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The Bounded Weak Star Topology and the General Strict Topology

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Let B be a Banach algebra with an approximate identity (e_α) such that $\sup \| e_\alpha \| = 1$, let X be a left Banach B-module with $\| x \| = \sup \{\| bx \| : b \in B, \| b \| \leqslant 1\}$, and let β denote the strict topology induced on X by B. We show that every linear subspace of X having β-compact unit ball is a conjugate Banach space whose bounded weak star topology coincides with β. This result is applied to some common conjugate Banach spaces, namely Banach spaces with boundedly complete bases, and the spaces $L^p(G)$ $(1 < p \leqslant \infty)$, G a compact Abelian group. As a by-product we obtain a new representation for the strict topology on the space of bounded analytic functions on the open unit disk.

INTRODUCTION

The bounded weak star topology induced on a conjugate Banach space by its predual is the strongest topology which agrees on bounded sets with the weak star topology [3, V.5.3]. It is well-known that the Banach space $H^\infty(D)$ of bounded analytic functions on the plane region D is the dual of a quotient space of the bounded Borel measures on D; and Rubel and Ryff [8] have recently shown that the bounded weak star topology induced on $H^\infty(D)$ by this duality coincides with the strict topology. More generally [13], if S is a locally compact Hausdorff space and $C(S)$ is the space of bounded continuous complex valued function on S, then any linear subspace E of $C(S)$ whose unit ball is strictly compact is the dual of a quotient space of the bounded Borel measures on S, and the bounded weak star topology thus induced on E is the strict topology.

The purpose of this paper is to extend these results to the general strict topology introduced by Sentilles and Taylor [12], and to give applications to some common conjugate Banach spaces. In particular, we identify the bounded weak star topology on any Banach space with a boundedly complete basis; and consider the spaces $L^p(G)$ $(1 < p \leqslant \infty)$, G a compact Abelian group. As a by-product we obtain a new way of representing the strict topology on the space of bounded analytic functions on the open unit disc.

1. The General Strict Topology

In this section, we record some terminology and notation, generally following that of [12]. We state, mostly without proof, those results of [12] needed in the sequel.

A net $(e_\alpha : \alpha \in A)$ in a Banach algebra B is called an *approximate identity* if $\| e_\alpha b - b \| \to 0$ and $\| b e_\alpha - b \| \to 0$ for every b in B. A Banach space X which is a left module over B is called a *left Banach B-module* if $\| bx \| \leqslant \| b \| \| x \|$ for every b in B and x in X. If X is a left Banach B-module, then its Banach space dual X' is a right Banach B-module under the product $(x', b) \to x'b$, where $x'b(x) = x'(bx)$ for each x in X.

Until further notice, B will denote a Banach algebra with an approximate identity $(e_\alpha : \alpha \in A)$ satisfying

$$\sup \| e_\alpha \| = 1, \tag{1.1}$$

and X will be a left Banach B-module such that

$$\| x \| = \sup\{\| bx \| : b \in B, \| b \| \leqslant 1\}. \tag{1.2}$$

We remark that condition (1.2) was not assumed in [12]. Instead the right side of the equation was introduced as a new, possibly inequivalent norm on X, and a number of the results below were obtained for this new norm rather than for the original one.

The *essential part* X_e of X is the closed linear subspace of X spanned by the set

$$BX = \{bx : b \in B, x \in X\}.$$

If $X = X_e$, then X is called *essential.* A direct consequence of the definition and (1.1) is that

$$X_e = \text{norm closure of } BX = \{x \in X : \| e_\alpha x - x \| \to 0\}.$$

Thus X_e is the set of elements in X on which (e_α) acts as a left approximate identity, and it follows immediately from this that both X_e

and B are essential B-modules. We now state an important factorization theorem which plays a crucial role both here and in [12]. In particular, it shows immediately that $X_e = BX$.

THEOREM A (see [12, Theorem 2.1]). *If X is essential, Z is a bounded subset of X, and $\| e_\alpha z - z \| \to 0$ uniformly on Z, then there exists b in B and a bounded subset W of X such that $Z = bW$.*

With obvious modifications, the same theorem holds for right Banach B-modules. Indeed, it is the right version which will be needed in the proof of our main result (Theorem 1).

The *strict topology* β induced on X by B is the locally convex topology given by the family of seminorms

$$x \to \| bx \| \quad (b \text{ in } B).$$

Condition (1.2) insures that the strict topology is Hausdorff. A useful companion to β is the topology κ defined by the seminorms

$$x \to \| e_\alpha x \| \quad (\alpha \text{ in } A).$$

Clearly, κ is a locally convex, Hausdorff topology, and $\kappa \subset \beta \subset$ norm topology.

The following "classical" example helps place these topologies in perspective (cf. [12, p. 146]). Let $X = C(S)$, S locally compact and Hausdorff; and let $B = C_0(S)$, those functions in $C(S)$ which vanish at infinity. Then X is a Banach B-module under pointwise multiplication, (1.2) is satisfied, and Urysohn's Lemma assures that B has an approximate identity satisfying (1.1). In this example, β is the original strict topology introduced by Buck [1], while κ is the topology of uniform convergence on compact subsets of S.

The next theorem summarizes some fundamental properties of the strict topology. These were first proved for the classical case by Buck [1], and, in general, by Sentilles and Taylor.

THEOREM B [12, Section 3]. (1) *β and κ coincide on the norm bounded subsets of X.*

(2) *A subset of X is norm bounded if and only if it is strictly bounded.*

(3) *The unit ball of X_e is strictly dense in that of X.*

(4) *The family of sets*

$$\{x \in X : \| bx \| \leqslant 1\} \quad (b \text{ in } B)$$

is a local base for the strict topology.

Parts (1) and (3) follow more or less directly from the fact that (e_α) is an approximate identity for B satisfying (1.1), while (4) is a consequence of the right version of Theorem A (applied to B, viewed as a right B-module). Part (2) follows from (1.2) and the uniform boundedness principle.

Since the strict topology is weaker than the norm topology, the strict dual of X can be regarded as a linear subspace of X'. Let M denote the strict dual of X taken in the X' norm. Then the map which associates to each x in X the linear functional $\lambda \to \lambda(x)$ (λ in M) is a one-to-one continuous linear transformation taking X into M'. We call this map the *canonical imbedding* of X into M', and see easily from (1.2) that it is an isometry. Note that according to Theorem B(2), M is the strong dual [10, p. 141] of (X, β), so the above mapping is, in fact, the canonical imbedding of (X, β) into its strong bidual [10, p. 143].

Since M is a normed space, M' is a Banach space. In the classical example M is the space of bounded Borel measures on S, which suggests that M is always a Banach space. This is in fact the case, and follows from the next result, which is a direct consequence of Theorem B(3) and (4) and plays an important role in the applications of our main theorem.

THEOREM C [12, Theorem 4.1 (1)]. *A linear functional x' on X is strictly continuous if and only if there exists y' in $(X_e)'$ and b in B such that*

$$x'(x) = y'(bx) \quad (x \text{ in } X).$$

Moreover, the restriction mapping $x' \to x'|_{X_e}$ is an isometric module isomorphism of M onto the essential part of $(X_e)'$.

COROLLARY. *M is a right essential Banach B-module.*

2. MAIN THEOREM

In this section, E is a linear subspace of X, and E^0 is the annihilator of E in M (the strict dual of X taken in the X' norm). E_1 denotes the unit ball of E. We now state our main result.

THEOREM 1. *Suppose E is a linear subspace of X whose unit ball is* *compact in the strict topology. Then the mapping which associates to each e in E the linear functional*

$$m + E^0 \to m(e) \quad (m \text{ in } M)$$

on M/E^0 is an isometric isomorphism of E onto the dual of M/E^0; and the bounded weak star topology thus induced on E coincides with the strict topology.

Proof (Cf. [8, Section 2] and [13, Theorem 2]). Recall from Section 1 that the canonical imbedding of X into M' is an isometry. Let $\alpha = w(X, M)$, the weak topology induced on X by M. Then $\alpha \subset \beta$, so it follows from elementary topology and the strict compactness of E_1 that α and β coincide on E_1. Thus E_1 is α-compact, hence $w(M', M)$-compact; and it follows from the Krein–Smulian theorem [3; V.5.7] that E is $w(M', M)$-closed in M'. Standard Banach space theory now shows that the mapping in question is an isometric isomorphism of E onto $(M/E^0)'$.

Let bw^* denote the bounded weak star topology induced on E by M/E^0. Since α coincides on E with $w(E, M/E^0)$, and $\beta = \alpha$ on E_1 (hence on all bounded subsets of E), we see that $\beta \subset bw^*$. To prove the reverse inequality, we need the fact that the bounded weak star topology on a conjugate Banach space is the topology of uniform convergence on norm null sequences in the predual [3, V.5.4]. If $(m_j + E^0)$ is a sequence in M/E^0 which converges to zero in norm, then (m_j) can be chosen from M such that $\| m_j \| \to 0$; from which it follows readily that $\sup_j \| m_j e_\alpha - m_j \| \to 0$. Since M is an essential right Banach B-module, the right version of Theorem A insures that $m_j = n_j b$ (all j), where $b \in B$ and (n_j) is a bounded sequence in M, say, $\| n_j \| \leqslant C$ for all j. Denoting the pairing between E and M/E^0 by $\langle , \rangle$, we see that for each e in E,

$$\sup_j |\langle m_j + E^0, e \rangle| = \sup_j | n_j(be)| \leqslant C \| be \|.$$

Thus $\beta \supset bw^*$ on E, and the proof is complete.

In [13], this theorem was proven for $X = C(S)$, $B = C_0(S)$. In this case, the unit ball of E is strictly compact if and only if it is a (closed) normal family. The theorem of Rubel and Ryff mentioned in the Introduction corresponds to the case where S is a plane region and E is the space of bounded analytic functions on S. Theorem 1 is also related to the work of Dorroh [2] and Sentilles [11] on localization of the strict topology.

3. Application to Bases

In this section, X will denote a Banach space with a basis $(x_n : n \geqslant 0)$. Thus for each x in X, there is a unique scalar sequence $(x_n'(x) : n \geqslant 0)$ such that $x = \sum x_n'(x)\, x_n$, where the series converges in the norm of X. It is well-known that the coordinate functionals x_n' are continuous linear functionals on X, and that X can be (equivalently) renormed such that

$$\| x \| = \sup_n \left\| \sum_{k=0}^{n} x_k'(x)\, x_k \right\| \tag{3.1}$$

(for this and other background material on bases, see [6, Chap. 3]).

The basis (x_n) is called *boundedly complete* [6, p. 36] if $\sum \beta_n x_n$ converges for each scalar sequence (β_n) such that

$$\sup_n \left\| \sum_{k=0}^{n} \beta_k x_k \right\| < \infty.$$

It is known [6, p. 37, Theorem 11 and Corollary 12] that if (x_n) is boundedly complete, then X is isomorphic to the dual of the closed linear span of (x_n'), where the pairing between the spaces is the obvious one. The purpose of this section is to represent the bounded weak star topology induced on X by this duality in terms of the multiplier algebra of X with respect to (x_n).

Following [7], we define a *multiplier of X (with respect to (x_n))* to be a scalar sequence $\alpha = (\alpha_n)$ such that for each x in X the series $\sum \alpha_n x_n'(x)\, x_n$ converges in the norm of X. Under coordinatewise operations the set of all such multipliers is an algebra called the *multiplier algebra of X (with respect to (x_n))*, and denoted by $\mu(X)$. For α in $\mu(X)$ and x in X, we write

$$\alpha x = \sum \alpha_n x_n'(x)\, x_n . \tag{3.2}$$

It follows from the closed graph theorem and the continuity of coordinate functionals that for each α in $\mu(X)$ the mapping $x \to \alpha x$ is a bounded linear transformation on X. Straightforward arguments show that the operator norm

$$\| \alpha \| = \sup\{\| \alpha x \| : \| x \| \leqslant 1\}$$

makes $\mu(X)$ into a commutative Banach algebra, and X (under the product (3.2)) into a left Banach $\mu(X)$-module.

The sequence e_n having 1 in the first n places and zero elsewhere is clearly a multiplier of X ($n = 0, 1, 2,\ldots$). Let $\mu_0(X)$ denote the closed linear span of $(e_n : n \geqslant 0)$. The main result of this section is the following.

THEOREM 2. *Suppose X is a Banach space with a boundedly complete basis (x_n). Then x is isomorphic to the dual of the closed linear span of (x_n') where the spaces are paired in the obvious way*; *and the bounded weak star topology thus induced on X is given by the seminorms*

$$x \to \left\| \sum \alpha_n x_n'(x)\, x_n \right\|, \tag{3.3}$$

where (α_n) ranges through $\mu_0(X)$.

Proof. Renorm X in accordance with (3.1). Then $\| e_n \| = 1$ ($n = 0, 1, 2,\ldots$) so (e_n) satisfies (1.1). The linear subspace spanned by (e_n) is an algebra for which (e_n) is an approximate identity, and it follows from this and (1.1) that the same is true for its closure $\mu_0(X)$. It is clear that X is a left Banach $\mu_0(X)$-module. Since (x_n) is a basis for X we have $\| x - e_n x \| \to 0$ for each x in X; so X is essential. It follows from this and (1.1) that X satisfies (1.2) with $B = \mu_0(X)$, hence the conditions of Section 1 are fulfilled.

Let β denote the strict topology induced on X by $\mu_0(X)$, and let κ be the topology induced by the seminorms $x \to \| e_n x \|$ ($n = 0, 1, 2,\ldots$). It follows from Theorem C that the strict dual M of X is isometrically isomorphic to $(X')_e$. Since $x_n' = x_n' e_n$ ($n = 0, 1, 2,\ldots$), each x_n' belongs to $(X')_e$; hence the closed linear span of (x_n') is contained in $(X')_e$. Conversely, if $y' \in (X')_e$, then $\| y' - y' e_k \| \to 0$. But $y' e_k$ is in the linear span of (x_n') ($k = 0, 1, 2,\ldots$); so y is in its closure. Thus $(X')_e$ is the closed linear span of (x_n').

In view of Theorem 1, the proof will be complete if we show that the unit ball X_1 of X is strictly compact. Since $\beta = \kappa$ on X_1 (Theorem B), it is enough to show that X_1 is κ-compact. Note that κ is simply the topology of coordinatewise convergence on X and (although this is not essential) is metrizable. If $x \in X_1$, then, by (3.1), $\| x_n'(x)\, x_n \| \leqslant 2$; hence $| x_n'(x) | \leqslant 2 \| x_n \|^{-1}$. Thus if K_n denotes the set of complex numbers of modulus $\leqslant 2 \| x_n \|^{-1}$, then the coordinate mapping $\phi : x \to (x_n'(x))$ is a homeomorphism taking (X_1, κ) into the compact product space $P = \prod_n K_n$. We claim that $\phi(X_1)$ is closed in P. To see this, suppose (z_k) is a sequence in X_1, and suppose the corresponding coordinate sequences converge in P to a sequence (α_n); that is, $\lim_k x_n'(z_k) = \alpha_n$ ($n = 0, 1, 2,\ldots$).

By condition (3.1),

$$\left\| \sum_{n=0}^{N} x_n'(z_k)\, x_n \right\| \leqslant \| z_k \| \leqslant 1.$$

So

$$\left\| \sum_{n=0}^{N} \alpha_n x_n \right\| = \lim_k \left\| \sum_{n=0}^{N} x_n'(z_k)\, x_n \right\| \leqslant 1$$

for $N = 0, 1, 2,\ldots$. It follows from the bounded completeness of (x_n) that $\sum \alpha_n x_n$ converges in norm to an element x in X_1 . Clearly, $x_n'(x) = \alpha_n$ $(n = 0, 1, 2,\ldots)$; so $\phi(x) = (\alpha_n)$. Thus $\phi(X_1)$ is closed in P, hence compact, and so X_1 is κ-compact, and the proof is complete.

The usefulness of this theorem in any given situation depends on how well $\mu_0(X)$ can be described. If the basis (x_n) is *unconditional* (for each x in X, every rearrangement of the series $\sum x_n'(x)\, x_n$ converges), then every bounded scalar sequence is a multiplier [6, Chap. 2, Section 1, Theorem 3]; and the proof of the following corollary shows that $\mu_0(X) = c_0$, the space of scalar sequences which converge to zero.

COROLLARY. *Suppose X is a Banach space with an unconditional, boundedly complete basis (x_n). Then the bounded weak star topolgy induced on X by the closed linear span of (x_n') is given by the seminorms* (3.3), *where (α_n) ranges through c_0 .*

Proof. Since (x_n) is unconditional, $l^\infty \subset \mu(X)$. Consideration of the products αx_n shows that $| \alpha_n | \leqslant \| \alpha \|$ for each α in $\mu(X)$. Thus $\mu(X) = l^\infty$, and $\| \alpha \|_\infty \leqslant \| \alpha \|$ for each α in $\mu(X)$. The interior mapping principle guarantees that the norms $\| \cdot \|$ and $\| \cdot \|_\infty$ are equivalent. So $\mu_0(X) = c_0$, and the result follows from Theorem 2.

It was shown in [13] that the bounded weak star topology on $l^\infty = (l^1)'$ coincides with the strict topology induced by c_0 (this is an immediate consequence of Theorem 1). The last corollary shows that the same is true for the bounded weak star topology on $l^p = (l^q)'$ $(1 < p < \infty, p^{-1} + q^{-1} = 1)$, and $l^1 = (c_0)'$.

With few changes the proof of Theorem 2 yields a more general result which applies directly to l^∞ as well as l^p $(1 \leqslant p < \infty)$. We state the result without proof. If (x_n , x_n') is a total biorthogonal sequence in X (i.e., a biorthogonal sequence with (x_n') total over X), define $\mu(X)$ to be the set of all scalar sequences (α_n) such that for each x in X there is y in X such that $x_n'(y) = \alpha_n x_n'(x)$ $(n = 0, 1, 2,\ldots)$ (see [7, Definition 1.1 and Corollary 3.3]). Define $\mu_0(X)$ as before.

THEOREM 2'. *Suppose* (x_n, x_n') *is a total biorthogonal sequence in* X *such that*

(1) X *has an equivalent norm satisfying* (3.1), *and*

(2) *for every scalar sequence* (β_n) *such that* $\sup_n \| \sum_{k=0}^{n} \beta_k x_k \| < \infty$, *there exists* x *in* X *with* $x_k'(x) = \beta_k$ $(k = 0, 1, 2, \ldots)$.

Then X *is isomorphic to the dual of the closed linear span of* (x_n'), *and the bounded weak star topology thus induced on* X *is given by the seminorms* (3.3), *where* (α_n) *ranges through* $\mu_0(X)$.

We remark that conditions (1) and (2) imply that the unit ball of X is complete in the topology of pointwise convergence on (x_n'), i.e., that X is boundedly complete in the sense of Johnson [14, Definition II.2]. If, in addition, (x_n) spans a dense linear subspace of X, then the duality between X and the closed linear span of (x_n') becomes a special case of [14, Theorem II.5].

4. APPLICATION TO $L^p(G)$

Let G be a locally compact Abelian group, and let dx denote Haar measure on G. Under convolution the spaces $L^p(G)$ $(1 \leqslant p \leqslant \infty)$ are Banach algebras and Banach $L^1(G)$-modules satisfying (1.2). Let q denote the index conjugate to $p : p^{-1} + q^{-1} = 1$.

THEOREM 3. *The following are equivalent*:

(a) G *is compact*

(b) *The bounded weak star topology induced on* $L^p(G)$ *by* $L^q(G)$ $(1 < p \leqslant \infty)$ *coincides with the strict topology induced by* $L^1(G)$.

Proof. Let $L^p = L^p(G)$ $(1 \leqslant p \leqslant \infty)$, and let (U_α) be a base for the neighborhoods of 0 in G. By Urysohn's Lemma there is for each α a nonnegative continuous function e_α on G such that $e_\alpha = 0$ off U_α and $\| e_\alpha \|_1 = 1$. By [4, Theorem 20.15], $\| e_\alpha * f - f \| \to 0$ for each f in L^p $(1 \leqslant p < \infty)$, so (e_α) is an approximate identity for L^1 satisfying (1.1), and L^p is an essential L^1-module. It follows from Theorem C that the strict dual of L^p $(1 < p < \infty)$ is L^q, where the spaces are paired by integration (cf. [12, Section 5]).

Suppose G is compact. Then $(L^\infty)_e = C(G)$; so the strict dual of L^∞ is $M(G)_e = L^1(G)$, again in the integration pairing. Fix $1 < p \leqslant \infty$. We claim that the unit ball S of L^p is strictly compact. By Theorem B it is enough to show that S is κ-compact. Let $S_\alpha = S * e_\alpha$. Since S is weak star compact and the map $f \to f * e_\alpha$ (f in L^p) is weak star continuous, S_α is weak star compact, and hence norm closed in L^p. But the members of S_α are continuous on G, uniformly bounded by $\| e_\alpha \|_\infty$, and have moduli of continuity bounded by that of e_α. It, therefore, follows from Ascoli's theorem that S_α is relatively compact in $C(G)$, hence norm compact in L^p (here we use the fact that dx is a finite measure, since G is compact). Thus the norm and weak star topologies coincide on each S_α. Consequently, the product space $P = \prod_\alpha S_\alpha$ is compact, and the map $f \to (f * e_\alpha)$ is a homeomorphism of (S, κ) into P which remains continuous even when S has the weak star topology. The image of S in P is therefore compact; so S is κ-compact, and (b) follows from Theorem 1.

Conversely, if G is not compact, then there is a sequence (x_n) in G and a compact neighborhood U of 0 such that the sets $U_n = x_n + U$ are pairwise disjoint. Let f be the characteristic function of U, and let f_n be the characteristic function of U_n, so $f_n(x) = f(x - x_n)$. Then $\| f_n \|_p = \| f \|_p$ for all n, so (f_n) is a bounded sequence in L^p. By Holder's inequality, there exists $C > 0$ such that for each g in L^q,

$$\left| \int f_n g \, dx \right| = \left| \int_{U_n} g \, dx \right| \leqslant C \left(\int_{U_n} | g |^q \, dx \right)^{1/q}.$$

The right term tends to zero because

$$\sum_n \int_{U_n} | g |^q \, dx \leqslant \| g \|_q^q < \infty.$$

So (f_n) converges to zero in the weak star topology. However, for each ϕ in L^1,

$$f_n * \phi(x) = f * \phi(x - x_n) \quad (x \text{ in } G).$$

So $\| f_n * \phi \|_p = \| f * \phi \|_p$ for all n. Thus (f_n) does not converge to zero in the strict topology. It follows that the weak star and strict topologies differ on the bounded set (f_n), so the strict topology cannot be the bounded weak star topology on L^p, and the proof is complete.

The same theorem and proof remain valid for $M(G)$, the space of bounded Borel measures on G, considered as the dual of $C_0(G)$.

COROLLARY 1. *If G is a compact Abelian group, then $L^p(G)$ ($1 < p \leqslant \infty$) in the bounded weak star topology induced by $L^q(G)$ is a topological algebra under convolution. The same is true for $M(G)$ in the bounded weak star topology induced by $C(G)$.*

Proof. From Theorem A, each ϕ in $L^1(G)$ can be written as $\phi = \phi_1 * \phi_2$, where $\phi_1, \phi_2 \in L^1(G)$. Thus if $f, g \in L^p(G)$ or $M(G)$, then

$$\|(f * g) * \phi\| = \|(f * \phi_1) * (g * \phi_2)\| \leqslant \|f * \phi_1\| \, \|g * \phi_2\|,$$

and the result follows.

We close with an application of Theorem 3 to bounded analytic functions. Let H^∞ denote the space of bounded analytic functions in the open unit disk U. Let T denote the unit circle. Then, for each f in H^∞, the radial limit

$$\lim_{r \to 1-} f(re^{i\theta}) = \tilde{f}(re^{i\theta})$$

exists a.e. $d\theta$. Moreover $\tilde{f} \in L^\infty(T)$, and the mapping $f \to \tilde{f}$ is an isometric isomorphism taking H^∞ onto a weak star closed subspace of $L^\infty(T)$ [5, pp. 34–39]. In the statement of the following corollary, β will denote the strict topology induced on H^∞ by $C_0(U)$; that is, the topology given by the seminorms

$$f \to \sup_{|z|<1} |f(z)\, k(z)|, \tag{4.1}$$

where k ranges through $C_0(U)$.

COROLLARY. *The topology β is also described by the seminorms*

$$f \to \|\tilde{f} * \phi\|_\infty \quad (\phi \text{ in } L^1(T)).$$

Hence (H^∞, β) is a topological algebra under convolution.

Proof. A result of Rubel and Shields [9, Theorem 4.23 and its proof] shows that $w(H^\infty, L^1(T)) = w(H^\infty, M(U))$, where the spaces are paired by integration in the obvious way. It follows that the bounded weak star topology that H^∞ has as the dual of a quotient of $L^1(T)$ coincides with the one it has as the dual of a quotient of $M(U)$. But the latter topology is β, as we have pointed out before; and the former is the restriction to H^∞ of the bounded weak star topology induced on $L^\infty(T)$ by $L^1(T)$. The result now follows from Theorem 3.

It is clear from (4.1) that (H^∞, β) is also a topological algebra under pointwise multiplication (for a detailed study of this algebra see [9, Section 5]). Thus H^∞ in its bounded weak star topology is a topological algebra under both convolution and pointwise multiplication. From Theorem 3, $L^\infty(T)$ in its bounded weak star topology is a topological algebra under convolution, and it is natural to ask if the same is true for pointwise multiplication.

PROPOSITION. *In its bounded weak star topology, $L^\infty(T)$ is not a topological algebra under pointwise multiplication.*

Proof. Let $f_n(e^{i\theta}) = \cos n\theta$ ($n = 0, 1, 2,...$). Then (f_n) is bounded in $L^\infty(T)$, and the Riemann–Lebesgue lemma insures that $f_n \to 0$ in the weak star topology, and hence in the bounded weak star topology. But the 0-th Fourier coefficient of f_n^2 is $1/2$ ($n = 1, 2,...$); so (f_n^2) does not converge to 0 in the bounded weak star topology (Fourier coefficients are bounded weak star continuous linear functionals), and the proof is complete.

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