals of evaluation at the points of $U$ are all continuous on $H^p$ ( [**13**], Lemma, page 36).

Another important result is Fatou's Radial Limit Theorem ( [**13**] Theorems 2.2, 2.6): if $0 < p \leqq \infty$ and $f \in H^p$ then the *radial limit*

$$f^*(\zeta) = \lim_{r \to 1-} f(r\zeta)$$

exists for almost every $\zeta \in \partial U$, and the map $f \to f^*$ is a linear isometry of $H^p$ onto a closed subspace of $L^p(\partial U)$. We remark that throughout this paper "almost everywhere" refers to linear Lebesgue measure on $\partial U$. The radial convergence in Fatou's Lemma can be replaced by "non-tangential convergence": for each $f$ in $H^p$, for almost every $\zeta$ in $\partial U$ the limit of $f(z)$ exists as $z \to \zeta$ through any triangle $\Delta$ in $U$ with a vertex at $\zeta$ ( [**13**], Theorem 2.2).

2.2. *Composition operators on $H^p$.* According to Littlewood's Subordination Theorem ( [**13**], Theorem 1.7, page 10), if $\varphi$ is holomorphic on $U$ with $\varphi(U) \subset U$ and $\varphi(0) = 0$, then for each $0 < p < \infty$ and $0 \leqq r < 1$:

$$\int_0^{2\pi} |f(\varphi(re^{i\theta}))|^p d\theta \leqq \int_0^{2\pi} |f(re^{i\theta})|^p d\theta$$

for every function $f$ holomorphic in $U$. Thus the induced composition operator $C_\varphi$ is bounded on $H^p$, with norm $\leqq 1$ ($= 1$ actually, since $C_\varphi$ fixes constant functions). It is easy to check that each conformal automorphism of $U$ induces a bounded composition operator on $H^p$; so every $\varphi$ holomorphic on $U$ with $\varphi(U) \subset U$ induces a bounded composition operator on $H^p$ even if $\varphi(0) \neq 0$, (see [28], [29], Chapter III; and [11], Theorem 2.1 for estimates of the norms of such operators).

2.3. *Compact composition operators*. By definition $C_\varphi$ is compact on $H^p$ if and only if it takes the unit ball of $H^p$, which is not compact, into a set whose closure is compact. Our problem is to relate this property to the manner in which $\varphi$ takes the unit disc into itself. To consider an extreme case, suppose $\varphi(U)$ has compact closure in $U$; that is, $\varphi(U) \subset rU$ for some $0 \leqq r < 1$. Then $C_\varphi$ is compact on $H^p$, even if $p = \infty$. This follows easily from the fact that the unit ball of $H^p$ is a normal family (Section 2.1). It is also easy to see that for $H^\infty$ these are the only functions $\varphi$ which induce compact composition operators ( [29] Theorem 2.8, page 28). Thus from now on we consider only the case $p < \infty$, where the situation is much more interesting: now $\varphi$ can have radial limits of modulus one and still induce a compact composition operator. This will be the case if, for example, $\varphi$ maps $U$ onto (or into) a polygon inscribed in the unit circle ( [30], Corollary 3.2); and even into a region with a little smoothness where it touches the boundary (see [30], Corollary 4.4 and Corollary 3.10 of this paper). However too much smoothness, even at just one point, is not allowed: for example the map $\varphi(z) = (1 + z)/2$ maps $U$ onto a region in $U$ which touches the unit circle at just one point, yet it induces a non-compact composition operator ( [29], page 23). In Section 3 of this paper we give a single result (Corollary 3.10) which explains all these examples. What is important right now is their intuitive message: "$C_\varphi$ will be compact on $H^p$ if and only if $\varphi$ squeezes the unit disc rather sharply into itself".

Our goal in this paper is to make quantitative sense out of this intuitive principle. As we have already mentioned, the crucial link between the geometry of $\varphi$ and the compactness of $C_\varphi$ turns out to be the angular derivative of $\varphi$, to which we turn next.

2.4. *The angular derivative*. We previously defined $\varphi$ to have an angular derivative at $\zeta \in \partial U$ if there exists $\omega \in \partial U$ such that the limit

$$\varphi'(\zeta) = \lim \frac{\omega - \varphi(z)}{\zeta - z},$$

where $z \to \zeta$ non-tangentially through $U$, exists and is finite. When this happens it is clear that $\varphi$ has non-tangential limit $\omega$ at $\zeta$. Thus the angular derivative of $\varphi$ has a chance to exist only at those points on $\partial U$ at which $\varphi$ has an angular limit of modulus one. What is not so obvious is that the existence of the angular derivative of $\varphi$ at $\zeta$ is equivalent to the existence of the angular limit of the complex derivative $\varphi'(z)$ at $\zeta$. This forms part of the Julia-Caratheodory Theorem, to which we turn next. In order to state this result efficiently we need some notation. For $\zeta \in \partial U$ let

$$d(\zeta) = \liminf_{z \to \zeta} \frac{1 - |\varphi(z)|}{1 - |z|}$$

where $z$ is allowed to tend unrestrictedly to $\zeta$ through $U$. By the Schwarz Lemma $d(\zeta) > 0$ (if $\varphi(0) = 0$ it is $\geqq 1$).

2.5. *The Julia-Caratheodory Theorem.* Suppose $\zeta \in \partial U$ is fixed. Then the following three conditions on $\varphi$ are equivalent:

(i) $d(\zeta) < \infty$.

(ii) $\varphi$ has an angular derivative at $\zeta$.

(iii) $\varphi'$ has an angular limit at $\zeta$, and $\varphi$ itself has an angular limit of modulus 1 at $\zeta$.

Moreover the quantities in (ii) and (iii) coincide when they exist. If (i) holds (so in particular $\varphi$ has an angular limit $\omega$ of modulus 1 at $\zeta$) and if $\varphi$ is normalized so that $\omega = \zeta$; then in addition the quantities in (i), (ii), and (iii) coincide.

Proofs of this result can be found, for example, in section 5.3 of **[23]**, and (for the unit ball of $\mathbf{C}^N$) in **[27]** Section 8.5. The heart of the proof is a geometric result known as Julia's Lemma, which we will also need in the sequel. It is a sort of Schwarz Lemma, but with a boundary point playing the role of the origin. For its statement we need some notation. If $0 < a < \infty$ and $\zeta \in \partial U$, let

$$K(\zeta, a) = \{z \in U : |\zeta - z|^2 < a(1 - |z|^2)\}.$$

A little calculation shows that $K(\zeta, a)$ is the disc of radius $a/(1 + a)$ in $U$ that is tangent to $\partial U$ at $\zeta$.

2.6. Julia's Lemma (**[4]**, page 8, **[6]**). *Suppose $\zeta \in \partial U$ and $d = d(\zeta) < \infty$. Then for every $0 < a < \infty$:*

$$\varphi(K(\zeta, a)) \subset K(\zeta, ad).$$

The second part of the Julia-Caratheodory Theorem insures that if $\varphi$ has an angular derivative at $\zeta$, then it carries any curve in $U$ which ends non-tangentially at $\zeta$ into another non-tangential curve ending at $\varphi^*(\zeta)$ and making the same angle with $\partial U$ as the original curve. That is, $\varphi$ must be "conformal" at $\zeta$: it cannot therefore move points near the unit circle in the vicinity of $\zeta$ too far from that circle. For example $\varphi$ cannot map $U$ into a polygon with a vertex at $\zeta$. This is the intuition behind the necessary condition for compactness discovered by Shapiro and Taylor:

2.7. Proposition (**[30]**, Theorem 2.1). *Suppose $0 < p < \infty$. If $C_\varphi$ is compact on $H^p$ then the angular derivative of $\varphi$ exits at no point of $\partial U$.*

Thus the existence of the angular derivative at even a single point of the unit circle is enough to defeat compactness. In Section 5 we will give a different proof of this result in a more general setting. For the most part however we will be concerned with its converse.

We close this section by recording a well known reformulation of the definition of compactness for composition operators. Its proof, which we omit, follows directly from the fact that bounded subsets of $H^p$ are normal families.

2.8. PROPOSITION ( [**29**], Theorem 2.5). *Necessary and sufficient for the compactness of $C_\varphi$ on $H^p$ is the following: for every sequence $(f_n)$ bounded in $H^p$ and convergent to zero uniformly on compact subsets of $U$, the image sequence $(f_n \circ \varphi)$ converges to zero in the $H^p$ metric.*

**3. Main results: statements and applications.** In this section we introduce the weighted Bergman and Dirichlet spaces of functions holomorphic in $U$, state our main results, and derive their major consequences. Proofs of the main results (Theorems 3.5 and 3.10) will be given in Section 5. The purpose of this section is to show how they extend and simplify earlier work, and how they lead to the construction of examples which illustrate the different nature of the compactness problem for $H^p$ and for Bergman spaces.

3.1. *Notation.* Here $\lambda$ will denote normalized Lebesgue area measure on $U$; and for $\alpha > -1$, $\lambda_\alpha$ will denote the finite measure defined on $U$ by

$$d\lambda_\alpha(z) = (1 - |z|^2)^\alpha d\lambda(z).$$

3.2. *Bergman and Dirichlet spaces.* For $\alpha > -1$ and $0 < p < \infty$ the weighted Bergman space $A^p_\alpha$ is the collection of functions $f$ holomorphic in $U$ for which

$$\|f\|^p_{p,\alpha} = \int_U |f|^p d\lambda_\alpha < \infty.$$

The *weighted Dirichlet* space $D_\alpha(\alpha > -1)$ is the collection of $f$ holomorphic in $U$ for which the complex derivative $f'$ belongs to $A^2_\alpha$. It is well known that $A^p_\alpha$ is a complete linear metric space for all values of $p$, a Banach space if $p \geqq 1$, and a Hilbert space if $p = 2$. The space $D_\alpha$ is a Hilbert space in the norm $\|\cdot\|_{D_\alpha}$ defined by:

$$\|f\|^2_{D_\alpha} = |f(0)|^2 + \int_U |f'|^2 d\lambda_\alpha.$$

For all of these spaces the unit ball is a normal family, and point evaluation is continuous. A standard power series computation using integration in polar coordinates and the orthonormality of the monomials $z^n$ in $L^2(\partial U)$ shows that a function $f(z) = \sum a_n z^n$ holomorphic in $U$ belongs to $A^2_\alpha$ if and only if

$$\sum (n + 1)^{-1-\alpha}|a_n|^2 < \infty,$$

and to $D_\alpha$ if and only if

$$\sum (n+1)^{1-\alpha}|a_n|^2 < \infty,$$

and that the series above define norms equivalent to the original ones. Thus we have:

3.3. PROPOSITION. *$D_1 = H^2$ and if $\alpha > -1$ then $D_{\alpha+2} = A^2_\alpha$. In these equalities there is also equivalence of norms. If $-1 < \alpha < 1$ then $D_\alpha \subset H^2$, the injection map being continuous.*

The next result follows upon integrating both sides of the inequality in Littlewood's Subordination Theorem (Section 2.2), and making the appropriate remarks about automorphisms of $U$.

3.4. PROPOSITION. *Every composition operator $C_\varphi$ is bounded on $A^p_\alpha$ for all $0 < p < \infty$ and $\alpha > -1$.*

By Propositions 3.3 and 3.4, every composition operator on $D_\alpha$ is bounded as long as $\alpha \geqq 1$. However for $\alpha < 1$ this is no longer the case: now $D_\alpha$ is properly contained in $H^2$, and an obvious necessary condition for $C_\varphi$ to be bounded on $D_\alpha$ is that $\varphi$ itself belong to $D_\alpha$ (since $C_\varphi$ takes the identity map on $U$ to $\varphi$). Not all $\varphi$'s have this property, and even some that do fail to induce bounded operators on $D_\alpha$. We will say more about this problem at the end of Section 3. Right now we proceed to state our principal results.

3.5. THEOREM. *Suppose $0 < p < \infty$ and $\alpha > -1$ are given. Then $C_\varphi$ is compact on $A^p_\alpha$ if and only if $\varphi$ has no angular derivative at any point of $\partial U$.*

According to this result the compactness of $C_\varphi$ on $A^p_\alpha$ depends neither on $p$ nor $\alpha$. In fact the independence of $p$ is an important step in the proof of Theorem 3.5: it allows us to concentrate on the case $p = 2$ and exploit the equation $D_{\alpha+2} = A^2_\alpha$ (Proposition 3.3). This phenomenon of independence of $p$ was first noted for Hardy spaces in the disc by Shapiro and Taylor ( **[30]**, Theorem 6.1) who proved it by classical methods having no counterparts in either $A^p$ or the Hardy spaces in special variables. Our technique involves Carleson measures (Section 4): it was first introduced by MacCluer **[20]** to get the same result for the $H^p$ spaces of the unit ball in $\mathbf{C}^N$. Thus the surprising aspect of Theorem 3.5 is not so much that compactness on $A^p_\alpha$ does not depend on $p$, but rather that it does not depend on $\alpha$.

Theorem 3.5 allows us to construct examples which illustrate the differences between the compactness problems for $A^p_\alpha$ and for $H^p$. Before presenting our first one we recall that an *inner function* is a holomorphic function on $U$ with modulus $\leqq 1$ every where on $U$ and radial limit of modulus 1 at almost every point of $\partial U$. It is well known, and not difficult to prove, that no inner function can induce a compact composition operator on any $H^p$ space (see Proposition 3.7). This stands in sharp contrast with the next result.

3.6. *Example.* There exists an inner function $\varphi$ such that $C_\varphi$ is compact on $A^p_\alpha$ for all $0 < p < \infty$ and $\alpha > -1$.

*Proof.* Our example will be a singular inner function ( [**13**], page 24)

$$(1) \quad \varphi(z) = \exp \int_{\partial U} \frac{z + \zeta}{z - \zeta} d\mu(\zeta)$$

where $\mu$ is a positive finite Borel measure on $\partial U$ that is singular with respect to linear Lebesgue measure. A theorem of M. Riesz ( [**25**]; see also [**3**], Theorem 2, page 117) asserts that such a function $\varphi$ has an angular derivative at the point $\omega \in \partial U$ if and only if

$$(2) \quad \int_{\partial U} \frac{d\mu(\zeta)}{|\zeta - \omega|^2} < \infty.$$

Thus, in view of Theorem 3.5, we will be done if we can construct a $\mu$ which fails inequality (2) at every $\omega \in \partial U$.

This is not difficult. Choose positive numbers $(\mu_n)$ such that $\sum \mu_n < \infty$, but $\sum \sqrt{\mu_n} = \infty$. Let $(I_n)$ be a sequence of consecutive arcs on $\partial U$ with length $I_n = \sqrt{\mu_n}$, and let $\zeta_n$ be the center of $I_n$. The measure we seek is the positive atomic measure

$$\mu = \sum \mu_n \delta_n$$

where $\delta_n$ is the unit mass at $\zeta_n$. To see that $\mu$ fails property (2) at each $\omega \in \partial U$ we need only observe that each such $\omega$ belongs to infinitely many intervals $I_n$, hence

$$|\omega - \zeta_n| < \sqrt{\mu_n}$$

for infinitely many $n$, so the series

$$\sum \mu_n/|\zeta_n - \omega|^2$$

diverges since infinitely many terms are $> 1$. But this series is just the integral on the left-hand side of (2), so the proof is complete.

As we mentioned previously, $C_\varphi$ will never be compact on $H^p$ when $\varphi$ is an inner function. The above example therefore furnishes a negative answer to the question of Shapiro and Taylor about the possibility of a converse to Proposition 2.7. What is behind all of this is yet another necessary condition for $H^p$-compactness of $C_\varphi$. This one is due to Schwartz ( [**29**], Theorem 3.6). In the interests of completeness we will also sketch the proof. Recall that $\varphi^*(\zeta)$ is the radial limit of $\varphi$ at $\zeta \in \partial U$.

3.7. PROPOSITION. *Fix* $0 < p < \infty$. *If* $C_\varphi$ *is compact on* $H^p$ *then* $|\varphi^*(\zeta)| < 1$ *for almost every* $\zeta \in \partial U$.

*Proof.* Suppose $C_\varphi$ is compact, and let $E$ be the set of points $\zeta \in \partial U$ such that $|\varphi^*(\zeta)| = 1$. Define $e_n \in H^p$ for $n = 1, 2, 3, \ldots$ by

$$e_n(z) = z^n \quad (z \in U).$$

So $\|e_n\| = 1$ and $e_n \to 0$ uniformly on compact subsets of $U$. By Proposition 2.8, $C_\varphi e_n = \varphi^n$ tends to zero in the $H^p$ norm. Since this norm can be computed on the boundary of $U$ we have (letting $m$ be normalized Lebesgue measure on $\partial U$):

$$m(E) \leqq \int_{\partial U} |\varphi^*|^{np} dm = \|\varphi^n\|_p^p \to 0,$$

hence $m(E) = 0$, as desired.

In view of Example 3.6 and Proposition 3.7 it appears that Shapiro and Taylor should have asked if non-existence of the angular derivative and radial limits of modulus $< 1$ a.e. might characterize compactness. It would not have helped: this too is false, as shown by the following example.

3.8. *Example.* There exists $\varphi$ such that $C_\varphi$ is non-compact on each $H^p$ space, yet such that:

(i) $\varphi$ has an angular derivative at no point of $\partial U$, and
(ii) $|\varphi^*(\zeta)| < 1$ for a.e. $\zeta$ in $\partial U$.

*Proof.* Let $\psi_0$ denote the atomic inner function constructed in example 3.6. Let $a = \psi(0)$, and consider the inner function

$$\psi(z) = (a - \psi_0(z))/(1 - \bar{a}\psi_0(z)) \quad (z \in U),$$

which takes the origin to itself. We claim that the function $\varphi$ defined by

$$\varphi(z) = (1 + \psi(z))/2 \quad (z \in U)$$

is the desired example. Clearly $|\varphi^*(\zeta)| < 1$ for almost every $\zeta$ in $\partial U$, even though $\|\varphi\|_\infty = 1$. To see that $C_\varphi$ is not compact on $H^p$, write

$$\varphi = \chi \circ \psi \quad \text{where } \chi(z) = (1 + z)/2.$$

Then $C_\varphi = C_\psi C_\chi$ (note the order of the factors). As we have commented several times, $C_\chi$ is not compact on $H^p$ (Proposition 2.7, for example), so it takes the unit ball of $H^p$ into a set $A$ whose closure is not compact. But since $\psi$ is an inner function which fixes the origin, $C_\psi$ is an isometry of $H^p$ into itself (see, for example, [**24**], page 443), so $C_\psi(A)$ does not have compact closure either. But $C_\psi(A)$ is the image under $C_\varphi$ of the $H^p$ unit ball, hence $C_\varphi$ is not a compact operator on $H^p$.

On the other hand, Theorem 3.5 asserts that any $\varphi$ with no radial limit of modulus 1 must induce a compact operator on Bergman spaces:

3.9. COROLLARY. *If $\varphi$ has no radial limit of modulus* 1 *then $C_\varphi$ is compact on $A^p_\alpha$ for all $0 < p < \infty$ and $\alpha > -1$.*

*Proof.* Recall that if $\varphi$ has an angular derivative at $\zeta \in \partial U$, then $|\varphi^*(\zeta)| = 1$. Thus the hypothesis of the Corollary asserts that $\varphi$ has an angular derivative at no point of the unit circle; and the result follows from Theorem 3.5.

The above results show that for arbitrary $\varphi$ the compactness problem for $C_\varphi$ on $H^p$ is quite different from that on Bergman spaces. However our next result shows that a slight additional restriction on $\varphi$ causes all these differences to disappear. It is here that we encounter for the first time the weighted Dirichlet spaces $D_\alpha$ defined in Section 3.2. Recall that if $\alpha < 1$ then $D_\alpha$ is properly contained in $H^2$, and not every $\varphi$ induces a bounded composition operator on such a $D_\alpha$.

3.10. THEOREM. *Suppose the angular derivative of $\varphi$ exists at no point of $\partial U$. If in addition $C_\varphi$ is bounded on $D_\alpha$ for some $-1 < \alpha < 1$, then $C_\varphi$ is compact on $H^p$ for all $0 < p < \infty$.*

The change of variable formula for multiple integrals shows that if $\varphi$ is univalent, or just of bounded multiplicity, then $C_\varphi$ is a bounded operator on $D_0$, the standard unweighted Dirichlet space. So the most natural examples $\varphi$ do satisfy the additional restriction of Theorem 3.10, and for them the characterization of $H^p$-compactness of $C_\varphi$ is the same as for Bergman spaces. As a result, the main sufficient condition for compactness employed by Shapiro and Taylor ( [**30**], Theorem 2.4) can now be replaced by a precise and easily employed geometric criterion.

3.11. COROLLARY. *Suppose $\Omega$ is a domain in $U$ whose boundary is a Jordan curve which touches the unit circle only at the point* 1. *Suppose that in some neighborhood of* 1 *the boundary of $\Omega$ has Cartesian equation $1 - x = h(y)$ with $h$ positive and continuous for $0 < |y| < \delta$.* Suppose $\varphi$ maps U univalently onto $\Omega$ with $\varphi(1) = 1$. *Then $C_\varphi$ is compact on $H^p$ if and only if*

$$\int_{-\delta}^{\delta} y^{-2}h(y)dy = \infty.$$

*Remark.* Here the statement "$\varphi(z) = 1$" is not ambiguous, because $\varphi$ extends (uniquely) to a homeomorphism of the closed unit disc onto the closure of $\Omega$ (see for example [**26**], Section 14.19 and 14.20). The corollary itself follows immediately from the boundedness of $C_\varphi$ on $D_0$, Theorem 3.10, and the angular derivative criterion of Tsuji ( [**32**], Theorem IX.10, page 377), translated from the upper half-plane to the unit disc.

As an illustration of the usefulness of this last result, note that it yields all the known results mentioned in Section 2.3. The importance of univalent maps $\varphi$ in the study of composition operators is this: if $\varphi$

is univalent and $C_\varphi$ compact on $H^p$ (or $A^p_\alpha$ for that matter) then so is $C_\psi$ for any $\psi$ holomorphic on $U$ with $\psi(U) \subset \varphi(U)$. The reason is that $\psi = \varphi \circ \omega$ where $\omega = \varphi^{-1} \circ \psi$ is holomorphic on $U$ with $\omega(U) \subset U$. Thus $C_\psi = C_\omega C_\varphi$ where $C_\omega$ is bounded and $C_\varphi$ compact, hence $C_\psi$ itself is compact.

The following example shows that there is no corresponding result for non-compactness: we can have $\varphi(U) \supset \psi(U)$ with $\psi$ univalent and $C_\psi$ non-compact, yet $C_\varphi$ may be compact; even if $\psi(z) = z$.

3.12. *Example.* There exists $\varphi$ such that $\varphi(U) = U$, yet $C_\varphi$ is compact on $H^p$ for all $p < \infty$.

*Proof.* Let $\Omega$ be an infinite curvilinear strip in the left half-plane whose boundaries are asymptotic to the positive $y$-axis. It is not difficult to arrange $\Omega$ so that the exponential map takes it onto $U\backslash\{0\}$ while covering each point of this set no more than a fixed number of times; say twice. Let $F$ be a univalent map of $U$ onto $\Omega$. Then $\varphi_0 = e^F$ is almost the function we want: it maps $U$ onto $U\backslash\{0\}$ with multiplicity at most 2, and has no radial limit of modulus 1. Thus by Theorem 3.10, the operator $C_{\varphi_0}$ is compact on $H^p$.

We complete the proof by modifying $\varphi_0$, without destroying the compactness of $C_{\varphi_0}$, to include the value 0. Let $a \in U\backslash\{0\}$ be fixed, and set

$$\psi(z) = z\left(\frac{a - z}{1 - \bar{a}z}\right).$$

Then $\psi$ maps $U$ onto itself with multiplicity two, and

$$\psi(U\backslash\{0\}) = U.$$

Let $\varphi = \psi \circ \varphi_0$. Then $\varphi(U) = U$ and $C_\varphi = C_{\varphi_0} C_\psi$ is the product of a compact and a bounded operator, so it is compact.

Theorem 3.10 raises the question of determining when $C_\varphi$ is bounded on $D_\alpha$ for $-1 < \alpha < 1$. The results we have just proved show that this problem is not a trivial one.

3.12. PROPOSITION. *There exists a function $\varphi$ in*

$$\cap\{D_\alpha : \alpha > 1/2\}$$

*for which $C_\varphi$ is bounded on no space $D_\alpha$ for $\alpha < 1$.*

*Proof.* Our example is the function $\varphi$ constructed in example 3.6, with the weights $\mu_n$ chosen so that

$$\sum \mu_n^\alpha < \infty \quad \text{for each } \alpha > 1/2.$$

Ahern [**1**] has made a definitive study of the $D_\alpha$ classes to which such atomic inner functions can belong. Upon translating his Lemma 4.1 and Theorem 4.2 (page 333) into our notation we see that the singular inner function $\varphi$ associated with the atomic measure $\mu$ above belongs to $D_\alpha$ for every $\alpha > 1/2$. But it cannot belong to $D_{1/2}$ by the main result of [**3**] and by Lemma 4.1 of [**1**]. So certainly $C_\varphi$ is not bounded on $D_\alpha$ for $\alpha \leqq 1/2$. The same is true for $1/2 < \alpha < 1$, for if not then by Theorem 3.10 the operator $C_\varphi$ would be compact on $H^2$ since $\varphi$ is constructed to have angular derivative at no point of $\partial U$. But this contradicts the fact that $\varphi$ is an inner function (Proposition 3.7).

It remains for us to prove Theorems 3.5 and 3.10. This will be the subject of the next two sections.

**4. Carleson measures for $A^p_\alpha$.** In this section we study the relationship between compact composition operators on $A^p_\alpha$ and a special class of measures on the unit disc. Such Carleson measures have already been employed in [**20**] to study compact composition operators on the $H^p$ spaces of the unit ball of $\mathbf{C}^N$ for $N > 1$, and have in addition proven their utility in a variety of different situations in complex analysis (see [**7**], [**17**], [**22**], and [**31**] for example).

4.1. *Notation.* For $0 < \delta \leqq 2$ and $\zeta \in \partial U$ let

$$S(\zeta, \delta) = \{z \in U: |z - \zeta| < \delta\}.$$

The $\lambda_\alpha$-measure of the semidisc $S(\zeta, \delta)$ is easily seen to be comparable with $\delta^{\alpha+2}$ $(\alpha > -1)$. This furnishes some motivation for the definitions below.

4.2. *Definitions.* Suppose $\mu$ is a finite positive Borel measure on $U$, and $\alpha > -1$. We call $\mu$ an *$\alpha$-Carleson measure* if

$$\|\mu\|_\alpha = \sup \mu(S(\zeta, \delta))/\delta^{\alpha+2} < \infty,$$

where the supremum is extended over all $\zeta \in \partial U$ and $0 < \delta \leqq 2$. If in addition

$$\lim_{\delta \to 0} \sup_{\zeta \in \partial U} \mu(S(\zeta, \delta))/\delta^{\alpha+2} = 0,$$

then we call $\mu$ a *compact* $\alpha$-Carleson measure.

Informally then, $\mu$ is $\alpha$-Carleson if

$$\mu(S) = O(\lambda_\alpha(S)) \quad \text{for all } S = S(\zeta, \delta),$$

and compact if "$O$" can be replaced by "$o$". The next result explains the connection between these measures and the spaces $A^p_\alpha$.

4.3. THEOREM. *Fix $0 < p < \infty$ and $\alpha > -1$, and let $\mu$ be a finite positive Borel measure on $U$. Then*: (a) *$\mu$ is an $\alpha$-Carleson measure if and only if $A^p_\alpha \subset L^p(\mu)$. In this case the identity map*

$$I_\alpha : A^p_\alpha \to L^p(\mu)$$

*is a bounded linear operator with norm comparable with* $\|\mu\|_\alpha$.

(b) *If* $\mu$ *is an* $\alpha$*-Carleson measure then* $I_\alpha$ *is compact if and only if* $\mu_\alpha$ *is compact.*

*Remarks.* Part (a) was proved first for the case $\alpha = 0$ by Hastings [**15**], and for general $\alpha$ by Stegenga ( [**31**], Theorem 1.2). A general method for proving the "only if" direction has been provided by Luecking [**17**]. Part (b) occurs in the context of $H^p$ spaces in several variables in [**20**], and it has occurred in the work of several authors ( [**10**], [**21**], [**33**] for example) as a device for studying compact Toeplitz operators on Bergman spaces in both one and several variables. The proof below of part (b) is presented more in the interests of completeness than originality.

*Proof of Part* (b). Suppose $I_\alpha$ is a compact operator from $A^p_\alpha$ into $L^p(\mu)$. Fix any $\beta > (\alpha + 2)/p$ and for each $0 < \delta < 1$ and $\zeta \in \partial U$ define

$$f_\delta(z) = \frac{\delta^{\beta-(\alpha+2)/p}}{(\zeta - (1-\delta)z)^\beta}.$$

Then a routine calculation shows that these functions form a bounded subset of $A^p_\alpha$. Clearly they tend to zero uniformly on compact subsets of $U$ as $\delta \to 0$. Reasoning as in the proof of Proposition 3.7 we conclude that the compactness of $I_\alpha$ forces

$$(1) \quad \epsilon(\delta) = \int_U |f_\delta|^p d\mu \to 0$$

as $\delta \to 0$. Suppose $S = S(\zeta, \delta)$ for $0 < \delta < 1$. Then

$$|\zeta - (1-\delta)z| < 2\delta,$$

on $S$, so on that set,

$$|f_\delta(z)|^p > 1/(2^{\beta p}\delta^{\alpha+2})$$

hence

$$\mu(S)/(2^{\beta p}\delta^{\alpha+2}) \leqq \int_S |f_\delta|^p d\mu \leqq \epsilon(\delta).$$

This last inequality and (1) above show that $\mu$ is a compact $\alpha$-Carleson measure.

Suppose conversely that $\mu$ is such a measure. We wish to prove that $I_\alpha$ is compact. By now – familiar reasoning it is enough to show that each bounded sequence $(f_n)$ in $A^p_\alpha$ that is convergent to zero uniformly on compact subsets of $U$ must be norm-convergent to zero in $L^p(\mu)$. Fix $0 < \delta < 1$ and let $\mu_\delta$ be the restriction of the measure $\mu$ to the annulus $1 - \delta < |z| < 1$. Then it is easy to see that "$\alpha$-Carleson norm" of $\mu_\delta$ is

$$(2) \quad \|\mu_\delta\|_\alpha \leqq K \sup \mu(S(\zeta, t))/t^{\alpha+2}$$

where the supremum is extended over all $0 < t < \delta$ and $\zeta \in \partial U$; and $K$ is a positive constant which depends only on $\alpha$.

Since $\mu$ is a compact $\alpha$-Carleson measure, the right-hand side of (2) tends to zero as $\delta \to 0$. Denoting it by $\epsilon(\delta)$ we have

$$\int_U |f_n|^p d\mu = \int_{|z|<1-\delta} |f_n|^p d\mu + \int_U |f_n|^p d\mu_\delta$$

$$\leqq o(1) + K\epsilon(\delta) \, \|f_n\|_{p,\alpha}^p$$

as $\delta \to 0$, where $K$ is another constant depending only on $\alpha$ and $p$. Here the estimate of the first term comes from the uniform convergence of $(f_n)$ to zero on $|z| \leqq 1 - \delta$, and the estimate of second from part (a) of the present theorem. Since $\epsilon(\delta) \to 0$ as $\delta \to 0$, we are done.

Here is the Carleson measure characterization of the compact composition operators on the spaces $A_\alpha^p$.

4.4. Corollary. *Fix $0 < p < \infty$ and $\alpha > -1$. Then $C_\varphi$ is a compact operator on $A_\alpha^p$ if and only if the measure $\lambda_\alpha\varphi^{-1}$ is a compact $\alpha$-Carleson measure.*

*Proof.* By the change of variable formula of measure theory (see **[14]**, Theorem C, page 163) we have for each $f$ in $A_\alpha^p$

$$(2) \quad \int_U |f \circ \varphi|^p d\lambda_\alpha = \int |f|^p d\lambda_\alpha\varphi^{-1}$$

where $\lambda_\alpha\varphi^{-1}$ is the measure which assigns mass $\lambda_\alpha\{\varphi^{-1}(B)\}$ to each Borel set $B$ of the unit disc. Recall that each $C_\varphi$ is a bounded operator on $A_\alpha^p$ (Proposition 3.4), so by Theorem 4.3 and equation (2) we know that $\lambda_\alpha\varphi^{-1}$ is an $\alpha$-Carleson measure.

Now let $X$ denote the (usually incomplete) space $A_\alpha^p$ taken in the metric of $L^p(\lambda_\alpha\varphi^{-1})$. By equation (2) the operator $C_\varphi$ sets as an isometry $\widetilde{C}_\varphi$ of $X$ into $A_\alpha^p$. Thus $C_\varphi = \widetilde{C}_\varphi I_\alpha$ is compact if and only if $I_\alpha$ is. This observation and Theorem 4.3 complete the proof.

We have already commented on the importance of the next result, whose proof is an obvious consequence of Corollary 4.4.

4.5. Corollary. *Fix $\alpha > -1$. If $C_\varphi$ is compact on $A_\alpha^p$ for some $0 < p < \infty$ then it is compact on $A_\alpha^p$ for all such $p$.*

As we have mentioned previously, this result allows us to concentrate on the case $p = 2$, where Proposition 3.3 asserts that everything can be done in the context of weighted Dirichlet spaces.

**5. Composition operators on $D_\alpha$.** The main result of this section (Theorem 5.3) concerns compact composition operators on the Dirichlet spaces $D_\alpha$. It yields as a consequence the as yet unproven Theorems 3.5

and 3.10 about Hardy and Bergman spaces. We begin with the analogue for Dirichlet spaces of Corollary 4.4: the Carleson measure characterization of compact composition operators. As always, $\varphi(U) \subset U$.

5.1. PROPOSITION. *Suppose* $\alpha > -1$ *and* $\varphi \in D_\alpha$. *Let* $\mu_\alpha$ *be the measure defined on* $U$ *by*

$$\begin{aligned} d\mu_\alpha(z) &= |\varphi'(z)|^2 d\lambda_\alpha(z) \\ &= |\varphi'(z)|^2(1 - |z|^2)^\alpha d\lambda(z). \end{aligned}$$

*Then* $C_\varphi$ *is a bounded (respectively: compact) operator on* $D_\alpha$ *if and only if* $\mu_\alpha\varphi^{-1}$ *is an* $\alpha$*-Carleson (respectively: compact* $\alpha$*-Carleson) measure in* $U$.

Since the proof of this result is entirely similar to that of Corollary 4.4, we leave it to the reader; except to note that the new factor $|\varphi'(z)|^2$ in the measure arises from the derivative in the definition of the $D_\alpha$ norm and the chain rule.

Although the next result is not required for the proof of Theorem 5.3, it does help to illuminate both the hypotheses and the proof of that theorem. It is also of independent interest.

5.2. COMPARISON THEOREM. *Suppose* $-1 < \alpha < \beta$ *and* $C_\varphi$ *is a bounded (respectively: compact) operator on* $D_\alpha$. *Then the same is true of* $C_\varphi$ *on* $D_\beta$.

*Proof.* Suppose first that $C_\varphi$ is compact on $D_\alpha$. We may without loss of generality assume that $\varphi(0) = 0$, since each conformal automorphism of $U$ induces an isomorphic composition operator on $D_\alpha$. By the Schwarz Lemma:

(1) $\quad 1 - |\varphi(z)|^2 \geqq 1 - |z|^2 \quad (z \text{ in } U).$

Now recall the measure $\mu_\alpha$ of Proposition 5.1. According to that Proposition our hypothesis on $C_\varphi$ says that $\mu_\alpha\varphi^{-1}$ is a compact $\alpha$-Carleson measure. Thus we have for all $\zeta \in \partial U$ and $0 < \delta \leqq 2$:

(2) $\quad \mu_\alpha\varphi^{-1}(S(\zeta, \delta)) \leqq \epsilon(\delta)\delta^{\alpha+2}$

where $0 < \epsilon(\delta) \to 0$ as $\delta \to 0$. Our goal is to prove the same sort of inequality with $\alpha$ replaced on both sides by $\beta$.

Fix $S = S(\zeta, \delta)$. By inequality (1) we know that

(3) $\quad \varphi(z) \in S \Rightarrow 1 - |z| < \delta,$

so by the definition of $\mu_\beta$:

$$\begin{aligned} \mu_\beta\varphi^{-1}(S) &= \int_{\varphi^{-1}(S)} |\varphi'(z)|^2(1 - |z|^2)^\beta d\lambda(z) \\ &\leqq \delta^{\beta-\alpha} \int_{\varphi^{-1}(S)} |\varphi'(z)|^2(1 - |z|^2)^\alpha d\lambda(z) \quad \text{(by (3))} \\ &= \delta^{\beta-\alpha}\mu_\alpha\varphi^{-1}(S) \end{aligned}$$

$$\leqq \epsilon(\delta)\sigma^{\beta+2} \quad \text{(by (2))}.$$

Thus $\mu_\beta\varphi^{-1}$ is a compact $\alpha$-Carleson measure, hence by Proposition 5.2, $C_\varphi$ is a compact operator on $D_\beta$.

For boundedness the same proof applies, but now $\epsilon(\delta) \equiv$ constant.

5.3. MAIN THEOREM. *Suppose* $\alpha > -1$.

(a) *If* $C_\varphi$ *is a compact operator on* $D_\alpha$ *then* $\varphi$ *does not have an angular derivative at any point of* $\partial U$.

(b) *Suppose conversely that the angular derivative of* $\varphi$ *fails to exist at each point of* $\partial U$. *If in addition* $C_\varphi$ *is bounded on* $D_\gamma$ *for some* $-1 < \gamma < \alpha$, *then* $C_\varphi$ *is compact on* $D_\alpha$.

*Remarks.* Since $D_1 = H^2$, Theorem 3.10 for $p = 2$ is just a restatement of Theorem 5.3 for the case $\alpha = 1$. The same is true of Theorem 3.5 once we recall that if $\gamma \geqq 1$ then $C_\varphi$ is always bounded on $D_\gamma (= A^2_{\gamma-2}$ if $\gamma > 1$ and $H^2$ if $\gamma = 1)$. The fact that compactness of $C_\varphi$ on $H^p$ or $A^p_\alpha$ does not depend on the index $p$ (Theorem 4.5 and **[30]**, Theorem 6.1) finishes the proofs of Theorems 3.5 and 3.10.

We have already remarked that the case $\alpha = 1$ of Theorem 5.3(a) has occurred in the work of Shapiro and Taylor ( **[30]**, Theorem 2.1). For $\alpha = 2$ it is due to D. M. Boyd ( **[5]**, Theorem 3.4). Here we give a different, more geometric, proof.

*Proof of theorem.* (a) Suppose that $\varphi$ has an angular derivative at some point $\zeta \in \partial U$. We will show that $C_\varphi$ is not compact on $D_\alpha$. By rotating the disc if necessary we may assume that $\varphi^*(\zeta) = \zeta$, so by the second part of Theorem 2.5, $\varphi'(\zeta) = d$ with $0 < d < \infty$. Julia's Lemma (Theorem 2.6) now asserts that

(1) $\quad \varphi^{-1}(K(\zeta, ad)) \supset K(\zeta, a)$

for each $a > 0$, where we recall that $K(\zeta, a)$ is the open disc in $U$ of radius $a/(1 + a)$ that is tangent to $\partial U$ at $\zeta$.

Now suppose $0 < \delta \leqq 1$ is fixed, and write $S = S(\zeta, \delta)$. Our goal is to show that for some constant $c > 0$ which does not depend on $\delta$,

(2) $\quad \mu_\alpha\varphi^{-1}(S) \geqq c\delta^{\alpha+2}.$

This, by Proposition 5.1, will establish the non-compactness of $C_\varphi$.

To this end, fix a triangle $\Delta$, lying entirely in $U$ except for a vertex at $\zeta$, on which $|\varphi'| > d/2$. Such a triangle exists by Theorem 2.5(iii): of course its choice does not depend on $\delta$. Let $a = \delta/(2 - \delta)$ and let $K_0 = K(\zeta, a)$, so $S \supset K_0$. Let $K = K(\zeta, a/d)$ so by (1) above,

$$\varphi^{-1}(K_0) \supset K.$$

Using successively these containments and the lower bound for $|\varphi'|$ on $\Delta$:

$$\begin{aligned}\mu_\alpha\varphi^{-1}(S) &\geqq \mu_\alpha\varphi^{-1}(K_0) \geqq \mu_\alpha(K) \geqq \mu_\alpha(K \cap \Delta)\\ &= \int_{K\cap\Delta} |\varphi'(z)|^2(1 - |z|^2)^\alpha d\lambda(z)\\ &\geqq (d/2)^2 \int_{K\cap\Delta} (1 - |z|^2)^\alpha d\lambda(z).\end{aligned}$$

Now for $z \in \Delta$ we have $1 - |z|^2 \geqq c|1 - z|$, where the constant $c$ depends only on the geometry of $\Delta$. To simplify notation, let $c$ from now on denote a positive constant, which may change with each occurrence, but which is always independent of $\delta$. The last two inequalities yield:

(3) $\mu_\alpha\varphi^{-1}(S) \geqq c \int_{K\cap\Delta} |1 - z|^\alpha d\lambda(z).$

The integral on the right side of inequality (3) is easily evaluated for $\delta$ small enough by changing to polar coordinates, say $\rho$ and $\theta$, with the origin at the point $\zeta$, so now $|1 - z| = \rho$. The resulting calculation, the details of which we leave to the reader, shows that this integral is $\geqq$

$$c(\text{diameter of } K)^{\alpha+2} \geqq c\delta^{\alpha+2}.$$

The last two inequalities yield (2), which completes the proof of part (a).

(b) Suppose now that the hypotheses of part (b) are satisfied. For $0 < \delta < 2$ define

$$h(\delta) = \sup\left\{\frac{1 - |z|^2}{1 - |\varphi(z)|^2} : 1 - |z| \leqq \delta\right\}.$$

By Theorem 2.5 (Julia-Caratheodory), the hypothesis of non-existence of the angular derivative is equivalent to

(4) $\lim_{\delta\to 0} h(\delta) = 0.$

Without loss of generality we may assume that $\varphi(0) = 0$: in particular this implies $h(\delta) \leqq 1$ for all $\delta$, and it provides the following refinement of inequality (3) which appeared in the proof of the last theorem. Fix $S = S(\zeta, \delta)$ and suppose $\varphi(z) \in S$. Then by the Schwarz Lemma, as before:

$$\delta \geqq 1 - |\varphi(z)| \geqq 1 - |z|.$$

Thus the definition of $h(\delta)$ implies:

(5) $1 - |z|^2 \leqq (1 - |\varphi(z)|^2)h(\delta) \leqq 2\delta h(\delta)$

whenever $z \in \varphi^{-1}(S)$, so we have

$$\begin{aligned}\mu_\alpha\varphi^{-1}(S) &= \int_{\varphi^{-1}(S)} |\varphi'(z)|^2(1 - |z|^2)^\alpha d\lambda(z)\\ &\leqq (2\delta h(\delta))^{\alpha-\gamma} \int_{\varphi^{-1}(S)} |\varphi'(z)|^2(1 - |z|^2)^\gamma d\lambda(z) \quad \text{(by (5))}\\ &= \epsilon(\delta)\delta^{\alpha-\gamma}\mu_\gamma\varphi^{-1}(S)\end{aligned}$$

where

$$\epsilon(\delta) = [2h(\delta)]^{\alpha-\gamma} \to 0 \quad \text{as } \delta \to 0$$

by (4) and the fact that $\alpha > \gamma$. Now we use the hypothesis that $C_\varphi$ is $D_\gamma$-bounded, so by Proposition 5.1 there exists a constant $K$ independent of $\zeta$ and $\delta$ such that

$$\mu_\gamma\varphi^{-1}(S) \leqq K\delta^{\gamma+2}.$$

Thus

$$\mu_\alpha\varphi^{-1}(S) \leqq K\epsilon(\delta)\delta^{\alpha+2},$$

and since $\epsilon(\delta) \to 0$ as $\delta \to 0$, $\mu_\alpha$ is therefore a compact $\alpha$-Carleson measure. Thus $C_\varphi$ is compact on $D_\alpha$ by Proposition 5.1, and the proof is complete.

We note in closing that the sort of argument used to prove part (a) of Theorem 5.3 also yields quite simply an important known result about functions $\varphi$ which induce compact composition operators.

5.4. THEOREM. *If $\alpha > -1$ and $C_\varphi$ is compact on $D_\alpha$, then $\varphi$ has a fixed point in $U$.*

*Proof.* Suppose that $\varphi$ has no fixed point in $U$. Then the "approximate fixed point" argument which initiates the proof of the Denjoy-Wolff Theorem (see [6], for example) shows that there exists a point $\zeta \in U$ and a sequence $(z_n)$ in $U$ such that $z_n \to \zeta$, $\varphi(z_n) \to \zeta$, and

$$\lim_n \frac{1 - |\varphi(z_n)|}{1 - |z_n|} = a \leqq 1.$$

Thus Julia's Lemma applies, and as in the proof of Theorem 5.3 (a), the measure $\mu_\alpha\varphi^{-1}$ is not a compact $\alpha$-Carleson measure.

This result was first proved for $H^2$ (the case $\alpha = 1$ here) by Caughran and Schwartz [**8**], and by MacCluer [**19**] for the $H^p$ spaces of the unit ball of $\mathbf{C}^N$. Both these proofs rely on the full strength of the Denjoy-Wolff Theorem: our contribution is to point out that only Julia's Lemma is needed. The result itself plays an important role in the description of the spectrum of a compact composition operator. The proof given here works as well for the Hardy, Bergman, and Dirichlet spaces of the unit ball: the situation which we consider in the next section.

**6. Results in several variables.** In this final section we shift the setting from the unit disc to the open unit ball $B_N$ of $\mathbf{C}^N$ when $N > 1$. Here $\varphi$ denotes a holomorphic map from $B_N$ into itself, and the composition operator $C_\varphi$ acts on complex valued functions holomorphic on $B_N$. Unfortunately $C_\varphi$ need no longer map the Hardy and Bergman spaces of $B_N$ into themselves: see [**19**], Section 2; and Corollary 6.9 below. In fact Cima and Wogen [**11**] have constructed holomorphic homeomorphisms of $B_N$ into itself which induce unbounded composition operators on $H^p(B_N)$ for $p < \infty$. Their examples even extend homeomorphically to the closed ball. Thus it is already a major unsolved problem to decide which maps $\varphi$ induce bounded operators on these spaces.

Initial progress on the boundedness problem has been made by MacCluer [**20**] and Cima, Stanton, and Wogen [**9**]. MacCluer studies mappings $\varphi$ which take $B_N$ into *Koranyi approach regions*. These are the regions

$$D_a(\zeta) = \left\{ z \in B_N : |1 - \langle z, \zeta \rangle| < \frac{a}{2}(1 - |z|^2) \right\}$$

in $B_N$, where $a > 1$ and $\zeta$ (the "vertex") $\in \partial B_N$. In the function theory of $B_N$ these regions play the role of the non-tangential approach regions used in the study of boundary behavior of holomorphic functions on the unit disc (see [**27**], Chapter 5). In fact the intersection of $D_a(\zeta)$ with the complex line through $\zeta$ is just such a non-tangential approach region; however in the complex directions orthogonal to $\zeta$ the region looks like a sphere tangent to $\partial B$ at $\zeta$ ([**27**], Section 5.4.1).

The main result of [**20**] is that there exists a critical "aperature" $a(N)$ such that: (i) $C_\varphi$ is bounded on $H^p(B_N)$ $(0 < p < \infty)$ whenever

$$\varphi(B_N) \subset D_{a(N)}(\zeta) \quad \text{for some } \zeta;$$

(ii) $C_\varphi$ is compact on $H^p(B_N)$ $(0 < p < \infty)$ if

$$\varphi(B_N) \subset D_a(\zeta) \quad \text{for some } 1 < a < a(N);$$

and (iii) these results are sharp in the sense that non-compact $C_\varphi$'s exist in case (i), and if $a > a(N)$ then there exist maps $\varphi$ such that $\varphi(B) \subset D_a(\zeta)$ yet $C_\varphi$ is not bounded on any $H^p(B_N)$ $(0 < p < \infty)$.

Carleson measure arguments play a crucial role in [**20**], as they do here; and they are also employed there to show that boundedness or compactness of $C_\varphi$ on $H^p(B_N)$ does not depend on $0 < p < \infty$. Cima, Stanton, and Wogen [**9**] prove that $C_\varphi$ is bounded for univalent maps

$\varphi$ with bounded Fréchet derivative and Jacobian bounded away from zero in $B_N$. Unfortunately the restriction of boundedness on $\varphi'$, just as in the case $N = 1$, does not allow $C_\varphi$ to be compact (see Theorem 6.6 below). In the first part of this section we obtain the conclusion of Cima, Stanton, and Wogen for the Hardy, Bergman, and certain of the Dirichlet spaces of $B_N$ under the weaker hypothesis that the inverse of $\varphi$ have bounded Fréchet derivative on $\varphi(B_N)$. For such $\varphi$ we characterize the compactness of $C_\varphi$ as in the case of the unit disc: in terms of the non-existence of the "angular derivative".

The situation for general $\varphi$ differs markedly however from that in one variable. In the second half of this section we present a class of examples which show that non-existence of the angular derivative need not imply compactness, or even boundedness, for $C_\varphi$ on the weighted Bergman spaces $A^p_\alpha(B_N)$; and certainly not on $H^p(B_N)$. We show in addition that for different values of $\alpha$ these Bergman spaces have different classes of compact composition operators, again in contrast with the situation for $N = 1$ (Theorem 3.5).

6.1. *Notation.* We follow primarily the notation of [**27**]. In particular $\langle , \rangle$ denotes the complex inner product on $\mathbf{C}^N$, $\sigma$ the rotation-invariant Borel probability measure on $\partial B$, and $\nu$ the normalized Lebesgue volume measure on $B_N$ itself. For $\alpha > -1$ write $\nu_\alpha$ for the weighted measure

$$d\nu_\alpha(z) = (1 - |z|^2)^\alpha d\nu(z)$$

on $B_N$. The *weighted Bergman* space $A^p_\alpha(B_N)$ $(0 < p < \infty, \alpha > -1)$ is the collection of functions holomorphic on $B_N$ which belong to $L^p(\nu_\alpha)$.

For $f$ holomorphic on $B_N$ we write $\nabla f(z)$ for the complex gradient of $f$ at $z$:

$$\nabla f(z) = (D_1 f(z), \ldots, D_N f(z))$$

where $D_j = \partial/\partial z_j$. The weighted Dirichlet space $D_\alpha(B_N)$ is the collection of functions holomorphic in $B_N$ for which $\nabla F$ belongs to $A^2_\alpha(B_N)$. All of these spaces are metrized exactly as in the case $N = 1$ (Section 3.2); and we continue to employ the notation $\|\cdot\|_{p,\alpha}$ for the norm of $A^p_\alpha(B_N)$, and $\|\cdot\|_{D_\alpha}$ for that of $D_\alpha$. In the same vein we use the symbol $\|\cdot\|_p$ to denote the norm in both $H^p(B_N)$ and in $L^p(\sigma) = L^p(\partial B_N)$. As in the case $N = 1$, the Hardy and Bergman spaces of $B_N$ are, for $p = 2$, special cases of Dirichlet spaces.

6.2. PROPOSITION. *Suppose $f$ is holomorphic on $B_N$ and $f = \sum f_s$ is the expansion of $f$ in a series of holomorphic homogeneous polynomials: $f_s$ of degree $s$ $(s = 0, 1, 2, \ldots)$. Then*

(a) $f \in A^2_\alpha(B_N)$ *if and only if*

$$\sum_{s=0}^{\infty} (s + 1)^{-1-\alpha} \|f_s\|_2^2 < \infty,$$

*and*

(b) $f \in D_\alpha(B_N)$ *if and only if*

$$\sum_{s=0}^{\infty} (s + 1)^{1-\alpha} \|f_s\|_2^2 < \infty.$$

*In particular, Proposition* 3.3 *remains true for* $N > 1$.

6.3. *Carleson measures.* We note without proof that the Carleson measure conditions for boundedness and compactness of $C_\varphi$ on $A_\alpha^p$ and $D_\alpha$ remain true in several variables, with $\nu_\alpha$ in the role of $\lambda_\alpha$, and now

$$S(\zeta, \delta) = \{z \in B_N : |1 - \langle z, \zeta \rangle| < \delta\}$$

(see [**10**] and [**17**], for example). The corresponding results for $H^p(B_N)$ have been developed by MacCluer [**20**]. In particular the notions of boundedness and compactness for a composition operator $C_\varphi$ on $H^p(B_N)$ or $A_\alpha^p(B_N)$ are independent of $p < \infty$. We can now state our main result on boundedness.

6.4. THEOREM. *Suppose* $\varphi : B_N \to B_N$ *is univalent, and that the Fréchet derivative of* $\varphi^{-1}$ *is bounded on* $\varphi(B_N)$. *Then for every* $\alpha \geqq 0$ *the composition operator* $C_\varphi$ *is bounded on* $D_\alpha(B_N)$. *For* $\alpha > -1$ *and* $0 < p < \infty$ *it is also bounded on* $H^p(B_N)$ *and* $A_\alpha^p(B_N)$.

*Proof.* By our remarks about Carleson measures we need only prove the Bergman and Hardy space results for $p = 2$, so in view of Proposition 6.2 it is only Dirichlet spaces which need to be considered. Because $\varphi$ is univalent we will not have to use Carleson measures to prove the result for $D_\alpha(B_N)$.

Let $\varphi'$ denote the Fréchet derivative of $\varphi$. We may without loss of generality suppose that $\varphi(0) = 0$ since, as in the case $N = 1$, the biholomorphic mappings of $B_N$ onto itself act transitively and induce bounded composition operators (see [**27**], Chapter 2 and Section 5.6.3). Thus for $f \in D_\alpha(B_N)$:

$$(1) \quad \|C_\varphi f\|_{D_\alpha}^2 = |f(0)|^2 + \int_{B_N} |\nabla(f \cdot \varphi)|^2 d\nu_\alpha.$$

Just as in the case $N = 1$, the linear functional of evaluation at a point of $B_N$ is continuous on $D_\alpha$, so we need only estimate the integral on the right side of equation (1), which we denote by $I$. By the Chain Rule:

$$I = \int_{B_N} |\nabla f(\varphi(z))\varphi'(z)|^2 (1 - |z|^2)^\alpha d\nu(z)$$

$$\leqq \int_{B_N} |\nabla f(\varphi(z))|^2 \|\varphi'(z)\|^2 (1 - |z|^2)^\alpha d\nu(z)$$

where $\varphi'(z)$ is the Fréchet derivative of $\varphi$ at $z$, operating as a linear transformation on $\mathbf{C}^N$, and $\|\varphi'(z)\|$ is its Hilbert-Schmidt norm (the square root of the sum of the squares of its matrix entries $D_i\varphi_j(z)$, where $\varphi_i$ is the $i^{\text{th}}$ coordinate function of the mapping $\varphi$).

Let $J\varphi(z)$ be the determinant of $\varphi'(z)$, i.e., the complex Jacobian of $\varphi$. Since $\varphi$ is univalent, $J\varphi$ is never zero on $B_N$. Write

(2) $\Omega(z) = \|\varphi'(z)\|^2 / |J\varphi(z)|^2 \quad (z \in B_N)$.

Then by the last inequality on the integral $I$ and the change of variable formula:

$$I \leqq \int_{\varphi(B_N)} \Omega(\varphi^{-1}(w)) |\nabla f(w)|^2 (1 - |\varphi^{-1}(w)|^2)^\alpha d\nu(w)$$

$$\leqq \int_{\varphi(B_N)} \Omega(\varphi^{-1}(w)) |\nabla f(w)|^2 d\nu_\alpha(w)$$

where the last inequality follows from the Schwarz Lemma for $B_N$ ( **[27]**, Section 8.1), our hypothesis $\varphi(0) = 0$, and the fact that $\alpha \geqq 0$. Thus we need only prove that $\Omega$ is bounded on $B$.

It is here that we use the boundedness of $(\varphi^{-1})$. Let $w = \varphi(z)$ and $\psi = \varphi^{-1}$, so $z = \psi(w)$. By the chain rule

$$\psi'(w)^{-1} = \varphi'(z)$$

so $J\psi(w) = 1/J\varphi(z)$. Thus

$$\Omega(z) = |J\psi(w)|^2 \|\psi'(w)^{-1}\|^2 \quad (z \in B_N,\ w = \varphi(z)).$$

Let $A$ denote the matrix of $\psi'(w)$ with respect to the standard basis of $\mathbf{C}^N$, and $A_{ij}$ the $(i, j)$-cofactor of $A$. Thus

$$A^{-1} = (A_{j,i})_{i,j=1}^n / \det A,$$

so

(3)
$$\begin{aligned} |\Omega(z)| &= |\det A|^2 \|A^{-1}\|^2 \\ &= \sum_{i,j=1}^N |A_{ij}|^2 \\ &\leqq N^2 (\max |a_{ij}|)^{2(N-1)} \quad (1 \leqq i, j \leqq N) \\ &\leqq N^2 \|A\|^{2(N-1)} \\ &= N^2 \|\psi'(w)\|^{2(N-1)}. \end{aligned}$$

Our hypothesis is that $\|\psi'\|$ is bounded on $\varphi(B_N)$. Thus $\Omega$ is bounded on $B_N$, and our proof is complete.

*Remark.* The proof above works. of course under the formally weaker hypothesis that $\Omega$ be bounded on $B_N$, and the same will be true of the compactness result to follow. It might therefore be of interest to study further the significance of the condition that $\Omega$ be bounded. When $N = 2$, equation (3) shows that it is actually equivalent to the boundedness of $(\varphi^{-1})'$, since the cofactor matrix $(A_{ji})$ is then just a signed rearrangement of $A$.

For operators of the sort discussed in the theorem above, our next result characterizes compactness in terms of the "angular derivative" of $\varphi$. We say that $\varphi$ has an *angular derivative* (perhaps "admissible derivative" would be more appropriate) at $\zeta \in \partial B_N$ if there exists $\omega \in \partial B_N$ such that the difference quotient

$$\frac{\langle \omega - \varphi(z), \zeta \rangle}{\langle \omega - z, \zeta \rangle} \quad (z \in B_N)$$

has a limit as $z \to \zeta$ through every Koranyi approach region $D_\alpha(\zeta)$ (i.e., has an admissible limit in the language of [27], Chapter 5). As in the case $N = 1$, existence of the angular derivative at $\zeta$ implies that the complex derivative of $\varphi$ in the direction $\zeta$ has a limit as $z$ tends to $\zeta$ in an appropriately restricted manner, and there is also a quantitative criterion for the existence of the angular derivative. All this is set out in [27], Theorem 8.5.6. We state here only the part of this result which is needed for the proof of our theorem.

6.5. LEMMA. *Suppose $\zeta \in \partial B_N$. If*

$$\underline{\lim}_{z \to \zeta} \frac{1 - |\varphi(z)|^2}{1 - |z|^2} < \infty$$

*where $z$ tends unrestrictedly through $B_N$ to $\zeta$, then the angular derivative of $\varphi$ exists at $\zeta$.*

For the composition operators appearing in Theorem 6.4 we now characterize those which are compact.

6.6. THEOREM. *Suppose $\varphi$ is univalent and $\varphi^{-1}$ has bounded derivative on $\varphi(B_N)$. Fix $\alpha \geqq 0$. Then $C_\varphi$ is compact on $D_\alpha$ if and only if the angular derivative of $\varphi$ exists at no point of $\partial B_N$. The same result holds for $H^p(B_N)$ and $A^p_\alpha(B_N)$ for $0 < p < \infty$ and all $\alpha > -1$.*

*Proof.* As before, we need only consider $D_\alpha(B_N)$. Suppose $\varphi$ has no angular derivative on $B_N$. For $w \in \partial B_N$ define

$$h(w) = \begin{cases} \dfrac{1 - |\varphi^{-1}(w)|^2}{1 - |w|^2} & (w \in \varphi(B_N)) \\ 0 & (w \in B_N \setminus \varphi(B_N)). \end{cases}$$

Even though this is not quite the function $h$ which occurs in the proof of

Theorem 5.3, it will play the same role. Since $\varphi$ has no angular derivative it follows from Lemma 6.5 that $h$ is continuous on $B_N$ and that $h(w) \to 0$ as $|w| \to 1-$. Now we proceed exactly as in the proof of Theorem 6.4, but we use the definition of $h$ instead of the Schwarz Lemma to compare $1 - |\varphi^{-1}(w)|^2$ with $1 - |w|^2$. The result is:

$$\|C_\varphi f\|_{D_\alpha}^2 \leqq |f(0)|^2 + \int_{B_N} \Omega(\varphi^{-1}(w)) |\nabla f(w)|^2 h(w)^\alpha d\nu_\alpha(w)$$

$$\leqq |f(0)|^2 + \|\Omega\|_\infty \int_{B_N} |\nabla f(w)|^2 h(w)^\alpha d\nu_\alpha(w),$$

for each $f \in D_\alpha(B_N)$. Here $\Omega$ is the function used in the proof of Theorem 6.4, and $\|\Omega\|_\infty$ is its supremum over $B_N$, which is finite by the last part of the proof of that theorem. The Schwarz Lemma insures that $h$ is also bounded on $B_N$ (we may, as before, assume $\varphi(0) = 0$), and so $h \in C_0(B)$. An easy estimate now shows that if $(f_j)$ is a sequence of functions in the unit ball of $D_\alpha(B_N)$ which converges to zero uniformly on compact subsets of $B_N$, then

$$\|C_\varphi f\|_{D_\alpha} \to 0.$$

Thus $C_\varphi$ is compact on $D_\alpha(B_N)$ by the analogue for that space of Proposition 2.8.

The converse is in fact more general: for any map $\varphi$, existence of the angular derivative at just one point of $\partial B$ insures that $C_\varphi$ is not compact on $D_\alpha(B_N)$. This was essentially proved for $H^p(B_N)$ by MacCluer ([**19**], Lemma 1.6 and Theorem 1.7), and that proof can be made to work in the Dirichlet setting. Alternatively the Carleson measure argument employed to get the corresponding part of Theorem 5.3 works in the present situation with no essential changes, since an appropriate generalization of Julia's Lemma (Section 2.6) is available in higher dimensions (see [**27**], Theorem 8.5.3). We leave the details to the reader.

6.7. *Complications in higher dimensions.* We close this paper by showing that Theorems 6.4 and 6.6 do not hold for general holomorphic self-maps of $B_N$. Our counter-examples also show that for different values of $\alpha$ the Bergman spaces $A_\alpha^p(B_N)$ have different classes of compact composition operators, in sharp contrast with the situation in the unit disc (Theorem 3.5).

The operators we are going to produce are constructed from the holomorphic monomial

$$\pi(z) = N^{N/2} z_1 z_2 \ldots z_N$$

where $z = (z_1, \ldots, z_N) \in \mathbf{C}^N$. The Arithmetic-Geometric Mean Inequality shows that $\pi$ maps $B$ onto $U$ and $\bar{B}$ onto $\bar{U}$, with $\pi^{-1}(\partial U)$ the torus

$$|z_1| = \ldots = |z_n| = N^{-1/2}$$

in $\partial B_N$. The properties of this monomial, viewed as a mapping of $B_N$ onto $U$ were studied by Ahern [2], who used it to transfer the bad boundary behavior of functions in the Bergman spaces of $U$ to members of $H^p(B_N)$ near the above-mentioned torus in $\partial B_N$.

For $0 < \beta < 1$ define the conformal map $\psi_\beta$ of $U$ into itself by

$$\psi_\beta(w) = 1 - (1 - w)^\beta \quad (w \in U).$$

It is easy to check that $\psi_\beta(U)$ is a non-tangential region in $U$ with vertex at the point 1, whose boundary curves make an angle $\pi\beta/2$ at 1 with the unit interval.

The composition operators we consider here are the ones induced by the holomorphic maps

$$\varphi_\beta : B_N \to B_N$$

defined by

$$\varphi_\beta(z) = (\psi_\beta(\pi(z)), 0')$$

where $0'$ denotes the origin of $\mathbf{C}^{N-1}$. Clearly $\varphi_\beta$ does not have an angular derivative at the point $(1, 0')$ of $\partial B_N$, and since this is the only boundary point at which $\varphi_\alpha$ has length 1, it follows that $\varphi_\beta$ has an angular derivative at no boundary point of $B_N$. Our main result about composition operators induced by the maps $\varphi_\beta$ is stated in terms of the critical index

$$\alpha_0 = \alpha_0(N, \beta) = \frac{\left(\beta - \frac{1}{2}\right)(N + 1) - 1}{1 - \beta}.$$

Note that $\alpha_0$ may be $\leqq -1$ for small values of $0 < \beta < 1$. What is important for our purposes is that this does not happen for

$$(N + 1)/2N < \beta < 1.$$

6.8. THEOREM. *For $N > 1$, $0 < \beta < 1$, and $0 < p < \infty$: $C_{\varphi_\beta}$ is bounded on $A^p_\alpha(B_N)$ if and only if $\alpha \geqq \alpha_0$. It is compact if and only if $\alpha > \alpha_0$.*

Since for $N > 1$ the index $\alpha_0(N, \beta)$ covers the interval $(-1, \infty)$ as $\beta$ ranges from $(N + 1)/2N$ to 1, this theorem yields immediately:

6.9. COROLLARY. *Suppose $N > 1$, $0 < p < \infty$ and $\alpha > -1$. Then there exists a map $\varphi$ with no angular derivative at any point of $\partial B_N$ such that $C_\varphi$ is bounded on $A^p_\alpha(B_N)$ but not compact. There also exists such a $\varphi$ for which $C_\varphi$ is not bounded on $A^p_\alpha(B_N)$. Moreover, for different values of $\alpha$ the spaces $A^p_\alpha(B_N)$ have different classes of compact composition operators.*

The proof of Theorem 6.8 is a calculation involving Carleson measures.

We have already defined the $N$-dimensional "Carleson regions" $S(\zeta, \delta)$ in Section 6.3. In the unit ball the notions of "$\alpha$-Carleson measure" and "compact $\alpha$-Carleson measure" are the same as those for the disc (definition 4.2), except that $\delta^{\alpha+2}$ is replaced by $\delta^{\alpha+N+1}$ on the right side of the inequality in those definitions. The reason for this is:

6.10. LEMMA. *$\nu_\alpha(S(\zeta, \delta)) \sim \delta^{\alpha+N+1}$ for $\alpha > -1$.*

The notation here means that there exists $0 < K = K(\alpha) < \infty$ such that

$$K^{-1}\delta^{\alpha+N+1} \leqq \nu_\alpha(S(\zeta, \delta)) \leqq K\delta^{\alpha+N+1}$$

for all $0 < \delta \leqq 2$. Note that $\nu_\alpha(S(\zeta, \delta))$ does not depend on $\zeta \in \partial B_N$ since $\nu_\alpha$ is rotation-invariant. In the sequel we will use the "$\sim$" notation to describe a variety of situations involving functions and measures, leaving it to the reader to supply the exact meaning.

*Proof of lemma.* For $\zeta \in \partial B_N$ and $0 < \delta \leqq 2$ let

$$Q(\zeta, \delta) = \{\eta \in \partial B_N : |1 - \langle\zeta, \eta\rangle| < \delta\}$$

and

$$\mathscr{S}(\zeta, \delta) = \{z \in B_N : 1 - |z| < \delta/2 \text{ and } z/|z| \in Q(\zeta, \delta)\}.$$

Then

$$S(\zeta, \delta/2) \subset \mathscr{S}(\zeta, \delta) \subset S(\zeta, 3\delta/2)$$

so it is enough to consider instead the regions $\mathscr{S}(\zeta, \delta)$. The result now follows immediately from the formula for integration over $B_N$ in polar coordinates ( [**27**], Section 1.4.3), and the fact that

$$\sigma(Q(\zeta, \delta)) \sim \delta^N$$

( [**27**], Proposition 5.1.4).

We will prove Theorem 6.8 by analyzing the "Carleson" nature of the measure $\nu_\alpha\varphi_\beta^{-1}$, that is, by estimating the quantity

$$\nu_\alpha\varphi_\beta^{-1}(S(\zeta, \delta))$$

as $\zeta$ ranges through $\partial B_N$ and $\delta$ through $(0, 2]$. In order to do this we need two more preliminary results. The first one, due to Ahern ( [**2**], Theorem 1), identifies the measure $\sigma\pi^{-1}$ on $U$ (recall that the monomial $\pi(z)$ defined in Section 6.7 maps $B$ onto $U$).

6.11. PROPOSITION. *$\sigma\pi^{-1} \sim \lambda_{(N-3)/2}$ for $N > 1$.*

We are going to use this result of Ahern to estimate $\nu_\alpha\pi^{-1}$. This will require the following.

6.12. LEMMA. *Suppose $\alpha, \gamma > -1$. If $f$ is continuous on the closed unit interval, then*

$$\int_0^1 \int_0^1 f(xy)(1 - x)^{\gamma}(1 - y)^{\alpha} y^{\gamma+1} dx dy$$

$$= B(\alpha + 1, \gamma + 1) \int_0^1 f(t)(1 - t)^{\alpha+\gamma+1} dt,$$

*where $B$ (,) is the beta function.*

*Proof.* Write the integral on the left as an iterated integral with the $x$-integration done first. In this inner integral make the substitution $t = xy$, interchange the order of integration, and in the new inner integral introduce the new variable of integration $s$ defined by $1 - y = (1 - t)s$. The result then follows immediately.

6.12. *Proof of Theorem* 6.8. By our previous remarks on the relationship between Carleson measures and composition operators, the theorem is equivalent to the estimate

(4) $\quad \nu_{\alpha}\varphi_{\beta}^{-1}(S(\zeta, \delta)) \sim \delta^{\gamma}$

where $\gamma = (\alpha + (N + 3)/2)/\beta$.

Since $\varphi(B_N)$ lies in a Koranyi approach region with vertex at $e_1$, it follows from Lemma 2.1 of [**20**] that we need only consider $\zeta = e_1$ in (4). Now

$$\varphi_{\beta}^{-1}(S(e_1, \delta)) = \pi^{-1}\psi_{\beta}^{-1}(S_1(1, \delta))$$
$$= \pi^{-1}(S_1(1, \delta^{\beta}))$$

where $S_1(1, \delta)$ is the intersection of $U$ and the open disc of radius $\delta$ centered at 1 ($= S(1, \delta)$ in the notation of Sections 4 and 5). Thus

(5) $\quad \nu_{\alpha}\varphi_{\beta}^{-1}(S(e_1, \delta)) = \nu_{\alpha}\pi^{-1}(S_1(1, \delta^{1/\beta}))$.

We are going to show that

(6) $\quad \nu_{\alpha}\pi^{-1} \sim \lambda_{\alpha+(N-1)/2}$ on $\{z \in B: 1/2 \leqq |z| < 1\}$,

which, in view of equation (5) will yield (4) for $0 < \delta < 1/2$ which will, of course, be enough for our purposes.

To prove (6) let $g$ be a continuous function on the closed unit ball which vanishes on the closed ball

$$\{z \in B: |z| \leqq 1/2\}.$$

This will allow us to freely alter powers of $r$ in the calculation below at the expense of changing equal signs to "$\sim$". Letting the index $\gamma$ of Lemma 6.12 be $(N - 3)/2$:

$$\int_{B_N} g(\pi(z))(1 - |z|^2)^\alpha d\nu(z)$$

$$= 2N \int_0^1 \int_{\partial B_N} g(\pi(r\zeta))d\sigma(\zeta)(1 - r^2)^\alpha r^{2N-1}dr$$

$$= 2N \int_0^1 \int_{\partial B_N} g(r^N\pi(\zeta))d\sigma(\zeta)(1 - r^2)^\alpha r^{2N-1}dr$$

$$\sim \int_0^1 \int_{\partial B_N} g(r\pi(\zeta))d\sigma(\zeta)(1 - r)^\alpha dr$$

$$\sim \int_0^1 \int_U g(rw)(1 - |w|)^\gamma d\lambda(w)(1 - r)^\alpha dr$$

(by Proposition 6.11)

$$\sim \int_0^{2\pi} \left\{ \int_0^1 \int_0^1 g(r\rho e^{i\theta})(1 - \rho)^\gamma (1 - r)^\alpha \rho^{\gamma+1} d\rho dr \right\} d\theta (w = \rho e^{i\theta}).$$

Now we can apply Lemma 6.11 to the integral in braces to obtain:

$$\int_B g \circ \pi d\nu_\alpha \sim \int_0^{2\pi} \int_0^1 g(te^{i\theta})(1 - t)^{\alpha+\gamma+1} dt d\theta$$

$$\sim \int_U g d\lambda_{\alpha+\gamma+1}.$$

The constants hidden in the notation above to not depend on the function $g$, so the estimate (6) holds, and the proof is complete.

**7. Additional remarks.** Regarding Example 3.6, Kenneth Stephenson (*Isometries of the Nevanlinna Class*, Indiana Univ. Math. J. *26* (1977), 307-324, Lemma 5.9) has given essentially the same construction of an atomic singular inner function for which the angular derivative exists at no point of the unit circle.

For more information on concepts related to the notion of Carleson measure, see Daniel Luecking's paper. *Forward and reverse Carleson inequalities for functions in Bergman spaces and their derivatives*, American J. Math *107* (1985), 85-111.

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*University of Virginia,*
*Charlottesville, Virginia;*
*Michigan State University,*
*East Lansing, Michigan*