EXTENSION OF LINEAR FUNCTIONALS ON F-SPACES WITH BASES

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1. Introduction. A linear topological space is said to have the Hahn-Banach Extension Property (HBEP) if every continuous linear functional on a closed subspace has a continuous linear extension to the whole space. Duren, Romberg, and Shields [4, §7] give an example, due to A. Shuchat, of a non-locally convex space with the HBEP; and ask if this can happen in a non-locally convex F-space. Here we show that the answer is negative for F-spaces with a basis. For this class of spaces, then, the HBEP and local convexity are equivalent. The proof is in §3, with the necessary background material occupying §2.

2. Background material. An *F-space* is a complete linear metric space over the real or complex field. If E is an F-space, there is a complete translation invariant metric d in E for which the functional $||x|| = d(x, 0)$ is an *F-norm*, that is:

(a) $||x|| \geq 0$ for all x in E, and $||x|| = 0$ iff $x = 0$,
(b) $||x + y|| \leq ||x|| + ||y||$,
(c) $||\alpha x|| \leq ||x||$ whenever $|\alpha| \leq 1$,
(d) $\lim_{n\to\infty} ||x/n|| = 0$ for each x in E,
(e) the metric $d(x, y) = ||x - y||$ is complete.

Conversely, if E is a real or complex linear space, and $||\cdot||$ is an F-norm on E, then $d(x, y) = ||x - y||$ defines a metric under which E becomes an F-space (see Kelley–Namioka [5; 52]). We say two F-norms on E are *equivalent* if they induce the same topology on E.

The interior mapping principle and the principle of uniform boundedness hold for F-spaces (see Dunford and Schwartz [3, Chapter II]).

From now on, E denotes an F-space whose topology is induced by an F-norm $||\cdot||$. E' is the (continuous) dual of E. A sequence $\{e_n\}_0^\infty$ in E is called a *basis* if to each $x \in E$ there corresponds a unique sequence $\{\xi_n(x)\}_0^\infty$ of scalars such that the series $\sum_{n=0}^\infty \xi_n(x)e_n$ converges in E to x. The coordinate functionals ξ_n are clearly linear, and are continuous (see Corollary to Proposition 1). A sequence $\{e_n\}$ in E is called *basic* if it is a basis for the closed subspace it spans. The following result is essentially proved by Arsove [1].

PROPOSITION 1. *Suppose $\{e_k\}_0^\infty$ is a basis in E. Then*

Received October 30, 1968. This work represents part of the author's dissertation at the University of Michigan. The author wishes to thank Professor Allen Shields for his guidance.

$$||x||_1 = \sup \left|\left| \sum_{k=n}^{m} \xi_k(x)e_k \right|\right| \qquad (0 \leq n \leq m)$$

defines an F-norm on E equivalent to $||\cdot||$.

Proof. Equivalence follows from the interior mapping principle, once we know that $||\cdot||_1$ is an F-norm. Clearly $||\cdot||_1$ satisfies (a)–(c). Arsove [1, Theorem 2] shows that it satisfies (e), so it remains to verify (d). If $x \in E$ and $\epsilon > 0$, we can choose $N > 0$ such that

$$\left|\left| \sum_{k=M}^{L} \xi_k(x)e_k \right|\right| < \epsilon/2$$

whenever $L \geq M > N$. Thus

$$\left|\left| \sum_{k=N+1}^{\infty} \xi_k(x)e_k \right|\right|_1 \leq \epsilon/2.$$

Since $||\cdot||_1$ satisfies (c), we also have

$$\left|\left| \sum_{k=N+1}^{\infty} \xi_k(x/n)e_k \right|\right|_1 \leq \epsilon/2,$$

so

$$||x/n||_1 \leq \left|\left| \sum_{k=0}^{N} \xi_k(x/n)e_k \right|\right|_1 + \epsilon/2 \qquad (n = 1, 2, \cdots).$$

$||\cdot||$ is an F-norm, so whenever $0 \leq L \leq M \leq N$ and n is sufficiently large, we have

$$\left|\left| \sum_{k=L}^{M} \xi_k(x/n)e_k \right|\right| = \left|\left| n^{-1}\left(\sum_{k=L}^{M} \xi_k(x)e_k \right|\right| < \epsilon/2;$$

whereupon

$$\left|\left| \sum_{k=0}^{N} \xi_k(x/n)e_k \right|\right|_1 < \epsilon/2,$$

and

$$||x/n||_1 < \epsilon. \qquad ///$$

The new norm has the following useful property:

$$\left|\left| \sum_{k=n}^{m} \xi_k(x)e_k \right|\right|_1 \leq ||x||_1 \qquad (x \in E; 0 \leq n \leq m), \tag{1}$$

from which it is easy to obtain

COROLLARY 1. *If* $\{e_n\}_0^\infty$ *is a basis for E, then the coordinate functionals are continuous.*

PROPOSITION 2. *If* $\{e_n\}_0^\infty$ *is a basis for E,* $\{n_k\}_{k=0}^\infty$ *is a strictly increasing* *sequence of non-negative integers,* $\{\alpha_n\}_0^\infty$ *is a sequence of scalars, and*

$$f_k = \sum_{n=n_k+1}^{n_{k+1}} \alpha_n e_n \neq 0 \qquad (k = 0, 1, 2, \cdots),$$

then $\{f_k\}$ *is a basic sequence.*

Proof. We may suppose $\|\cdot\|$ has property (1). Suppose x is in N, the closed subspace spanned by $\{f_k\}$. Then $x = \lim x_n$, where

$$x_n = \sum_{k=0}^{\infty} \beta_{n,k} f_k \qquad (n = 0, 1, 2, \cdots),$$

and for each n, $\beta_{n,k} = 0$ except for finitely many k. From (1)

$$\|(\beta_{n,k} - \beta_{m,k}) f_k\| \leq \|x_m - x_n\|,$$

so from the continuity of scalar multiplication $\{\beta_{n,k}\}_{n=0}^\infty$ is a Cauchy sequence $(k = 0, 1, 2, \cdots)$. Let

$$\beta_k = \lim_{n\to\infty} \beta_{n,k} \qquad (k = 0, 1, 2, \cdots).$$

Suppose $n_k + 1 \leq j \leq n_{k+1}$. Then for $n \geq 0$,

$$\xi_j(x_n) = \xi_j(\beta_{n,k} f_k) = \beta_{n,k}\xi_j(f_k),$$

so

$$\xi_j(x) = \lim_n \xi_j(x_n) = \beta_k \xi_j(f_k).$$

It follows that

$$\sum_{j=n_k+1}^{n_{k+1}} \xi_j(x) e_j = \beta_k \sum_{j=n_k+1}^{n_{k+1}} \xi_j(f_k) e_j = \beta_k f_k.$$

Thus for each $x \in N$, insertion of parentheses in the convergent series $x = \sum_{j=0}^\infty \xi_j(x) e_j$ yields the convergent series $x = \sum_{k=0}^\infty \beta_k f_k$. This representation of x is clearly unique, so $\{f_k\}$ is a basis for N. ///

We shall need the notion of *Mackey topology*. Let X be a real or complex linear space and Y a subspace of its algebraic dual. Then there is a (unique) strongest locally convex topology on E for which Y is the (continuous) dual. This topology is called *the Mackey topology of the dual pair* (X, Y), and is denoted by $m(X, Y)$ [5, §18]. A locally convex topology τ is called a *Mackey topology* if $\tau = m(X, X')$, where X' is the τ-dual of X. Every pseudometrizable locally convex topology is Mackey [5; 210]. A consequence of this is

PROPOSITION 3. *Let E be an F-space and let V_n denote the convex hull of* $\{x \in E: \|x\| < n^{-1}\}$ $(n = 1, 2, \cdots)$. *Then* $\{V_n\}_1^\infty$ *is a local base for* $m(E, E')$. *Consequently* $m(E, E')$ *is weaker than the original topology and is pseudometrizable.*

Proof. It is easy to check that $\{V_n\}_1^\infty$ is a local base for a vector topology τ on E weaker than the original topology τ_0. Thus every τ-continuous linear functional is τ_0-continuous. Suppose λ is τ_0-continuous. Then λ is bounded on some neighborhood $\{x \in E: \|x\| < n^{-1}\}$, and has the same bound on its convex hull V_n. Therefore λ is τ-continuous. Thus the τ-dual of E is E'. Since τ has a countable local base, it is pseudometrizable, hence Mackey. Thus $\tau = m(E, E')$. ///

It is easy to see that $m(E, E')$ is metrizable if and only if E' separates points of E. In particular this happens when E has a basis, since the coordinate functionals are continuous and separate points. In the other direction, if $E = L^p([0, 1])$ $(0 < p < 1)$, with

$$\|f\| = \int_0^1 |f(t)|^p \, dt,$$

then the convex hull of $\{f \in E: \|f\| < n^{-1}\}$ $(n = 1, 2, \cdots)$ is E itself. Thus $m(E, E')$ is the indiscrete topology, and $E' = \{0\}$.

We close this section with a technical lemma.

Lemma 1. *Suppose E has a basis $\{e_n\}$ and $\|\cdot\|$ has property* (1). *Let $\{V_n\}$ be as in Proposition* 3. *Then for each n, the Minkowski functional p_n of V_n also has property* (1).

Proof. Given $x \in E$, $0 \le k \le l$, and $p = p_n$ we must show that

$$p\left(\sum_{j=k}^{l} \xi_j(x) e_j\right) \le p(x).$$

If $\alpha > p(x)$ then $\alpha^{-1}x \in V_n$, so

$$\alpha^{-1}x = \sum_{i=1}^{N} \lambda_i x_i ,$$

where $\lambda_i \ge 0$, $\sum_{i=1}^{N} \lambda_i = 1$, and $\|x_i\| < n^{-1}$ $(i = 1, 2, \cdots, N)$. Thus

$$\alpha^{-1} \sum_{j=k}^{l} \xi_j(x) e_j = \sum_{i=1}^{N} \lambda_i \left(\sum_{j=k}^{l} \xi_j(x_i) e_j\right).$$

From (1) we have

$$\left\|\sum_{j=k}^{l} \xi_j(x_i) e_j\right\| \le \|x_i\| < n^{-1} \qquad (i = 1, 2, \cdots, N),$$

so $\alpha^{-1} \sum_{j=k}^{l} \xi_j(x) e_j \in V_n$. Therefore

$$p\left(\sum_{j=k}^{l} \xi_j(x) e_j\right) < \alpha. \qquad ///$$

3. The Hahn-Banach Extension Property.

In this section we show that the HBEP fails in every non-locally convex F-space with a basis. We isolate the crucial element of the proof:

Proposition 4. *Suppose $\{f_n\}$ is a basic sequence in E with*

Note: Conclusion of proof of Prop. 4 persist under following hypothesis: $\exists$ sequence $(f_n) \subset E$ and bi-orthog seq of cts lin fns (φ_n) on $F = \overline{sp}(f_n)$ $\ni$

(1) $f_n \to 0$ $m(E, E')$

(2) $(\varphi_n(x))_1^\infty \in \ell^\infty$ $\forall x \in F$.

(2) $$\inf \|f_n\| = \delta > 0$$

and

(3) $$\lim f_n = 0 \quad in \quad m(E, E').$$

Then there is a continuous linear functional on the closed linear span of $\{f_n\}$ that cannot be extended to E.

Proof. Let N be the closed linear span of $\{f_n\}$. If $f \in N$, then $f = \sum \varphi_n(f) f_n$, where $\{\varphi_n\}$ are the coordinate functionals for $\{f_n\}$. We claim that $\lim \varphi_n(f) = 0$ $(n \to \infty)$. If not, there is an integer $N > 0$ and a subsequence $\{n_k\}$ of non-negative integers such that

$$|\varphi_{n_k}(f)| > N^{-1} \qquad (k = 0, 1, \cdots).$$

Let $\|\cdot\|_1$ be an F-norm in N equivalent to $\|\cdot\|$ and having property (1) relative to the basis $\{f_n\}$. Then whenever $n \leq n_k \leq m$,

$$\left\| \sum_{i=n}^{m} \varphi_i(f) f_i \right\|_1 \geq \|\varphi_{n_k}(f) f_{n_k}\|_1$$
$$\geq \|N^{-1} f_{n_k}\|_1$$
$$\geq N^{-1} \|f_{n_k}\|_1 \geq N^{-1} \delta_1 ,$$

where $\delta_1 = \inf \|f_n\|_1 > 0$. This contradicts the convergence of $\sum \varphi_n(f) f_n$, proving our assertion.

EASIER: $\sum \varphi_n(f) f_n$ co → $\varphi_n(f) f_n$ → 0 by subtracting from partial sums, ∴ $\varphi_n(f) \to 0$ since $\|f_n\| > \delta$

Let $V_1 \supset V_2 \supset \cdots$ be the local base for $m(E, E')$ described in Proposition 3, and let p_n be the Minkowski functional of V_n. Then $p_1 \leq p_2 \leq \cdots$, and

$$\lim_n p_k(f_n) = 0 \qquad (k = 1, 2, \cdots),$$

so there is a subsequence $\{n_k\}$ such that

$$\lim_k p_k(f_{n_k}) = 0.$$

Choose a sequence of scalars $\{\lambda_n\} \in l^1$ with

(4) $$|\lambda_{n_k}| \neq O(p_k(f_{n_k})) \qquad (k \to \infty).$$

Since $\varphi_n(f) \to 0$ for each $f \in N$,

$$\lambda(f) = \sum_{n=0}^{\infty} \lambda_n \varphi_n(f) \qquad (f \in N)$$

defines a linear functional on N which is continuous by the principle of uniform boundedness. Don't need UBP here. Can estimate directly.

We claim λ has no continuous linear extension to E. For if $\tilde{\lambda}$ were such an extension, there would be an integer $N \geq 0$ and a constant $C > 0$ such that

$$|\tilde{\lambda}(x)| \leq C p_N(x) \qquad (x \in E).$$

But $\bar{\lambda}(f_n) = \lambda(f_n) = \lambda_n$, hence

$$|\lambda_n| = O(p_N(f_n)).$$

For $k \geq N$, $p_k \geq p_n$, hence

$$|\lambda_{n_k}| = O(p_k(f_{n_k})),$$

which contradicts (4). ///

THEOREM 1. *If E is a non-locally convex F-space with a basis, then E does not have the* HBEP.

Proof. Let $\{e_n\}$ be the basis and suppose $\|\cdot\|$ obeys (1). In view of Proposition 3 $m, = m(E, E')$ is strictly weaker than the original topology τ on E. We will show that there is a sequence $\{\beta_j\}$ of scalars such that $\{\sum_{j=0}^{k} \beta_j e_j\}_{k=0}^{\infty}$ is m-Cauchy but not τ-Cauchy. Temporarily granting this, we find an increasing sequence $\{n_k\}$ of non-negative integers such that

$$f_k = \sum_{j=n_k+1}^{n_{k+1}} \beta_j e_j \qquad (k = 0, 1, 2, \cdots)$$

satisfies (2) and (3) of Proposition 4. $\{f_n\}$ is a basic sequence by Proposition 2, so it follows from Proposition 3 that its closed linear span has a continuous linear functional that cannot be extended to the whole space.

To finish the proof, suppose no such sequence of scalars exists. We claim m is complete. Suppose $\{x_n\}$ is m-Cauchy. Since the coordinate functionals are m-continuous, there is a sequence $\{\alpha_j\}$ of scalars such that

$$\lim_{n\to\infty} \xi_j(x_n) = \alpha_j \qquad (j = 0, 1, 2, \cdots).$$

Let $\{p_n\}$ be the m-seminorms mentioned in Lemma 1, so each $p \in \{p_n\}$ obeys (1). Consequently

$$\begin{aligned} p\Big(\sum_{j=k}^{l} \alpha_j e_j\Big) &\leq p\Big(\sum_{j=k}^{l} [\alpha_j - \xi_j(x_n)]e_j\Big) + p\Big(\sum_{j=k}^{l} \xi_j(x_n)e_j\Big) \\ &\leq \lim_{m\to\infty} p\Big(\sum_{j=k}^{l} [\xi_j(x_m) - \xi_j(x_n)]e_j\Big) + p\Big(\sum_{j=k}^{l} \xi_j(x_n)e_j\Big) \\ &\leq \liminf_{m\to\infty} p(x_m - x_n) + p\Big(\sum_{j=k}^{l} \xi_j(x_n)e_j\Big). \end{aligned}$$

Since $\{x_n\}$ is m-Cauchy, the first term on the right can be made small by choosing n large. Since $\sum \xi_j(x_n)e_j$ is τ-convergent, hence m-convergent, the second term can be made small for all sufficiently large k and l. Therefore $\{\sum_{j=0}^{k} \alpha_j e_j\}_{k=0}^{\infty}$ is m-Cauchy. By the hypothesis of this paragraph it is τ-Cauchy, hence both τ and m convergent to some $x \in E$. Now

$$p(x - x_n) = \sup_{k,l\geq 0} p\Big(\sum_{j=k}^{l} [\alpha_j - \xi_j(x_n)]e_j\Big)$$

$$\leq \limsup_{m\to\infty} \sup_{k,l\geq 0} p\left(\sum_{i=k}^{l} [\xi_i(x_m) - \xi_i(x_n)]e_i\right)$$

$$\leq \limsup_{m\to\infty} p(x_m - x_n),$$

and it follows that $x = m\text{-}\lim x_n$. Thus m is complete, and an application of the interior mapping principle shows that the identity mapping taking (E, τ) onto (E, m) is a homeomorphism. This contradicts the fact that τ is not locally convex. ///

We conjecture that the HBEP fails for every non-locally convex F-space. In such a space there is always a sequence which tends to 0 in the Mackey topology but not in the original topology. If it were true that every such sequence contained a basic subsequence, the conjecture would follow immediately from Proposition 4. The following result has been obtained by Bessaga and Pełczyński [2] for *locally bounded* spaces, that is, spaces having a bounded neighborhood of 0. Such spaces are metrizable, in fact p-normable for some $0 < p \leq 1$ (see [6]).

PROPOSITION 5. *Suppose E is a locally bounded F-space with a basis. Let $\{\xi_n\}$ be the coordinate functionals for this basis and d the metric in E. If $\{f_n\}$ is a sequence in E with*

$$\inf_n d(f_n, 0) > 0 \quad \text{and} \quad \lim_n \xi_i(f_n) = 0$$

($i = 0, 1, 2, \cdots$), *then $\{f_n\}$ contains a basic subsequence.*

An immediate consequence of Propositions 4 and 5 is

THEOREM 2. *Suppose E is an F-space embedded in a locally bounded F-space with a basis. If E is not locally convex, then E does not have the* HBEP.

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Easier proof of Th. 1

Lemma. E an F-space with basis (e_n), coord fns (ξ_n). If (b_n) a seq. in E with $\lim_n \xi_k(b_n) = 0$ $(k = 0, 1, 2 \cdots)$ then given (ε_k) pos $\neq 0$ $\exists$ subseq (b_{n_k}) & block basis (δ_k) $\ni$ $\|b_{n_k} - \delta_k\| < \varepsilon_k$.

Pf of Th 1. If E not loc cx then $\exists (b_n)_1^\infty \subset E \ni b_n \to 0$ in (E, E') but $\inf_n \|b_n\| = \delta > 0$. Passing to subseq & using Lemma, $\exists$ block basis $(\delta_n) \ni \|b_n - \delta_n\| <$ ∴ $\|\delta_n\| \geq \delta/2$ but $\delta_n \to 0$ in (E, E'). Also Prop 4 on sp (δ_n). ///

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