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Composition operators ♡ Toeplitz operators

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Abstract. The study of composition operators, most notably on the Hardy space H^2, often leads to issues involving Toeplitz operators acting on that space. Toeplitz operators arise, for example, in Littlewood's original proof that composition operators are bounded on H^2, in the computation of their adjoints, and in questions about their "normality." In addition, there are interesting questions about just how "Toeplitz" a composition operator can be. This article reviews some recent work on these matters.

1. Operators

All of the work I am about to describe takes place in the Hardy space H^2 of the open unit disc $\mathbb{U}$ of the complex plane. More precisely, H^2 is the collection of functions $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ holomorphic in $\mathbb{U}$ with

$$\|f\|^2 \ := \ \sum_{n=0}^{\infty} |\hat{f}(n)|^2 \ < \ \infty \,. \tag{1.1}$$

The functional $\|\cdot\|$ so defined is a norm that makes H^2 into a Hilbert space isometrically isomorphic, via the map that associates a function $f \in H^2$ with its sequence of Maclaurin coefficients, to the sequence space ℓ^2.

Two fundamental operations that preserve analyticity are multiplication and composition, and these give rise to the most natural linear transformations on spaces of holomorphic functions: analytic Toeplitz operators and composition operators.

1.1. Analytic Toeplitz operators. For $b \in H^\infty$, the space of bounded holomorphic functions on $\mathbb{U}$, the operator T_b defined on H^2 by multiplication by b,

$$(T_b f)(z) = b(z) \cdot f(z) \qquad (f \in H^2, z \in \mathbb{U})$$

is called the *analytic Toeplitz operator* T_b *with symbol* b. Easily the best known such operator is the one induced by the the identity function $b(z) \equiv z$; it is affectionately denoted by "T_z", or sometimes by "S", and called the *forward shift* on H^2 because its action on an H^2 function shifts the Maclaurin series coefficents one unit to the right, placing a zero in the empty initial position.

Clearly T_z is a contraction on H^2, but for general $b \in H^\infty$ it is not clear, given our definition of H^2 in terms of Maclaurin coefficients, that T_b even maps H^2 into itself. Fortunately it does, and this is easy to prove. For this, let L^2 denote the space

$L^2(dm)$ where m is normalized arclength measure on the unit circle. For $n \in \mathbb{Z}$ and $\zeta \in \partial\mathbb{U}$ let $e_n(\zeta) = \zeta^n$. Then $\{e_n : n \in \mathbb{Z}\}$ is clearly an orthonormal family in L^2, so for $0 \le r < 1$ and $f \in \mathrm{Hol}(\mathbb{U})$ (the space of all functions holomorphic on $\mathbb{U}$), uniform convergence of Maclaurin series yields

$$\sum_{n=0}^{\infty} |\hat{f}(n)|^2 r^n = \int_{\partial\mathbb{U}} |f(r\zeta)|^2 \, dm(\zeta)$$

which, along with a monotone convergence argument, shows that:

$$\sum_{n=0}^{\infty} |\hat{f}(n)|^2 = \lim_{r\to 1-} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 \, d\theta \,, \tag{1.2}$$

where, temporarily, the value "∞" is allowed. Thus $f \in H^2$ if and only if the limit on the right is finite, and it follows easily that, for $b \in H^\infty$ the operator T_b maps H^2 into itself, with

$$\|T_b\| \le \|b\|_\infty := \sup\{|b(z)| : z \in \mathbb{U}\}. \tag{1.3}$$

In fact there is equality here:

$$\|T_b\| = \|b\|_\infty \qquad (b \in H^\infty). \tag{1.4}$$

Although this additional precision will not figure in what follows, the idea behind its proof will be important, so here is the argument. For each point $a \in \mathbb{U}$, the *reproducing kernel for* a is the function K_a defined by

$$K_a(z) \;=\; \frac{1}{1-a^*z} \;=\; \sum_{n=0}^{\infty} a^{*n} z^n \qquad (z \in \mathbb{U}) \tag{1.5a}$$

where here, and henceforth, if a is a complex number then a^* denotes its complex conjugate. The power series representation in (1.5a) makes it clear that K_a belongs to H^2 with

$$\|K_a\| \;=\; \frac{1}{\sqrt{1-|a|^2}} \qquad (a \in \mathbb{U}). \tag{1.5b}$$

The "reproducing kernel" terminology comes from the fact that for each $a \in \mathbb{U}$ and $f \in H^2$ the function K_a "reproduces the value of f at a" in the following sense:

$$f(a) = \langle f,\, K_a \rangle, \tag{1.5c}$$

a formula that follows easily from the power series representation of K_a.

A property of reproducing kernels that will be important to us is that *they are eigenvectors of analytic Toeplitz operators.* More precisely, for each $a \in \mathbb{U}$ and each bounded holomorphic function b on $\mathbb{U}$:

$$T_b^* K_a = b(a)^* K_a \tag{1.5d}$$

To prove this, fix $f \in H^2$ and calculate:

$$\langle f,\, T_b^* K_a \rangle = \langle T_b f,\, K_a \rangle = \langle bf,\, K_a \rangle = b(a) f(a) = \langle f,\, b(a)^* K_a \rangle \,.$$

which establishes (1.5d).

Now to prove (1.4) recall from (1.3) that only the direction "$\ge$" is at issue. For this one need only take norms on both sides of (1.5d), thus obtaining, for each $a \in \mathbb{U}$:

$$|b(a)|\, \|K_a\| = \|T_b^* K_a\| \le \|T_b^*\|\, \|K_a\|$$

from which it follows that $\|b\|_\infty \le \|T_b^*\| = \|T_b\|$, as desired. □

Analytic Toeplitz operators are but a special case of a more general class of operators on H^2 that will surface explicitly in §5, at which point we will see that *adjoints of Toeplitz operators are also Toeplitz operators.* The best known of the non-analytic Toeplitz operators is T_z^*, the adjoint of the forward shift. Its effect on an H^2 function is to shift the Maclaurin coefficient sequence one unit to the left, dropping off the term of index zero. For this reason T_z^* is often called *the backward shift*, frequently denoted by B.

1.2. Composition operators. For φ a holomorphic function on $\mathbb{U}$ with values in $\mathbb{U}$ (i.e., a "holomorphic selfmap of $\mathbb{U}$"), define C_φ on Hol($\mathbb{U}$) by $C_\varphi f = f \circ \varphi$. Then C_φ is a linear transformation of Hol($\mathbb{U}$), and is even continuous if that space is given its natural topology of uniform convergence on compact sets. Not so obvious, and indeed quite remarkable, is the fact that $C_\varphi(H^2) \subset H^2$. This result is essentially due to Littlewood [**15**] (see also [**21**, Ch. 1] and, for a different proof, [**12**, Ch. 1]).

Here is Littlewood's original proof of the boundedness of composition operators on H^2. The following special case, called *Littlewood's subordination principal*, is the key. It says that holomorphic selfmaps of $\mathbb{U}$ that fix the origin induce composition operators that act contractively on H^2.

Theorem 1.1. [**15**, 1925] *If φ is a holomorphic selfmap of $\mathbb{U}$ with $\varphi(0) = 0$, then*

$$\|f \circ \varphi\| \le \|f\| \qquad \text{for every } f \in H^2.$$

Proof. It is easy to check that the backward shift T_z^*, discussed at the end of §1.1, has this representation on H^2:

$$T_z^* f(z) = \frac{f(z) - f(0)}{z} \qquad (f \in H^2, z \in \mathbb{U})\,. \tag{1.6}$$

Thus for each $f \in H^2$:

$$f(z) = \hat{f}(0) + z\, T_z^* f(z) \qquad (z \in \mathbb{U}). \tag{1.7}$$

Suppose now that f is a polynomial, so there is no question about $f \circ \varphi$ belonging to H^2. It follows from (1.7) that

$$C_\varphi f = \hat{f}(0) + T_\varphi\, C_\varphi T_z^* f. \tag{1.8}$$

Since $\varphi(0) = 0$, the two summands on the right-hand side of the (1.8) are orthogonal in H^2, so

$$\|C_\varphi f\|^2 = |\hat{f}(0)|^2 + \|T_\varphi C_\varphi T_z^* f\|^2$$

Since $\|\varphi\|_\infty \le 1$ the Toeplitz operator T_φ is, by (1.3), a contraction on H^2, so we arrive at this key inequality:

$$\|f \circ \varphi\|^2 \le |\hat{f}(0)|^2 + \|C_\varphi T_z^* f\|^2\,. \tag{1.9}$$

Upon applying (1.9) with $T_z^* f$ in place of f, and noting that $\widehat{T_z^* f}(0) = \hat{f}(1)$, we obtain

$$\|(T_z^* f) \circ \varphi\|^2 \le |\hat{f}(1))|^2 + \|(T_z^{*2} f) \circ \varphi\|^2\,,$$

which, upon substitution into (1.9) yields

$$\|f \circ \varphi\|^2 \le |\hat{f}(0)|^2 + |\hat{f}(1)|^2 + \|(C_\varphi T_z^{*2} f)\|^2\,.$$

Now let n denote the degree of the polynomial f. Then $T_z^{*n+1} f = 0$, so we obtain at the $n+1$-st iteration of the substitution process described above:

$$\begin{aligned} \|C_\varphi f\|^2 &\leq |\hat{f}(0)|^2 + |\hat{f}(1)|^2 + \cdots + |\hat{f}(n)|^2 + \|C_\varphi T_z^{*n+1} f\|^2 \\ &= |\hat{f}(0)|^2 + |\hat{f}(1)|^2 + \cdots + |\hat{f}(n)|^2. \end{aligned}$$

Thus $\|C_\varphi f\| \leq \|f\|$ for each polynomial f; it is is a routine exercise to extend this inequality to all functions in H^2. □

COROLLARY 1.2. *For any holomorphic selfmap φ of $\mathbb{U}$, the composition operator C_φ is a bounded operator on H^2.*

PROOF. We need only consider the case where φ does not fix the origin. For this let $a = \varphi(0)$ and set

$$\alpha(z) = \frac{a - z}{1 - a^* z} \qquad (z \in \mathbb{C}\backslash\{1/a^*\})$$

(recalling that a^* denotes the complex conjugate of a). Then α is a conformal automorphism of $\mathbb{U}$ that is its own inverse, hence $\psi = \alpha \circ \varphi$ is a holomorphic selfmap of $\mathbb{U}$ that fixes the origin, and

$$\varphi = \alpha \circ \psi. \tag{1.10}$$

A straightforward change of variable shows that the composition operator C_α is bounded on H^2 (see, e.g., [**21**, page 16]). The factorization (1.10) translates to one at the operator level: $C_\varphi = C_\psi C_\alpha$. Since $\psi(0) = 0$ the operator C_ψ is, by Theorem 1.1, bounded on H^2 hence C_φ is the product of two bounded operators, so is itself bounded. □

We saw in §1.1 that every reproducing kernel is an eigenfunction for adjoints of analytic Toeplitz operators (see equation (1.5d)). The situation for composition operators is different, but equally intriguing and, as we will soon see, useful.

PROPOSITION 1.3. *$C_\varphi^* K_a = K_{\varphi(a)}$ for any holomorphic selfmap φ of $\mathbb{U}$ and any point $a \in \mathbb{U}$.*

PROOF. For $f \in H^2$:

$$\langle f, C_\varphi^* K_a \rangle = \langle C_\varphi f, K_a \rangle = f(\varphi(a)) = \langle f, K_{\varphi(a)} \rangle. \qquad \square$$

1.3. What's to come. Here is an outline of the rest of the paper.

- §2. A discussion of what is perhaps the most obvious connection between our two classes of operators: *composition operators intertwine certain pairs of Toeplitz operators.*
- §3. The role of Toeplitz operators in the computation of composition operator *adjoints.*
- §4. The role of Toeplitz operators in questions concerning the *normality* of composition operators.
- §5. How much "toeplitzness" can a composition operator have?

2. Intertwining

We saw in the last section that analytic Toeplitz operators occur naturally in Littlewood's argument establishing the boundedness of composition operators on the Hardy space H^2. In that proof the transition from equation (1.7) to (1.8) was effected by the (not explicitly mentioned) intertwining relation $C_\varphi T_z = T_\varphi C_\varphi$. More generally, if $b \in H^\infty$ and φ is any holomorphic selfmap of $\mathbb{U}$, then

$$C_\varphi T_b = T_{b\circ\varphi} C_\varphi$$

i.e., C_φ *intertwines* T_b and $T_{b\circ\varphi}$.

For g and h in Hol($\mathbb{U}$), let's call h *subordinate* to g if $h = g \circ \varphi$, where φ is a holomorphic selfmap of $\mathbb{U}$ (this is slightly weaker than the classical terminology, which requires "subordinate" to also include the restriction that $\varphi(0) = 0$).

Definition. *For (bounded linear) operators S, T and X on a Hilbert space to say that X* intertwines *S with T means that $X \neq 0$ and $XS = TX$. When this happens we write "$S \propto_X T$."*

If we wish to de-emphasize the role of the intertwining operator X, we'll simply write "$S \propto T$." Thus the gist of the last paragraph can be summarized: *For bounded holomorphic functions g and h on $\mathbb{U}$:*

$$h \text{ subordinate to } g \implies T_g \propto T_h.$$

This observation raises two questions, which we will explore in the remainder of this section. If $T_g \propto T_h$:

(a) Is h subordinate to g? Equivalently; can the intertwining be effected by a composition operator?

(b) Is $h(\mathbb{U}) \subset g(\mathbb{U})$? (This will be the case whenever h is subordinate to g.)

In studying these questions, the eigenvalue equation (1.5d) plays an essential role. Suppose, to get started, that $g, h \in H^\infty$ and $T_g \propto T_h$. Thus there is a bounded linear operator $X \neq 0$ on H^2 such that $XT_g = T_h X$, or equivalently, $X^* T_h^* = T_g^* X^*$. For any point $z \in \mathbb{U}$, apply both sides of this last equation to the reproducing kernel K_a, obtaining by (1.5d);

$$T_g^*(X^* K_a) = h(a)^*(X^* K_a) \qquad (a \in \mathbb{U}).$$

Thus *$h(a)^*$ is a T_g^*-eigenvalue whenever $X^* K_a \neq 0$.* Since the map $a \to K_a$ is an H^2-valued holomorphic function on $\mathbb{U}$, its set of zeros is a discrete subset of $\mathbb{U}$, hence $h(\mathbb{U})^*$ is contained in the spectrum of T_g^*, and therefore $h(\mathbb{U})$ is contained in the spectrum of T_g.

Now the spectrum of T_g is nothing more than the closure of $g(\mathbb{U})$. To see why, note first that by (1.5d) the spectrum of T_g^* contains $g(\mathbb{U})^*$, hence that of T_g contains $g(\mathbb{U})$, and therefore its closure. For the other direction, observe that if λ is not in the closure of $g(\mathbb{U})$ then $g - \lambda$ is bounded away from zero; i.e., its reciprocal belongs to H^∞. Thus the operator $T_g - \lambda I = T_{g-\lambda}$ is invertible, with inverse $T_{1/(g-\lambda)}$, and so λ is not in the spectrum of T_g. Upon taking complements we get the desired result.

Summarizing: *If g and h are bounded holomorphic functions on $\mathbb{U}$ with $T_g \propto T_h$, then $h(\mathbb{U})$ is contained in the closure of $g(\mathbb{U})$*; i.e., a point of $h(\mathbb{U})$ either lies in $g(\mathbb{U})$ or on its boundary.

If, in addition, h is not constant, then its image is an open subset of the plane, so we can say a bit more:

Proposition 2.1. *If g and h are nonconstant functions in H^∞ with $T_g \propto T_h$, then $h(\mathbb{U})$ lies in the interior of the closure of $g(\mathbb{U})$.*

This says that if a point of $h(\mathbb{U})$ does not lie in $g(\mathbb{U})$ then it lies on a piece of the boundary that is "entirely surrounded by" $g(\mathbb{U})$. In particular, if $g(\mathbb{U})$ is a Jordan domain (a domain whose boundary is a Jordan curve), such a point cannot lie on its boundary, and so we have:

Proposition 2.2. *If g and h are nonconstant and g maps $\mathbb{U}$ onto a Jordan domain then $T_g \propto T_h$ implies $h(\mathbb{U}) \subset g(\mathbb{U})$. If, moreover, g is univalent, then h is subordinate to g.*

Only the second sentence needs further comment. Whenever g is univalent and $h(\mathbb{U}) \subset g(\mathbb{U})$, then h is subordinate to g. Indeed, $h = g \circ \varphi$, where $\varphi = g^{-1} \circ h$ maps $\mathbb{U}$ holomorphically into itself. Thus if g maps $\mathbb{U}$ univalently onto a Jordan domain, then T_g can be intertwined with T_h by means of the composition operator C_φ, where $\varphi = g^{-1} \circ h$.

The conclusion of Proposition still holds with $g(\mathbb{U})$ any simply connected domain. The key to this improvement is the following strengthening of Proposition 2.1, which I present without proof. For its statement, let's agree to call a point in a plane set E *capacitarily isolated in* E whenever there is a neighborhood of the point whose intersection with E has logarithmic capacity zero.

Theorem 2.3. [**3**, Theorem 3.2] *If $g, h \in H^\infty$ and $T_g \propto T_h$, then for each $z \in \mathbb{U}$, either $h(z) \in \mathbb{U}$ or $h(z)$ is capacitarily isolated in $\partial g(\mathbb{U})$.*

Since no point of a nondegenerate continuum is capacitarily isolated therein, we have the following improvement of Proposition 2.2.

Corollary 2.4. [**3**, Corollary 3.3] *If $g, h \in H^\infty$ and $\partial g(\mathbb{U})$ consists entirely of nondegenerate continua, then $T_g \propto T_h$ implies $h(\mathbb{U}) \subset g(\mathbb{U})$.*

If $g \in H^\infty$ is any *covering map* then an argument involving the monodromy theorem shows that $h(\mathbb{U}) \subset g(\mathbb{U})$ implies that h is subordinate to g (see, for example, [**3**, Theorem 3.4]). Thus, for covering maps that take the unit disc onto bounded domains with only nondegenerate continua as boundary components, there is the following complete equivalence between intertwining, image containment, and subordination.

Corollary 2.5. *If $f, g \in H^\infty$ and g is a covering map for which every component of $\partial g(\mathbb{U})$ is a nondegenerate continuum, then the following are equivalent:*

(a) $T_g \propto T_h$.
(b) $h(\mathbb{U}) \subset g(\mathbb{U})$.
(c) *h is subordinate to g.*

The function g will, in particular, satisfy the hypotheses of this Corollary if it is univalent. To see that intertwining need not always imply subordination (i.e., intertwining by a composition operator), note that the following result, a special case of one due to Deddens [**10**, Corollary 2, page 861], shows that $T_{z^2} \propto T_z$, even though the function $h(z) \equiv z$ is clearly not subordinate to $g(z) \equiv z^2$.

Theorem 2.6. *If g is an inner function with $g(0) = 0$ then $T_g \propto T_h$ for every holomorphic selfmap h of $\mathbb{U}$.*

In the above statement, an *inner function* is a holomorphic function on $\mathbb{U}$, with all values lying in $\mathbb{U}$, whose radial limits have modulus 1 at almost every point of $\partial\mathbb{U}$.

REMARK. With a little extra work one can weaken the hypothesis that g fix the origin to: "$0 \in g(\mathbb{U})$." However Deddens' result does not require even this. At full strength it shows that intertwining need not imply image-containment. For example, if g is a nonconstant inner function with $g(\mathbb{U}) \neq \mathbb{U}$ (e.g., g could be the unit singular function, or more generally the covering map taking $\mathbb{U}$ onto $\mathbb{U}\backslash K$ for any nonvoid compact set of logarithmic capacity zero [**6**, page 37]), and $h(z) \equiv z$, then $T_g \propto T_h$ even though $h(\mathbb{U}) = \mathbb{U}$ is not contained in $g(\mathbb{U})$ (which $= \mathbb{U}\backslash K$). For a different proof of Deddens' result see [**3**, Theorem 5.1].

PROOF OF THEOREM. Being a holomorphic selfmap of the unit disc, g induces a composition operator C_g on H^2. I claim that

$$C_g^* T_g = T_z C_g^* \tag{2.1}$$

i.e., the adjoint of C_g intertwines T_g with T_z. This will imply the general result, since for any holomorphic selfmap h of $\mathbb{U}$, application of C_h to both sides of (2.1) will yield

$$C_h C_g^* T_g = C_h T_z C_g^* = T_h C_h C_g^* \,,$$

so the operator $C_h C_g^* \neq 0$ intertwines T_g and T_h, as desired.

To prove (2.1), fix f and h in H^2 and observe that

$$\begin{aligned}
\langle T_z C_g^* f,\, h\rangle &= \langle f,\, C_g T_z^* h\rangle \;=\; \langle f,\, C_g\, \frac{h-h(0)}{z}\rangle \\
&= \langle f,\, \frac{h\circ g - h(0)}{g}\rangle \;=\; \langle f,\, g^*(h\circ g - h(0))\rangle \\
&= \langle g\cdot f,\, h\circ g - h(0)\rangle \;=\; \langle g\cdot f,\, h\circ g\rangle - h(0)^*\langle g\cdot f,\, 1\rangle \\
&= \langle T_g f,\, C_g h\rangle - h(0)g(0)f(0) \;=\; \langle C_g^* T_g f,\, h\rangle\,,
\end{aligned}$$

where the last equality of the second line uses the fact that $1/g = g^*$ at a.e. point of $\partial\mathbb{U}$ (since g is inner), that of the third line uses the fact that the constant function 1 is the reproducing kernel for the origin, and the final equality uses the hypotheses $g(0) = 0$. This establishes (2.1). □

3. Adjoints

What is the adjoint of a composition operator C_φ? By definition it is the operator C_φ^* given by the equation

$$\langle C_\varphi^* f,\, g\rangle = \langle g,\, C_\varphi\rangle \qquad (f, g \in H^2),$$

from which we derived Proposition 1.3 expressing the fact that such adjoints permute reproducing kernels. Since the reproducing kernels span a dense subspace of H^2 one might regard the conclusion of Proposition 1.3 as a formula for C_φ^*. If, however, one seeks a formula valid for all functions in H^2, then the story is different; no general formula is known that neatly expresses C_φ^* in terms of natural operators related to φ. This section will present some of what is known, and in the process give some idea of why the problem is so difficult.

The first general result on the adjoint problem was obtained about twenty years ago by Carl Cowen [**8**], who discovered a formula for the adjoints of composition operators induced by linear fractional selfmaps of $\mathbb{U}$. In this formula analytic Toeplitz operators play a crucial role. Recently a number of authors have set out to extend Cowen's result to larger classes of composition operators [**9, 16, 17**], with Hammond, Moorhouse and Robbins [**14**]—building on ideas in [**9**]—succeeding in obtaining a formula for C_φ^* when φ is a *rational* selfmap of $\mathbb{U}$. Once again a sort of analytic Toeplitz operator appears, but now the situation is complicated by branching behavior. In this section I will give a proof of Cowen's adjoint formula, derive a couple of variants, outline a "simple" proof that Paul Bourdon and I found [**4**] of the Hammond-Moorehouse-Robbins formula, and discuss some consequences.

3.1. The adjoint of a linear fractional composition operator. Until further notice φ denotes a nonconstant linear fractional map that takes the unit disc into itself; explicitly

$$\varphi(z) = \frac{az+b}{cz+d} \tag{3.1}$$

with $ad - bc \neq 0$ (guaranteeing nonconstancy). It is useful to view φ as a homeomorphism of the Riemann sphere $\hat{\mathbb{C}}$ by setting $\varphi(\infty) = a/c$ and $\varphi(-d/c) = \infty$, with the non-constancy condition on the coefficients guaranteeing that these definitions are unambiguous. The condition "$\varphi(\mathbb{U}) \subset \mathbb{U}$" can be characterized nicely in terms of the complex coefficients a, b, c, d of φ (see [**7**, Proposition A and Lemma 1]), but we will not need to worry about this here.

Cowen's formula for C_φ^* involves three functions constructed from the coefficients of φ. The first of these is another linear fractional map σ, defined by:

$$\sigma(z) := \frac{a^* z - c^*}{-b^* z + d^*}\,. \tag{3.2a}$$

Upon letting $\rho : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ denote the mapping of inversion in the unit circle ($\rho(z) = 1/z^*$, with $\rho(\infty) = 0$ and $\rho(0) = \infty$) and doing a bit of algebra, we discover that

$$\sigma = \rho \circ \varphi^{-1} \circ \rho \tag{3.2b}$$

where φ^{-1} is the compositional inverse of $\varphi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. This makes it easy to see that σ also maps the unit disc into itself, and so induces a composition operator on H^2.

The other two functions that show up in Cowen's formula are:

$$h(z) = cz + d \tag{3.2c}$$

and

$$g(z) = \frac{1}{-b^* z + d^*}\ , \tag{3.2d}$$

both of which lie in H^∞: the first obviously so, and the second because its pole $-d^*/b^* = \rho(\varphi(0))$ lies outside the closure of the unit disc. Thus g and h both induce analytic Toeplitz operators on H^2. Here, finally, is Cowen's theorem.

THEOREM 3.1. [**8**, Theorem 2, pp. 153–154] *Suppose φ is a nonconstant linear fractional selfmap of $\mathbb{U}$, given by* (3.1)*, with σ, h, and g given by equations* (3.2)*. Then on H^2:*

$$C_\varphi^* = T_g C_\sigma T_h^*\,. \tag{3.3}$$

Proof. For $f \in H^2$ and $z \in \mathbb{U}$ we have from the reproducing-kernel formula (1.5c) that

$$C_\varphi^* f(z) = \langle C_\varphi^* f,\, K_z\rangle = \langle f,\, C_\varphi K_z\rangle \tag{3.4}$$

Now for each $w \in \mathbb{U}$:

$$\begin{aligned} C_\varphi K_z(w) &= K_z(\varphi(w)) = \frac{1}{1 - z^*\varphi(w)} \\ &= \frac{1}{1 - z^*\left(\frac{aw+b}{cw+d}\right)} \\ &= \frac{cw+d}{(-bz^*+d) - (az^*-c)w} \\ &= \frac{1}{-bz^*+d} \cdot (cw+d) \cdot \frac{1}{1 - \left(\frac{az^*-c}{-bz^*+d}\right) w} \\ &= g(z)^* \cdot h(w) \cdot K_{\sigma(z)}(w)\,, \end{aligned}$$

where the last line is justified by the fact that $\sigma(z) \in \mathbb{U}$. Upon substituting this result into (3.4) we obtain

$$\begin{aligned} C_\varphi^* f(z) &= \langle f,\, g(z)^* \cdot T_h K_{\sigma(z)}\rangle = g(z) \cdot \langle T_h^* f,\, K_{\sigma(z)}\rangle \\ &= g(z) \cdot (T_h^* f)(\sigma(z)) = (T_g C_\sigma T_h^* f)(z)\,, \end{aligned}$$

thus establishing (3.3). □

Cowen's theorem has led to important resuts on composition operators induced by linear fractional transformations. Cowen himself used it to study co-, sub-, and hypo-normality for such operators [**8**, pp. 156–159], and to prove a striking formula for the norm of a composition operator induced by a map of the form $\varphi(z) = az + b$ [**8**, Theorem 3, page 154]. For some further references to applications of Cowen's formula, see the first paragraph of [**4**, page 1996].

3.2. Variants of Cowen's formula. For $f \in H^2$ Cowen's formula (3.3) yields, for each $z \in \mathbb{U}$:

$$C_\varphi^* f(z) = g(z)[(c^* T_z^* + d^*) f](\sigma(z))\,. \tag{3.5}$$

Now supppose $f(0) = 0$. From (1.6) we have $T_z^* f(z) = f(z)/z$, so

$$(T_z^* f)(\sigma(z)) = \frac{f(\sigma(z))}{\sigma(z)}\,.$$

Upon substituting this last expression into (3.5), expressing everything in terms of the coefficients of φ, and doing some patient calculation, we obtain

$$C_\varphi^* f(z) = z\,\frac{\sigma'(z)}{\sigma(z)}\, f(\sigma(z)) = z\,\sigma'(z)\,(T_z^* f)(\sigma(z)) \qquad (f \in H^2, f(0) = 0), \tag{3.6}$$

where the first of these expressions is valid for all $z \in \mathbb{U}$ with $\sigma(z) \neq 0$, i.e., for $z \neq c^*/a^* = \rho(\varphi(\infty))$, and the second is valid for all $z \in \mathbb{U}$.

For general $f \in H^2$ the idea is to write $f(z) = [f(z) - f(0)] + f(0) \cdot K_0$, where K_0, the reproducing kernel for the origin, is just the constant function 1. Then an application of Proposition 1.3 yields

$$C_\varphi^* f = C_\varphi^*[f - f(0)] + f(0)\, C_\varphi^* K_0 = C_\varphi^*[f - f(0)] + f(0)\, K_{\varphi(0)}\,.$$

If we now apply the second formula of (3.6) with $f - f(0)$ in place of f, and use the fact that T_z^*, being the backward shift, annihilates constant functions, we obtain

$$C_\varphi^* f(z) = f(0)K_{\varphi(0)}(z) + z\sigma'(z)f(\sigma(z))\,.$$

Upon writing $\gamma(z) = z\sigma'(z)$ and using the notation $v \otimes w$ for the rank-one Hilbert space operator that takes the value $\langle h, w\rangle v$ at the vector h, we obtain our first alternate formula for C_φ^*.

Corollary 3.2. *For φ a linear fractional selfmap of $\mathbb{U}$ let $\sigma = \rho \circ \varphi^{-1} \circ \rho$. Then σ is a linear fractional selfmap of $\mathbb{U}$, and*

$$C_\varphi^* = (K_{\varphi(0)} \otimes 1) \;+\; T_\gamma C_\sigma T_z^*\,. \tag{3.7}$$

The same reasoning, using the first equality of (3.6) instead of the second one, yields for each $f \in H^2$:

$$C_\varphi^* f(z) \;=\; z\,\frac{\sigma'(z)}{\sigma(z)}\,f(\sigma(z)) \;-\; f(0)\left[K_{\varphi(0)}(z) - z\,\frac{\sigma'(z)}{\sigma(z)}\right],$$

at least for each z for which $\sigma(z) \neq 0$, i.e., for which $z \neq c^*/a^* = \rho(\varphi(\infty))$. More patient calculation with coefficients shows that the term in square brackets on the right-hand side of the last displayed equation is

$$-\frac{c^*}{a^*z - c^*} \;=\; \frac{1}{1 - \varphi^*(\infty)z}\,.$$

Thus we have a second variant of Cowen's Theorem:

Corollary 3.3. *If φ is a linear fractional selfmap of $\mathbb{U}$, $\sigma := \rho \circ \varphi^{-1} \circ \rho$, and $\Gamma(z) = z\sigma'(z)/\sigma(z)$, then for each $f \in H^2$:*

$$C_\varphi^* f(z) \;=\; \frac{f(0)}{1 - \varphi(\infty)^* z} \;+\; \Gamma(z)f(\sigma(z)) \qquad (z \in \mathbb{U}\backslash\{\rho(\varphi(\infty)\}\,. \tag{3.8a}$$

If, moreover, $\varphi(\infty) \in \mathbb{U}$, then $\Gamma \in H^\infty$, and

$$C_\varphi^* \;=\; (K_{\varphi(\infty)} \otimes 1) \;+\; T_\Gamma\, C_\sigma\,. \tag{3.8b}$$

Note that in this variant, if $|\varphi(\infty)| \geq 1$ then each summand on the right-hand side of (3.8a) will have a simple pole at $\rho(\varphi(\infty))$ in the closed unit disc. However (3.8a) assures us that for each $f \in H^2$ these poles will cancel each other, leaving the sum in H^2. In the best possible case, namely $\varphi(\infty) \in \mathbb{U}$, let us agree to call an operator of the form $T_\Gamma C_\sigma$, i.e., the product of an analytic Toeplitz operator with a composition operator, a *weighted composition operator*. Then (3.8b) can be summarized like this:

Corollary 3.4. *If φ is a linear fractional selfmap of $\mathbb{U}$ with $\varphi(\infty) \in \mathbb{U}$ then C_φ^* is a rank-one perturbation of a weighted composition operator.*

3.3. Extension to rational selfmaps of $\mathbb{U}$. Can Cowen's formula (3.3), or one of its variants (3.7) or (3.8a) be generalized—say to *rational* selfmaps of $\mathbb{U}$? The simplest such example, C_φ with $\varphi(z) \equiv z^2$, hints ominously at what lies ahead. For this φ it is easy to compute that if $f(z) = \sum_{n=0}^\infty \hat{f}(n)z^n \in H^2$ then

$$C_\varphi^* f(z) = \sum_{n=0}^\infty \hat{f}(2n)z^n \qquad (z \in \mathbb{U}),$$

which we might optimistically try to abbreviate as $C_\varphi^* = (C_\sigma + C_{-\sigma})/2$, where $\sigma(z) = \sqrt{z}$. Taken literally, of course, this formula for C_φ^* makes no sense, since holomorphic branches of the square root do not exist on the entire unit disc. However we can define a holomorphic square root σ on any simply connected subdomain V of $\mathbb{U}\backslash\{0\}$, and so our power series representation for C_φ^* can be rewritten at least "locally on V" as

$$C_\varphi^* f(z) = \frac{1}{2}[f(\sigma(z) + f(-\sigma(z))] \qquad (z \in V).$$

Keeping in mind this cautionary tale, let's suppose φ is a rational function that maps $\mathbb{U}$ into itself, and that d is its order (the largest of the orders of p and q, where $\varphi = p/q$, with p and q polynomials having no common nonconstant factor). Just as in the linear fractional case (the case $d = 1$), the idea will be to express, for fixed $z \in \mathbb{U}$, the rational function

$$R_z(w) := C_\varphi K_z(w) = \frac{1}{1 - z^*\varphi(w)}$$

in terms of reproducing kernels. For $d > 1$ this will involve expanding R_z in partial fractions, and trying to manipulate the result to reveal the presence of reproducing kernels. For this to work we'd best consider only those $z \in \mathbb{U}$ for which R_z has *finite simple poles*; this will exclude at most a finite subset E of points z in $\mathbb{U}$. Thus for $z \in \mathbb{U}\backslash E$ we have

$$R_z(w) \;=\; \alpha + \sum_{j=1}^{d} \frac{\beta_j(z)}{w - w_j} \tag{3.9a}$$

where the distinct simple poles $w_1, \ldots, w_d$ of R_z comprise the set $\varphi^{-1}(\{\rho(z)\})$, and $\alpha = R_z(\infty) = (1 - z^*\varphi(\infty))^{-1}$ (which we set equal to zero if $\varphi(\infty) = \infty$). In order to insure the finiteness of each of these poles we also assume that $z \neq \rho(\varphi(\infty))$. Thus for $\rho(\varphi(\infty)) \neq z \in \mathbb{U}\backslash E$ we have

$$R_z(w) \;=\; \frac{1}{1 - z^*\varphi(\infty)} \;-\; \sum_{j=1}^{d} \frac{\beta_j(z)/w_j}{1 - w/w_j}\;. \tag{3.9b}$$

To see that reproducing kernels do indeed lurk on the right-hand side of this last equation, note that since the rational function $R_z = K_z \circ \varphi$ is holomorphic in a neighborhood of the closed unit disc, all of its poles must lie outside that disc, hence their reflections $\{\rho(w_j)\}_{j=1}^d$ all lie in $\mathbb{U}$. Thus, for each index j, the fraction $(1 - w/w_j)^{-1}$ that occurs on the right-hand side of (3.9b) is the reproducing kernel $K_{\rho(w_j)}$, whereupon (3.9b) can be rewritten

$$R_z(w) \;=\; \frac{1}{1 - z^*\varphi(\infty)} \;-\; \sum_{j=1}^{d} \frac{\beta_j(z)}{w_j} K_{\rho(w_j)}\;. \tag{3.9c}$$

Upon substituting this expression for $R_z = C_\varphi K_z$ into (3.4) we obtain this:

If $z \in \mathbb{U}\backslash(E \cup \{\rho(\varphi(\infty))\})$, then for every $f \in H^2$,

$$C_\varphi^* f(z) \;=\; \langle f, R_z\rangle \;=\; \frac{f(0)}{1 - \varphi(\infty)^* z} \;-\; \sum_{j=1}^{d} \beta_j(z)^* \rho(w_j) f(\rho(w_j)) \tag{3.10}$$

where $w_1, \ldots, w_d$ are the d distinct poles of R_z.

What does this last equation mean? Recall that the poles of R_z constitute the set $\varphi^{-1}(\{\rho(z)\})$, so the set of points $\{\rho(w_j)\}_{j=1}^d$ that occurs in (3.10) is the inverse image of the singleton $\{z\}$ under the map $\varphi_e := \rho \circ \varphi \circ \rho$. Now φ_e is a rational function of degree d that maps

$$\mathbb{U}_e \;:=\; \hat{\mathbb{C}} \setminus \{\text{the closed unit disc}\}$$

into itself. Thus φ_e^{-1} takes subsets of $\mathbb{U}$ into $\mathbb{U}$ (providing another proof that the points $\rho(w_j)$ all lie in $\mathbb{U}$). Elementary function theory tells us that, since $\varphi_e^{-1}(\{z\}) = \{\rho(w_j)\}_{j=1}^d$ consists of d distinct points, z itself has a neighborhood $V \subset \mathbb{U}$ such that $\varphi_e^{-1}(V)$ is the union of a pairwise disjoint family of open subsets $W_1, \ldots, W_d$ of $\mathbb{U}$, with each V_j a neighborhood of $\rho(w_j)$, and φ_e a univalent map taking W_j onto V. Thus, for each index j, the restriction of φ_e to W_j has a holomorphic inverse σ_j taking V onto W_j. We call the functions σ_j *branches of* φ_e^{-1} *on* V. With this notation we can rewrite the result containing (3.10) as follows:

If $z \neq \rho(\varphi(\infty))$ lies in $\mathbb{U}\backslash E$, and $f \in H^2$, then

$$C_\varphi^* f(z) \;=\; \frac{f(0)}{1-\varphi(\infty)^* z} \;-\; \sum_{j=1}^{d} \beta_j(z)^* \sigma_j(z) f(\sigma_j(z)) \tag{3.11}$$

where $\sigma_1, \ldots, \sigma_d$ are d distinct branches of φ_e^{-1} defined on a neighborhood of z.

This formula bears an uncanny resemblance to (3.8a) of Corollary 3.3, our second variant of the formula for the linear fractional case. The generalization would be complete if only we could prove that

$$\beta_j^*(z) = \frac{z\sigma_j'(z)}{\sigma_j(z)^2} \qquad (j = 1, \ldots, d).$$

This is, in fact, the case. The computation starts out simply enough since, by our assumption on z, the poles of R_z are simple and the β_j's are the residues of R_z at these poles, but the rest requires more computation, for which I'll refer you to [**4**, §2.4]. When all is done, we obtain a generalization of Corollary 3.3, originally derived via a different method by Hammond, Moorhouse, and Robbins [**14**]. To state it succinctly let us say that a point z of the Riemann sphere is a *regular value* of a rational function R of degree $d > 0$ if the inverse image of that point under R contains d distinct points. As we mentioned above, in this case there is a neighborhood of z on which there exist d distinct branches of R^{-1}.

THEOREM 3.5. *Suppose φ is a rational selfmap of $\mathbb{U}$. Let $\varphi_e = \rho \circ \varphi \circ \rho$ and let $z \in \mathbb{U}\backslash\{\rho(\varphi(\infty))\}$ be a regular value of φ_e. Then there are d distinct branches $\sigma_1, \cdots, \sigma_d$ of φ_e^{-1} defined on a neighborhood of z, and for each $f \in H^2$:*

$$C_\varphi^* f(z) \;=\; \frac{1}{1-\varphi(\infty)^* z} \;+\; \sum_{j=1}^{d} \Gamma_j(z) f(\sigma_j(z)) \tag{3.12}$$

where $\Gamma_j(z) = z\sigma_j'(z)/\sigma_j(z)$ for $j = 1, \ldots, d$.

As in our discussion of Corollary 3.3, it may happen that some of the functions σ_j have zeros in $\mathbb{U}$, in which case there are pole-cancellation miracles that render the left hand side of (3.12) holomorphic on $\mathbb{U}$ (see [**4**, §2.3] for more details). There is a variant of this result that generalizes Corollary 3.2 to the case $d > 1$;

Corollary 3.6. [**4**, §3.1] *With the notation of Theorem 3.5, for each $z \in \mathbb{U}$ that is a regular value of φ_e,*

$$C_\varphi^* f(z) \;=\; f(0)K_{\varphi(0)} \;+\; \sum_{j=1}^{d} \gamma_j(z)(T_z^* f)(\sigma_j(z)) \qquad (f \in H^2),$$

where $\gamma_j(z) = z\sigma_j'(z)$.

It is tempting to write the above equation as

$$C_\varphi^* \;=\; (K_{\varphi(0)} \otimes 1) \;+\; \sum_{j=1}^{d} T_{\gamma_j} C_\sigma T_z^* \,, \tag{3.13}$$

and this will indeed be the case whenever each of the multipliers γ_j is holomorphic and bounded on $\mathbb{U}$. This in turn will happen whenever each of the branches of φ_e^{-1} extends holomorphically to the entire unit disc, which will be the case whenever each point of the disc is a regular value of φ_e, or equivalently, whenever each point of $\mathbb{U}_e$ is a regular point of φ (see [**4**, §1], for more details). If, in addition, $\varphi(\infty) \in \mathbb{U}$ then it's legitimate to write

$$C_\varphi^* \;=\; (K_{\varphi(\infty)} \otimes 1) \;+\; \sum_{j=1}^{d} T_{\Gamma_j} C_{\sigma_j} \,, \tag{3.14}$$

since $\sigma_j(z) = 0$ for some index j if and only if $\varphi(\infty) = \rho(z)$, which lies outside the closed unit disc as long as $z \in \mathbb{U}$. Thus, our assumption that $\varphi(\infty) \in \mathbb{U}$ also guarantees the boundedness of the multipliers $\Gamma_j(z)$ on $\mathbb{U}$.

Here is an example that shows how this sort of behavior can lead to interesting consequences.

Example. Let $\varphi(z) = \frac{1}{3-z-z^2}$. Then $\varphi_e(w) = 3 - w^{-1} - w^{-2}$ hence the two right-inverses of φ_e, obtained by solving the equation $z = \varphi_e(w)$, are

$$\sigma_1(z) := \frac{1+\sqrt{13-4z}}{2(3-z)} \qquad \text{and} \qquad \sigma_2(z) = \frac{1-\sqrt{13-4z}}{2(3-z)}$$

where "$\sqrt{}$" denotes the principal branch of the square root function. Thus σ_1 and σ_2 are holomorphic on $\mathbb{U}$, and—as we observed earlier—they automatically map $\mathbb{U}$ into itself. Since $\varphi(\infty) = 0 \in \mathbb{U}$, and $K_0 = 1$, (3.14) applies and yields

$$C_\varphi^* = (1 \otimes 1) + T_{\gamma_1} C_{\sigma_1} + T_{\gamma_2} C_{\sigma_2} \ .$$

More can be said about this example. Note that $\sigma_1(1) = 1$, but $\sigma_2(1) = -1/2$. In fact, upon rationalizing the denominator of σ_2 we see quickly that $|\sigma_2(z)| < 1/2$ for every $z \in \mathbb{U}$, hence *the composition operator that σ_2 induces on H^2 is compact* (see [**21**, §2.2] for example).

Upon recalling that we have agreed to call the product of an analytic Toeplitz operator and a composition operator a *weighted composition operator*, the essential point of the result just obtained can be summarized as follows:

> *If $\varphi(z) = (3 - z - z^2)^{-1}$ then C_φ^* is a compact perturbation of a weighted composition operator.*

This example is but a special case of the following generalization of Corollary 3.4; for the proof see [**4**, Corollary 15(b)].

Corollary 3.7. *Suppose φ is a rational selfmap of $\mathbb{U}$ that maps exactly one point of $\partial\mathbb{U}$ into $\partial\mathbb{U}$. If $\varphi(\infty) \in \mathbb{U}$ and every point of $\hat{\mathbb{C}}\backslash\mathbb{U}$ is regular for φ, then C_φ^* is a compact perturbation of a weighted composition operator.*

For further examples, see [**4**, §4,5].

4. Normality

Recall that an operator T on a Hilbert space is *normal* whenever it commutes with its adjoint, and is *essentially normal* if $T^*T - TT^*$ is compact. This section deals with the problem of determining normality and essential normality for composition operators. As has been the case in previous sections, Toeplitz operators will play an important role.

4.1. Normality. The normal composition operators were characterized more than forty years ago by Howard Schwartz [**20**]. Observe that, trivially, for each $a \in \mathbb{U}$ the dilation $\varphi_a(z) \equiv az$ induces a normal composition operator on H^2 (for example, its matrix, with respect to the orthonormal monomial basis for H^2, is diagonal). Schwartz proved that *these are the only ones.* His proof is both beautiful and unpublished; here it is.

The key is the following elementary "normality lemma:"

> *Suppose T is a normal operator on a Hilbert space and $Tf = \lambda f$ for some complex number λ and non-zero vector f. Then $T^*f = \lambda^* f$.*

To prove this it is only necessary, since $T - \lambda I$ is also normal, to consider the case $\lambda = 0$, in which case the statement reduces to: $\ker T = \ker T^*$. This follows readily from the easily proved fact that for T normal, $\|Tf\| = \|T^*f\|$ for any vector f.

Now suppose C_φ is normal on H^2. The goal is to prove that $\varphi = \varphi_a$ for some $a \in \mathbb{U}$. A good start might be to prove that $\varphi(0) = 0$, and this can be accomplished readily thanks to the "normality lemma." Indeed, $C_\varphi 1 = 1$, hence by normality $C_\varphi^* 1 = 1$, hence upon letting $u(z) \equiv z$,

$$0 = \langle u,\, 1\rangle = \langle u,\, C_\varphi^* 1\rangle = \langle C_\varphi u,\, 1\rangle = \langle \varphi,\, 1\rangle = \varphi(0)\,.$$

The proof is completed by showing that $\varphi(z) \equiv \varphi'(0)z$. For this, fix $f \in H^2$ and calculate (where once again u denotes the identity function on $\mathbb{U}$):

$$\langle f,\, C_\varphi^* u\rangle = \langle C_\varphi f,\, u\rangle = \widehat{f\circ\varphi}(1) = (f\circ\varphi)'(0) = f'(\varphi(0))\varphi'(0)\,.$$

Since we've already seen that $\varphi(0) = 0$, the result of this last calculation can be rewritten

$$\langle f,\, C_\varphi^* u\rangle = \langle \varphi'(0)f,\, u\rangle = \langle f,\, \varphi'(0)^* u\rangle\,.$$

Since $f \in H^2$ is arbitrary, this implies that $C_\varphi^* u = \varphi'(0)^* u$ so applying the normality lemma one more time we obtain the desired result: $C_\varphi u = \varphi'(0)u$. □

Note that the proof actually shows:

> *If C_φ commutes with its adjoint on just the two vectors 1 and u, then φ is a dilation (and so C_φ is normal).*

4.2. Essential normality. A Hilbert space operator T is *essentially normal* if its self-commutator $[T, T^*] := T^*T - TT^*$ is compact. Clearly normal and compact operators are essentially normal—let's call these the *trivially* essentially normal ones. For a *nontrivial* essentially normal operator, consider the forward shift T_z on H^2. As noted noted earlier, its adjoint is the backward shift, and a quick computation shows that the commutator $[T_z, T_z^*]$ is just the orthogonal projection of H^2 onto the subspace of constant functions. Thus T_z is essentially normal, and —essential normality being preserved by the taking of adjoints—the same is true of T_z^*.

For composition operators, the study of essential normality was begun about ten years ago by Nina Zorboska [**23**] who proved (among other things) that:

(a) *Among the conformal automorphisms of* $\mathbb{U}$*, only the* rotations *(i.e., the dilations* φ_a *with* $|a| = 1$*) induce composition operators that are essentially normal.* In short: the essentially normal invertible composition operators are precisely the unitary ones.

(b) *If* φ *is linear fractional and fixes a point of* $\mathbb{U}$*, but is neither an automorphism nor a dilation, then* C_φ *is not essentially normal.*

Zorboska asked if *any* composition operator could be nontrivially essentially normal. For the ones induced by linear fractional maps her results restrict the problem to consideration of the non-automorphisms that fix no point of $\mathbb{U}$, i.e., to the ones that are either parabolic (conjugate to a translation of of the plane) or hyperbolic (conjugate to a positive dilation). Her question was answered a few years later by Bourdon et. al. [**2**], who proved:

THEOREM 4.1. *For* $\varphi : \mathbb{U} \to \mathbb{U}$ *linear fractional, the operator* $C_\varphi : H^2 \to H^2$ *is nontrivially essentially normal if and only if* φ *is a parabolic nonautomorphism.*

I'll devote the rest of this section to giving a detailed outline of the proof. As mentioned above, thanks to the work of Zorboska we need only consider nonautomorphisms that are either parabolic or hyperbolic. Cowen's adjoint theorem leads, for any linear fractional selfmap of $\mathbb{U}$, to the following *commutator formula*:

$$[C_\varphi^*, C_\varphi] = T_g\,[C_\sigma, C_\varphi]\,T_h^* + T_g\,C_\sigma[T_h^*, C_\varphi] + (T_g - T_{g\circ\varphi})\,C_{\sigma\circ\varphi}\,T_h^*\,, \tag{4.1}$$

where g and h are as in equations (3.2).

Notice that in the first term on the right-hand side of (4.1), the Toeplitz operators T_g and T_h are both invertible, since both g and h are invertible in H^∞ (i.e., they are bounded with bounded reciprocals). What about the last two terms? It turns out that they are *compact*, which yield the following reduction of the essential normality problem.

LEMMA 4.2. *Suppose* φ *is a linear fractional selfmap of* $\mathbb{U}$*, not an automorphism, but with a fixed point on* $\partial\mathbb{U}$*. Then* C_φ *is essentially normal if and only if* $[C_\sigma, C_\varphi]$ *is compact on* H^2*.*

PROOF. As we just saw, it is enough to show that the second and third summands on the right-hand side of (4.1) are both compact. For the second one, it's enough to show that the commutator $[T_h^*, C_\varphi]$ is compact. Now $h = cz + d$ where c and d are complex numbers, so we see after a brief calculation that it is enough to show that $\Delta := [T_z^* C_\varphi, C_\varphi T_z^*]$ is compact. In fact, Δ is a *Hilbert-Schmidt* operator. For this it suffices to show that that $\sum \|\Delta u^n\|^2 < \infty$, where, as above, $u(z) \equiv z$.

A brief computation using (1.6) shows that

$$\Delta(u^n)(z) \;=\; T_z^*\varphi^n(z) \;-\; \varphi^{n-1}(z) \;=\; \frac{\varphi(z)^n - \varphi(0)^n}{z} - \varphi(z)^{n-1}$$

so, for $z \in \partial\mathbb{U}$

$$|\Delta(u^n)(z)| \;=\; |\varphi(z)^{n-1}(\varphi(z) - z) - \varphi(0)^n| \;\leq\; |\varphi(z)|^{n-1}|\varphi(z) - z| + |\varphi(0)|^n.$$

Now by (1.2), (1.1) and the fact that φ is analytic in a neighborhood of the closed unit disc, the H^2-norm of Δu^n is just its L^2-norm over the unit circle (in fact this is true or any H^2 function, see e.g. [**19**, Ch. 17], especially Theorem 17.11). Thus

$$\sum_{n=1}^{\infty} \|\Delta(u^n)\|^2 \;\leq\; 2\sum_{n=1}^{\infty}\int_{\partial\mathbb{U}} |\varphi|^{2(n-1)}|\varphi - u|^2\,dm + 2\sum_{n=1}^{\infty}|\varphi(0)|^{2n}\,,$$

where dm denotes arclength measure on $\partial\mathbb{U}$ normalized to have total mass 1. Upon summing the geometric series on the right-hand side of the last inequality we obtain

$$\sum_{n=1}^{\infty} \|\Delta(u^n)\|^2 \;\leq\; 2\int_{\partial\mathbb{U}} \frac{|\varphi - u|^2}{1 - |\varphi|^2}\,dm \;+\; \frac{2|\varphi(0)|}{1 - |\varphi(0)|^2}\,.$$

Thus, to show that Δ is a Hilbert-Schmidt operator on H^2 it will be enough to prove that the integral on the right-hand side of the last inequality is finite. In fact, *the integrand is bounded.* To see why, recall that φ is a non-automorphism with no fixed point in $\mathbb{U}$. Thus it has a fixed point on $\partial\mathbb{U}$ which we may, without loss of generality, assume is the point 1. Thus φ maps the unit disc onto a strictly smaller subdisc whose boundary is tangent at the point 1 to the unit circle. Consequently, as $z \to 1$ through the unit circle, $|\varphi(z) - z|^2$, the numerator of our integral, goes to zero like $|1 - z|^2$ while the denominator, which is essentially the distance from $\varphi(z)$ to the unit circle, does the same. *Conclusion:* The second summand on the right-hand side of (4.1) is a Hilbert-Schmidt operator, and hence compact.

As for the third term on the right-hand side of (4.1), its first two factors have the form $T_\gamma C_\psi$, where $\psi := \sigma \circ \varphi$ is a holomorphic selfmap of $\mathbb{U}$ with a fixed point also at 1 (φ is assumed to have fixed point at 1 and hence so does $\sigma := \rho \circ \varphi^{-1} \circ \rho$), so $\gamma := g - g \circ \varphi$ vanishes at 1 to order 1. Thus

$$\sum_{n=0}^{\infty} \|(T_\gamma C_\psi)\, u^n\|^2 \;=\; \sum_{n=0}^{\infty} \|\gamma \cdot \psi^n\|^2 \;=\; \int_{\partial\mathbb{U}} \frac{|\gamma|^2}{1 - |\psi|^2}\,dm$$

where in the integral on the right, the numerator vanishes at the fixed point 1 to order 2, while—just as in the previous case—the denominator does the same. Thus the integrand is bounded, so the integral is finite. Thus $T_\gamma C_\psi$ is a Hilbert-Schmidt operator, hence so is the third summand on the right-hand side of (4.1). □

Lemma 4.2 provides an answer Zorboska's question about the existence of composition operators that are nontrivially essentially normal.

PROPOSITION 4.3. *If $\varphi \in \mathrm{LFT}(\mathbb{U})$ is a parabolic non-automorphism then C_φ is nontrivially essentially normal.*

PROOF. Since φ is not a dilation, C_φ is, by Schwartz's characterization of normality for composition operators, not normal. Neither is it compact (see [**21**, page 31], for example). So we need only check that the commutator

$$[C_\varphi, C_\sigma] \;:=\; C_\varphi \circ C_\sigma \;-\; C_\sigma \circ C_\varphi \;=\; C_{\sigma\circ\varphi} - C_{\varphi\circ\sigma}$$

is compact. In fact, much more is true: $[C_\varphi, C_\sigma] = 0$. The point is that φ, being parabolic, has a unique fixed point on the Riemann sphere, which must necessarily lie on the unit circle (else φ could not take the unit disc into itself). The same is true of $\sigma = \rho \circ \varphi^{-1} \circ \rho$, which shares the same fixed point as φ. Now linear fractional maps with the same fixed point are easily seen to commute under composition, hence the commutator of C_φ and C_σ is the zero operator. □

Proposition 4.3 establishes half of Theorem 4.1. The other half, which—according to Zorboska's results—involves only the hyperbolic nonautomorphism case, requires a little more work. By Lemma 4.2 it suffices to show that the commutator of C_φ and C_σ is *not* compact. The problem is that σ, while also a hyperbolic nonautomorphism with the same boundary fixed point as φ, does not commute with φ. What saves the day is the fact that $\psi := \varphi \circ \sigma$ *and* $\chi := \sigma \circ \varphi$ *are parabolic and have the same fixed point*, hence *they* commute. This means that the composition operators they induce have a common one-parameter family of distinct eigenvectors, and this gives rise to a nondegenerate curve of eigenvalues for $[C_\varphi, C_\sigma] = C_\psi - C_\chi$ (see [**4**, Theorem 5.2]) for the details). Thus, by the Riesz theory of compact operators, $[C_\varphi, C_\sigma]$ cannot be compact. This completes the proof of Theorem 4.1. □

One might hope to be able to use the results of §3 to generalize Theorem 4.1 to, say, rationally induced composition operators. Generalizations of this sort are possible, but the arguments proceed, not via our previous results on "rational adjoints", but rather by using perturbation techniques. See [**4**, §7] for the details.

5. Toeplitzness

So far we have seen several examples in which analytic Toeplitz operators show up in the study of composition operators. This section will attempt something different by asking just how much "toeplitzness" a composition operator can possess. Here, for the first time, we will use the most general definition of Toeplitz operator. For this our setting is the *boundary* of the unit disc on which resides the space $L^2 = L^2(m)$ of §1.1, with m denoting Lebesgue (arclength) measure on $\partial\mathbb{U}$, normalized to have total mass 1.

In this context H^2 is the subspace of L^2 consisting of functions whose Fourier coefficients of negative index all vanish. It is clearly the image of L^2 under the orthogonal projection P that leaves exponentials of non-negative index unchanged, and annihilates the others. For a function f in this "boundary" H^2, the Poisson integral (or, equally well, the Cauchy integral) is a holomorphic function in the original "interior" H^2 space of the previous sections, while for any function f in that interior space, the radial limit $f(\zeta) := \lim_{r\to 1-} f(r\zeta)$ exists for m-a.e. $\zeta \in \partial\mathbb{U}$, and defines a function in our boundary version of H^2 with the same norm as the original one. Thus the map that takes f in our "holomorphic" H^2 to its radial limit establishes an isometry between it and the "boundary version" of H^2 (see [**12**, Theorem 2.6, page 21] or [**19**, Theorem 17.11, page 340] for more details).

From now on I'll not distinguish explicitly between the two spaces, but will instead rely on context to sort things out.

Definition. For $b \in L^\infty$, the space of (m-a.e. equivalence classes of) bounded measurable functions on $\partial\mathbb{U}$, the *Toeplitz operator with symbol* b is the operator T_b

defined on H^2 by:

$$T_b f = P(bf) \qquad (f \in H^2).$$

It is easy to check that if b is the radial limit function of a function in H^∞, then T_b, viewed now as an operator on the "interior" H^2 is the analytic Toepltiz operator featured in the previous sections. But now there is something new.

PROPOSITION 5.1. *If b is in L^∞ then $T_b^* = T_{b^*}$.*

PROOF. For $f, g \in H^2$:

$$\begin{aligned} \langle T_b^* f,\, g\rangle &= \langle f,\, T_b g\rangle \;=\; \langle f,\, P(bg)\rangle \;=\; \langle Pf,\, bg\rangle \\ &= \langle f,\, bg\rangle \;=\; \langle b^* f,\, g\rangle \;=\; \langle b^* f,\, Pg\rangle \\ &= \langle P(b^* f),\, g\rangle \;=\; \langle T_{b^*} f,\, g\rangle \end{aligned}$$

which establishes the result. □

See, for example, [**11**, Chapter 7] for this, and for many deeper properties of Toeplitz operators. As promised earlier, Proposition 5.1 reveals the backward shift T_z^*, as well as the operator T_h^* that appeared in Cowen's adjoint formula, to be Toeplitz operators.

5.1. The matrix of a Toeplitz operator. Let $u(z) \equiv z$. For $b \in L^\infty$ consider the matrix of the Toeplitz operator T_b with respect to the orthonormal basis $(u^n)_0^\infty$ of H^2. The n-th column is the sequence of Maclaurin coefficients of $P(u^n b)$, thus, upon denoting by $\hat{b}(n)$ the n-th Fourier coefficient of b ($n \in \mathbb{Z}$), we see that the matrix in question is

$$\begin{bmatrix} \hat{b}(0) & \hat{b}(-1) & \hat{b}(-2) & \hat{b}(-3) & \cdots \\ \hat{b}(1) & \hat{b}(0) & \hat{b}(-1) & \hat{b}(-2) & \cdots \\ \hat{b}(2) & \hat{b}(1) & \hat{b}(0) & \hat{b}(-1) & \cdots \\ \hat{b}(3) & \hat{b}(2) & \hat{b}(1) & \hat{b}(0) & \cdots \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \end{bmatrix}$$

which is *constant on diagonals.*

Now it is easy to check that, for an operator T on a Hilbert space, its matrix with respect to an orthonormal basis $(e_n)_0^\infty$ is constant on diagonals if and only if T satisfies the operator equation $S^*TS = T$, where S is the forward shift with respect to the basis. Brown and Halmos [**5**, 1963-4] showed that for bounded operators on H^2 with the monomial basis $(u^n)_0^\infty$ (so that now $S = T_z$), the equation $S^*TS = T$ *characterizes* the Toeplitz operators in the sense that whenever it is satisfied the constant value of the n-th diagonal of the matrix of T is the n-th Fourier coefficient of a function $b \in L^\infty$ ($n \in \mathbb{Z}$), and $T = T_b$.

5.2. Which composition operators are Toeplitz? The identity operator is the analytic Toeplitz operator T_1, and also the composition operator C_φ, where $\varphi(z) \equiv z$. Are there any others? The answer is "no."

PROPOSITION 5.2. *The only composition operator that is also Toeplitz is the identity operator.*

Proof. Suppose C_φ is a Toeplitz operator, so its matrix with respect to the orthonormal basis $(u^n)_0^\infty$ is constant on diagonals. Now this matrix has as its columns the Maclaurin coefficient sequences of the successive powers of φ^n for $n = 0, 1, 2, \ldots$. Constancy on the main diagonal shows that $\hat{\varphi}(1) = 1$, while constancy on the successive subdiagonals, yields $\hat{\varphi}(n) = 0$ for $n > 1$. From the first super-diagonal we obtain the equation

$$\hat{\varphi}(0) \;=\; \hat{\varphi^2}(1) \;=\; 2\hat{\varphi(1)}\hat{\varphi(2)}$$

hence $\hat{\varphi(0)} = 2\hat{\varphi(0)}$, and so $\hat{\varphi(0)} = 0$, i.e., $\varphi(z) \equiv z$, as promised □

More generally, a composition operator can only trivially be a *compact perturbation* of a Toeplitz operator.

Theorem 5.3. [**18**, Theorem 1.1] *If C_φ has compact difference with a Toeplitz operator, then C_φ is either compact (in which case the Toeplitz operator is zero) or the identity operator (in which case the compact operator is zero).*

Proof (for a special case). For a simpler argument that still captures the essentials, I'll consider only the case of *analytic* Toeplitz operators. So suppose that for some $b \in H^\infty$ and the operator $C_\varphi - T_b := \Delta$ is compact on H^2.

Fix $z \in \mathbb{U}$ and apply Δ^* to the reproducing kernel K_z. By (1.5d) and Proposition 1.3 the result is:

$$\Delta^* K_z \;=\; C_\varphi^* K_z \;-\; T_b^* K_z \;=\; K_{\varphi(z)} \;-\; b(z)^* K_z\,.$$

Now let $k_z = K_z/\|K_z\|$, the unit vector in the direction of K_z. Upon noting that $\|K_z\| = (1 - |z|^2)^{-1}$ we obtain from the last display:

$$\Delta^* k_z \;=\; \sqrt{1-|z|^2}K_{\varphi(z)} - b^*(z)k_z, \tag{5.1a}$$

whereupon

$$\langle \Delta^* k_z,\, k_z\rangle \;=\; \frac{1-|z|^2}{1-\varphi(z)^* z} \;-\; b^*(z)\;.$$

As $|z| \to 1-$ the unit vectors k_z converge to 0 pointwise on $\mathbb{U}$, so in fact they converge *weakly* to zero in H^2. Since Δ, and hence Δ^*, is compact, this implies that

$$\lim_{|z|\to 1-} \|\Delta^* k_z\| = 0$$

which, along with (5.1a), shows that for a.e. $\zeta \in \partial\mathbb{U}$

$$b(\zeta) \;=\; \lim_{r\to 1-} \frac{1-r^2}{1-r\zeta^*\varphi(r\zeta)}\,. \tag{5.1b}$$

If φ is the identity map on $\mathbb{U}$, then the fraction on the right-hand side of (5.1b) is $\equiv 1$ on $\partial\mathbb{U}$, hence $b \equiv 1$ on $\mathbb{U}$, and so T_b is the identity operator, as is C_φ, hence the compact operator Δ is 0.

If φ is *not* the identity map on $\mathbb{U}$ then the "boundary identity theorem" for bounded analytic functions (and more generally, for functions in H^2) asserts that if two such functions have radial limits that agree on a subset of $\partial\mathbb{U}$ having positive measure, then the functions agree everywhere on $\mathbb{U}$ (see e.g., [**19**, Theorem 17.18, page 345]). The consequence for us is that for a.e. $\zeta \in \partial\mathbb{U}$, the radial limit of φ at ζ is *not* ζ. Hence from (5.1b), $b(\zeta) = 0$. Since this is true for a.e. $\zeta \in \partial\mathbb{U}$, we have $b \equiv 0$ on $\mathbb{U}$. □

For general Toeplitz operators we no longer have the eigenfunction equation (1.5d), but the above proof can, with a bit more work, be modified to work; see [**18**, Theorem 1.1] for the details.

5.3. Which composition operators are *asymptotically* Toeplitz? We saw in §5.1 that an operator on H^2 is Toeplitz if and only if $S^*TS = T$, where S is the forward shift T_z. Building on this result, Barría and Halmos [**1**, 1982] generalized the notion of "Toeplitz" as follows:

DEFINITION. *To say an operator on H^2 is* asymptotically Toeplitz *means that the operator sequence $(S^{*n}TS^n)_0^\infty$ converges strongly (i.e., pointwise) on H^2.*

When this happens the limit operator, call it T_∞, is necessarily bounded, and must satisfy the equation $S^*T_\infty S = T_\infty$, so it is Toeplitz: $T_\infty = T_b$ for some $b \in L^\infty$ which we call the "asymptotic symbol" of the original operator T. The question of asymptotic toeplitzness for composition operators on H^2 turns out to be considerably more interesting than the "non-asymptotic" counterpart that was dispatched by Proposition 5.2. The following result shows that there are many nontrivial asymptotically-Toeplitz composition operators.

PROPOSITION 5.4. [**18**, Proposition 3.1] *Suppose φ is a holomorphic selfmap of $\mathbb{U}$ whose radial limits have modulus < 1 at a.e. point of $\partial\mathbb{U}$. Then C_φ is asymptotically Toeplitz on H^2.*

PROOF. S^*, the backward shift on H^2, is a contraction, so for each $f \in H^2$ and each integer $n \geq 0$:

$$\|S^{*n}C_\varphi S^n f\|^2 \leq \|C_\varphi S^n f\|^2 = \|\varphi^n f \circ \varphi\|^2 = \int_{\partial\mathbb{U}} |\varphi|^{2n} |f \circ \varphi|^2 \, dm \,.$$

Since $|\varphi| < 1$ a.e. on $\partial\mathbb{U}$ and since $f \circ \varphi \in H^2$ we see that the integrand on the right-hand side of the above display converges a.e. to 0 as $n \to \infty$, and is bounded for each n by the integrable function $|f \circ \varphi|^2$. Thus by dominated convergence, $\|S^{*n}C_\varphi Sf\|^2 \leq \|C_\varphi S^n f\| \to 0$ as $n \to \infty$, i.e. C_φ is asymptotically Toeplitz, with asymptotic symbol identically 0. □

Is this sufficient condition for the asymptotic Toeplitzness of composition operators also *necessary*? Not in general, since if φ is the identity map on $\mathbb{U}$ then $C_\varphi = I$ is Toeplitz, hence asymptotically so. However if φ is not the identity *and fixes the origin* then the answer is "yes".

PROPOSITION 5.5. [**18**, Proposition 3.2] *Suppose φ, not the identity map, fixes the origin, and that C_φ is asymptotically Toeplitz on H^2. Then $|\varphi| < 1$ a.e. on $\partial\mathbb{U}$.*

PROOF. Let's first assume only that φ is neither the identity nor a rotation, and that $\varphi(0) = 0$. Then $\varphi(z) = z\psi(z)$ where ψ is a nonconstant holomorphic selfmap of $\mathbb{U}$, so for each pair f, g of functions in H^2:

$$\begin{aligned} \langle S^{*n}C_\varphi S^n f, \, g\rangle &= \langle \varphi^n (f \circ \varphi), \, z^n g\rangle \\ &= \langle \psi^n (f \circ \varphi), \, g\rangle \\ &= \int_{\partial\mathbb{U}} \psi^n (f \circ \varphi) g^* \, dm \,. \end{aligned}$$

Since ψ is nonconstant it has modulus < 1 at each point of $\partial\mathbb{U}$, hence $\psi^n(f \circ g) \to 0$ pointwise on $\mathbb{U}$. This same sequence of functions is bounded in H^2, hence it converges weakly to zero in H^2, and so the operator sequence $(S^{*n}C_\varphi S^n)$ converges to zero in the weak operator topology.

Now suppose that C_φ is asymptotically Toeplitz and not the identity. It is easy to see that φ cannot be a rotation (the point being that the sequence of powers of any unimodular constant $\neq 1$ does not converge), so if, additionally, $\varphi(0) = 0$ then by the result of the last paragraph, we must have

$$\lim_{n\to\infty} \|S^{*n}C_\varphi S^n f\| = 0 \tag{5.2}$$

for each $f \in H^2$.

Let E denote the set of points of $\partial\mathbb{U}$ at which (the radial limit function of) φ has modulus < 1. The goal is to show that $m(E) = 0$. In fact, since $|\psi| = |\varphi|$ a.e. on $\partial\mathbb{U}$:

$$\begin{aligned} m(E) &\leq \int_{\partial\mathbb{U}} |\psi|^2\, dm = \int_{\partial\mathbb{U}} |S^{*n}z^n\psi^n|^2\, dm \\ &= \int_{\partial\mathbb{U}} |S^{*n}\varphi^n|^2\, dm = \|S^{*n}C_\varphi S^n 1\|^2 \end{aligned}$$

where, by (5.2) above, the last term converges to zero as $n \to \infty$. Thus $m(E) = 0$, as desired. □

Surprisingly, Proposition 5.5 *fails* if the assumption $\varphi(0) = 0$ is dropped: *There exist holomorphic selfmaps φ of $\mathbb{U}$ for which $|\varphi| = 1$ on a nontrivial subarc of $\partial\mathbb{U}$ having positive measure, yet for which C_φ is asymptotically Toeplitz* [**18**, Theorem 3.4]. In the other direction, if φ, not the identity map, is an *inner* function, then C_φ is *not* asymptotically Toeplitz [**18**, Theorem 3.3]. The problem of characterizing those holomorphic selfmaps φ for which C_φ is asymptotically Toeplitz is, as far as I know, still open.

The initial part of the proof of Proposition 5.5 suggests a further generalization of the notion of "toeplitzness". Say an operator T on a Hilbert space is *weakly asymptotically Toeplitz* if the sequence $(S^{*n}TS^n)$ converges in the *weak operator topolgy* i.e., if

$$\lim_{n\to\infty} \langle S^{*n}TS^n f,\, g\rangle$$

exists for every pair of vectors f, g in the space. With this vocabulary the first paragraph of the proof of Proposition 5.5 can be restated as follows:

> *If φ is a holomorphic selfmap of $\mathbb{U}$ that fixes the origin but is neither the identity nor a rotation, then C_φ is weakly asymptotically Toeplitz.*

I conjecture that in this result the requirement that φ fix the origin can be dropped, i.e., that *every composition operator not induced by a (non-identity) rotation is weakly asymptotically Toeplitz, with "weak asymptotic symbol" equal to zero.* This proble, too, appears to be open.

Having observed that the notion of "asymptotic toeplitzness" depends crucially on the mode of operator convergence involved in its definition, it makes sense to ask which composition operators are *uniformly* asymptotically Toeplitz, meaning that the sequence $(S^{*n}C_\varphi S^n)_0^\infty$ converges in the *operator norm topology.* This question has a definitive (but unfortunately deflating) answer; Feintuch [**13**, Theorem 2.4] has shown that an operator on H^2 is uniformly asymptotically Toeplitz if and only if it is a compact perturbation of a Toeplitz operator (see also [**18**, §1]).

Thus, according to Proposition 5.3 above, a composition operator is uniformly asymptotically Toeplitz if and only if it is either compact or the identity.

5.4. Which composition operators are *mean* asymptotically Toeplitz? Whenever a sequence fails to converge it makes sense to ask if its sequence of *averages* converges. Let's temporarily say that a bounded operator T on H^2 is *mean asymptotically Toeplitz* whenever the sequence

$$A_n(T) \;:=\; \frac{1}{n+1}\sum_{k=0}^{n} S^{*k}TS^k \qquad (n = 0, 1, 2, \ldots)$$

converges pointwise on H^2. In this setting all subtlety about the notion of asymptotic toeplitzness for composition operators disappears.

THEOREM 5.6. [**22**, Theorem 1] *Every composition operator is mean asymptotically Toeplitz.*

PROOF. Since the identity operator is Toeplitz, the desired conclusion holds for the case $\varphi(z) \equiv z$. Suppose therefore that φ is *not* the identity map on $\mathbb{U}$. Then the "boundary identity theorem" featured in the proof of (the special case of) Theorem 5.3 guarantees that for a.e. ζ on the unit circle, $\varphi(\zeta) \neq \zeta$. For $\zeta \in \partial\mathbb{U}$ let $\psi(\zeta) := \zeta^*\varphi(\zeta)$. Then $\psi(\zeta) \neq 1$ for a.e. $\zeta \in \partial\mathbb{U}$, and

$$S^{*k}C_\varphi S^k \;=\; T_{(z^*\varphi)^k}C_\varphi \;=\; T_{\psi^k}C_\varphi \qquad (k = 0, 1, 2, \ldots),$$

hence $A_n(C_\varphi) = T_{\Psi_n}$, where

$$\Psi_n \;:=\; \frac{1}{n+1}\sum_{k=0}^{n}\psi^k \;=\; \frac{1}{n+1}\,\frac{1-\psi^{n+1}}{1-\psi}\;.$$

Thus $\Psi_n \to 0$ at a.e. point of $\partial\mathbb{U}$; also by its definition as the average of functions bounded in modulus by 1 a.e., $|\Psi_n| \leq 1$ a.e. on $\partial\mathbb{U}$. Thus for each $f \in H^2$,

$$\|A_n(C_\varphi)\|^2 \;=\; \|T_{\Psi_n}C_\varphi f\|^2 \;=\; \int_{\partial\mathbb{U}} |P(\Psi_n(f\circ\varphi)|^2\,dm \;\leq\; \int_{\partial\mathbb{U}} |\Psi_n(f\circ\varphi)|^2\,dm\,.$$

By our observations above, at a.e. point of the unit circle, the integrand in the last line converges to zero and is bounded by $|f\circ\varphi|^2$, which is integrable because $f\circ\varphi \in H^2$. Thus by the Dominated Convergence Theorem $\|A_n(C_\varphi)\| \to 0$, and so C_φ is mean asymptotically Toeplitz. □

One could define "mean asymptotic Toeplitzness" with respect to other regular convergence methods (transformations that preserve convergent sequences, and perhaps introduce new ones). It turns out that for the largest useful class of such methods, the *strongly regular* ones—the methods whose matrices have convergent row variations—nothing new is gained: *Every composition operator is mean asymptotically Toeplitz with respect to any strongly regular convergence method* [**22**, Theorem 1]. In particular, this holds for any Cesaro method (C, α) $(\alpha > 0)$, our original definition of "mean asymptotic toeplitzness" being the special case $\alpha = 1$. The proof for general strongly regular matrices strongly resembles the one given above for this special case.

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