NOTES ON OSCILLATORY INTEGRALS

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Abstract. How does one get reasonable estimates of highly oscillatory integrals like $\int_{-\pi}^{\pi} e^{i\lambda\alpha(x)}\,dx$ as $\lambda \to \infty$, where, for example, $\alpha(x) = x^2$, or $\sin x$, or $\cot\frac{x}{2} + x$? Why would one want to? These notes will try to explain.

1. Introduction

The basic problem. We are interested here in the asymptotic behavior, as $\lambda \to \infty$, of the integral

$$I_w(\lambda) := \int_a^b e^{i\lambda\alpha(x)} w(x)\,dx \qquad (\lambda \in \mathbb{R}), \tag{1}$$

where $[a, b]$ is a (usually) finite closed interval of the real line, with $\alpha\colon [a, b] \to \mathbb{R}$ and $w : [a, b] \to \mathbb{C}$ being functions on $[a, b]$ that obey whatever hypotheses validate the arguments we want to use. For example, the very least we should require is that α be measurable and w integrable, in which case $I_w(\lambda)$ exists for every real λ, and is bounded:

$$|I_w(\lambda)| \le \int_a^b |w(x)|\,dx \qquad (\lambda \in \mathbb{R}).$$

We'll see below that if some smoothness is required of w and α then $I_w(\lambda)$ will tend to zero as $\lambda \to \infty$, and we'll be able to say something about how fast it does so.

Remarks. (a) *Why restrict to positive λ?* Replacing λ with $-\lambda$ in (1) simply replaces $I_w(\lambda)$ with $\overline{I_{\overline{w}}}$, where the overline denotes "complex conjugate." Since all properties we assume for w will also apply to its complex conjugate, $|I_w(\lambda)|$ will have the same asymptotic behavior as $\lambda \to -\infty$ as it does when $\lambda \to \infty$.

(b) *Phase.* I think of the function α as the *argument* of $e^{i\lambda\alpha(x)}$, hence the symbol "α", however more commonly it is called the *phase* of $e^{i\lambda\alpha(x)}$, and denoted instead by φ. A point of $[a, b]$ where α' vanishes is called a point of *stationary phase*. Such points turn out to be of critical (no pun intended) importance in determining the asymptotic behavior of $I(\lambda)$, since in small neighborhoods of such a point the integrand oscillates very little, which results in less cancellation in the integral.

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(c) *Weight.* I'll call the integrable function w the *weight* associated with the integral $I(\lambda)$—even though it's not required to be positive.

Fourier coefficients. This is the special case of (1) wherein $\alpha(x) \equiv -x$, $a = -\pi$, $b = \pi$ and λ is an integer (which we call n). More precisely: if w integrable on the interval $[-\pi, \pi]$, then for n an integer, the integral

$$\hat{w}(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} w(x) e^{-inx}\, dx \tag{2}$$

is called the n-th *Fourier coefficient* of w. According to the Riemann-Lebesgue Lemma [3, §5.14, p. 103], $\hat{w}(n) \to 0$ as $n \to \infty$ (and therefore as $n \to -\infty$ as well). If we assume more about w then we can say something about the rate at which $\hat{w}(n)$ tends to zero. For example, suppose w and is 2π periodic and continuously differentiable on $\mathbb{R}$. Then an integration by parts shows that

$$\hat{w}(n) = \frac{1}{in} \widehat{w'}(n) \qquad (n \in \mathbb{Z}),$$

with the periodicity of w insuring cancellation of the boundary terms in the integration-by-parts formula. This implies, (since, e.g., w' is integrable on $[-\pi, \pi]$), that $\hat{w}(n) = \mathrm{o}(1/n)$ as $\lambda \to \infty$.

Requiring more differentiability of w yields faster decay to zero for its Fourier coefficients. For example, if w is k-times continuously differentiable on $\mathbb{R}$ then we can repeat the integration-by-parts argument k times (with periodicity insuring that, each time, the boundary terms cancel) to obtain

Theorem 1.1. *If $w \in C^{(k)}(\mathbb{R})$ is 2π-periodic, then $\hat{w}(n) = o(1/n^k)$ as $n \to \infty$.*

Corollary 1.2. *If w is infinitely differentiable and 2π-periodic on $\mathbb{R}$, then*

$$\lim_{n \to \infty} n^k\, \hat{w}(n) = 0$$

for every $k \geq 0$.

Integration by parts redux. Let's apply integration by parts to the more general oscillatory integral (1). Suppose $w \in C^1([a,b])$ and $\alpha \in C^2([a,b])$. Upon noting that

$$e^{i\lambda\alpha(x)} = \frac{1}{\lambda i \alpha'(x)} \frac{d}{dx} e^{i\lambda\alpha(x)},$$

we have

$$I_w(\lambda) = \frac{1}{i\lambda} \int_a^b \frac{w(x)}{\alpha'(x)} \frac{d}{dx} e^{i\lambda\alpha(x)}\, dx\,.$$

Upon assuming α' doesn't vanish on $[a, b]$, writing

$$\mathscr{D}w(x) = \frac{d}{dx}\left(\frac{w(x)}{\alpha'(x)}\right),$$

and integrating by parts, we obtain the crucial formula

$$I_w(\lambda) = \frac{1}{i\lambda}\left[\frac{w(x)}{\alpha'(x)}e^{i\lambda\alpha(x)}\Big|_{x=a}^{b} - I_{\mathscr{D}w}(\lambda)\right] \tag{3}$$

Our hypotheses on w and α guarantee that $\mathscr{D}w$ is continuous on $[a,b]$ so $I_{\mathscr{D}w}(\lambda)$ is bounded for $\lambda \geq 0$. Thus we've proved:

Theorem 1.3. *Suppose $\alpha \in C^2([a,b])$, $w \in C^1([a,b])$, and α' never vanishes on $[a,b]$. Then $|I_w(\lambda)| = O(1/\lambda)$ as $\lambda \to \infty$.*

If the boundary terms cancel, e.g. if w, α, and α' take the same value at both points a and b, then (3) becomes simply

$$I_w(\lambda) = -\frac{1}{i\lambda}\, I_{\mathscr{D}w}(\lambda) \tag{4}$$

so if α and $\mathscr{D}w$ have further differentiability and boundary cancellation, we can expect to repeat the process and obtain the estimate $|I_w(\lambda)| = \mathrm{O}(1/\lambda^2)$ as $|\lambda| \to \infty$.

An induction argument shows that if $\alpha \in C^\infty([a,b])$ and α' never vanishes on $[a,b]$, then $\mathscr{D}^n w$ exists for every positive integer n, and is continuous on $[a,b]$. We can insure boundary cancellation by requiring the relevant functions and derivatives to have periodic extension to the whole real line. Thus:

Theorem 1.4. *Suppose w and α are infinitely differentiable functions on the real line that are periodic with period $b-a$, and that α' does not vanish anywhere. Then for every $n > 0$:*

$$|I_w(\lambda)| = O\left(\frac{1}{\lambda^n}\right) \qquad (\lambda \to \infty).$$

I leave it to the reader to state and prove the corresponding theorem that supplies the estimate $|I_w(\lambda)| = \mathrm{O}(1/\lambda^n)$ for a *fixed* integer $n > 0$. The point here is that a more careful induction shows that if $\alpha \in C^{(n+1)}([a,b])$ and $w \in C^{(n)}([a,b])$, then $\mathscr{D}^n w$ exists on $[a,b]$, and has the form $1/(\alpha')^{2n}$ times a sum of products involving α and its derivatives of order up to $n+1$, and w and its derivatives of order up to n.

A note of caution. Suppose $w \in C^\infty([a,b])$ and the boundary terms in our integration-by-parts argument do *not* cancel. Then all we can extract from that argument, infinite differentiability not withstanding, is:

$$|I(\lambda)| = \mathrm{O}\left(\frac{1}{\lambda}\right) \qquad (\lambda \to \infty).$$

For example, if $[a,b] = [-1,1]$ and $w \equiv 1$, then our integral is

$$I(\lambda) = \int_{-1}^{1} e^{i\lambda x}\,dx = \frac{2\sin\lambda}{\lambda} \qquad (\lambda \in \mathbb{R}), \tag{5}$$

and so $\limsup_{\lambda\to\infty} \lambda|I(\lambda)| = 2$.

2. Van der Corput's[1] Lemma(s)

The goal of this section is to improve (with an additional assumption) the estimate of Theorem 1.3, and to get information about what happens when the derivative of α is allowed to take the value zero. We do this by looking more closely at the boundary terms in the basic integration-by-parts formula (3) of the last section.

Nonstationary phase. Here we require α' to be continuous and never zero on $[a,b]$, so it will be bounded away from zero there, i.e.,

$$m := \inf_{x\in[a,b]} |\alpha'(x)| > 0.$$

We'll initially impose two further simplifying assumptions that seem, at first glance, to be quite special (spoiler alert: they're not): $\lambda = 1$ and $w \equiv 1$. These reduce (3) to

$$I := \int_a^b e^{i\alpha(x)}\,dx = \frac{e^{i\alpha(x)}}{\alpha'(x)}\bigg|_{x=a}^{b} - \int_a^b \frac{d}{dx}\left(\frac{1}{\alpha'(x)}\right) e^{i\alpha(x)}\,dx \tag{6}$$

whereupon

$$|I| \le \frac{1}{|\alpha'(b)|} + \frac{1}{|\alpha'(a)|} + \int_a^b \left|\frac{d}{dx}\frac{1}{\alpha'(x)}\right| dx$$

In order to keep the argument going, we make one further assumption:

Suppose α' is monotonic.

Without loss of generality we may assume α' is *decreasing* (else replace α with $-\alpha$, which, you'll recall, just replaces I by its complex conjugate). Since now $1/\alpha'$ is increasing, the absolute values in the integrand on the right-hand side of the last display disappear, hence that inequality can be rewritten

$$|I| \le \frac{1}{|\alpha'(b)|} + \frac{1}{|\alpha'(a)|} + \frac{1}{\alpha'(b)} - \frac{1}{\alpha'(a)}$$

Recall our assumption that $|\alpha'| \ge m$ on $[a,b]$. Thus, on $[a,b]$, either $\alpha' \ge m$ or $\alpha' \le -m$. In the former case the right-hand side of the last display is $2/\alpha'(b) \le 2/m$, and in the latter case it is $-2/\alpha'(a) \le 2/m$. We have just proved most of:

Lemma 2.1 (van der Corput's Lemma—Part 1)**.** *Suppose $\alpha \in C^2([a,b])$, and on $[a,b]$ the derivative α' is: (a) monotone, and (b) has absolute value $\ge m > 0$. Then for every $\lambda > 0$:*

$$\left|\int_a^b e^{i\lambda\alpha(x)}\,dx\right| \le \frac{2}{m\lambda} \tag{7}$$

[1] Johannes van der Corput, Dutch number theorist: 1890–1975.

Proof. The argument above establishes the case $\lambda = 1$. For general positive λ, just apply this special case with $\alpha(x)$ replaced by $\lambda\alpha(x/\lambda)$, and the interval of integration now taken to be $[\lambda a, \lambda b]$, and then do the obvious change of variable. □

Remark. The constant "2" on the right-hand side of (7) cannot be improved, as is shown by the result (5).

Note how the right-hand side of (7) does not involve either the function α or the length of the interval of integration. This allows us to easily extend that estimate to possibly unbounded arguments α.

Corollary 2.2. *Inequality* (7) *remains true if we simply assume that, on the open interval* (a, b) *the argument* α *is twice continuously differentiable, with monotonic derivative such that* $|\alpha'(x)| \geq m > 0$ *for all* $x \in (a, b)$.

Proof. For small positive ε, apply (7) on the closed interval $[a + \varepsilon, b - \varepsilon]$, and then let $\varepsilon \to 0$. □

Stationary phase. Let's continue to assume that $\alpha \in C^2([a, b])$, but now with $\alpha'(x_0) = 0$ for some point $x_0 \in [a, b]$. Suppose further that α'' is nonzero on $[a, b]$. Assume for the moment that α'' is positive: $\alpha'' \geq m > 0$ on $[a, b]$. Then for $\delta < \min\{x_0 - a, b - x_0\}$ (about which more will be said in a moment) we can write

$$I(\lambda) := \int_a^b e^{i\lambda\alpha(x)}\,dx = \int_a^{x_0-\delta} + \int_{x_0-\delta}^{x_0+\delta} + \int_{x_0+\delta}^b e^{i\lambda\alpha(x)}\,dx \tag{8}$$

Since $\alpha'' \geq m$ on $[a, b]$ we have α' monotonic on that interval, and of magnitude $\geq \delta m$ off the subinterval $[x_0 - \delta, x_0 + \delta]$. Thus the first and third integrals on the right-hand side of (8) each has magnitude bounded by $2/(m\delta\lambda)$. The magnitude of the second integral is trivially bounded by 2δ, hence for each $\lambda > 0$ we have

$$|I(\lambda)| \leq \frac{4}{m\delta\lambda} + 2\delta\ .$$

Thus, for all λ large enough, it makes sense to set $\delta = 1/\sqrt{m\lambda}$, from which we obtain the inequality $|I(\lambda)| \leq 6/\sqrt{m\lambda}$.

We have just proved most of:

Lemma 2.3 (van der Corput's Lemma—Part 2)**.** $\alpha \in C^2([a, b])$*, with* $|\alpha''| \geq m > 0$ *on* $[a, b]$. *If* α' *vanishes somewhere on* $[a, b]$*, then for all sufficiently large* λ*:*

$$|I(\lambda)| \leq \frac{6}{\sqrt{m\lambda}}\ .$$

Proof. The argument above handles the case $\alpha'' \geq m$ and $\alpha'(x_0) = 0$ at an interior point x_0 of $[a, b]$. If, instead, $\alpha'' \leq -m$ on $[a, b]$, then replace α by $-\alpha$, which restores the positivity of the second derivative at the expense of replacing $I(\lambda)$ by its complex conjugate. If α' vanishes at one of the endpoints of $[a, b]$, then the argument above works with the integral $I(\lambda)$ split into just two integrals, resulting in an upper bound of $4/\sqrt{m\lambda}$.

What happens if α' does not vanish on $[a, b]$? [2] Let's again assume $\alpha'' \geq m \geq 0$ on $[a, b]$. Then α' is strictly increasing, and so its absolute value must assume its minimum at one of the endpoints. If this minimum occurs at the left endpoint a, then $\alpha'(a) > 0$, and so $\alpha'(x) \geq \alpha'(a) + m(x - a) \geq m(x - a)$ for $x \in [a, b]$. Then, just as in the last paragraph, let's fix a small positive number δ, to be specified in a moment; then

$$I(\lambda) = \int_a^{a+\delta} + \int_{a+\delta}^b e^{i\lambda\alpha(x)}\, dx.$$

Now the first integral has magnitude $\leq \delta$. As for the second one, note that α' is increasing and $\geq m\delta$ on $[a + \delta, b]$, so by Theorem 2.1 this integral has magnitude $\leq 2/(m\delta\lambda)$. Thus upon setting $\delta = 1/\sqrt{m\lambda}$, we see that $|I(\lambda)| \leq 4/\sqrt{m\lambda}$ for all sufficiently large λ.

The same argument works if $|\alpha'|$ has its minimum at the right endpoint b; I leave the details to the reader. □

Corollary 2.4. *For every $a > 0$ and all λ sufficiently large:*

$$\left| \int_0^a e^{i\lambda x^2} dx \right| \;\leq\; \frac{3\sqrt{2}}{\sqrt{\lambda}}\,.$$

Proof. Let $\alpha(x) = x^2$, so $\alpha'' \equiv 2$, hence Lemma 2.3 asserts that our integral has absolute value $\leq 6/(\sqrt{2}\lambda)$. □

What's really going on here? Why does van der Corput's Lemma give our integrals a slower rate of decay in the stationary case than in the nonstationary one? Can this rate be improved?

To answer these questions, let's continue to focus on the integral of Corollary 2.4, i.e, on the case $\alpha(x) \equiv x^2$. Fix $\lambda > 0$ and set $\delta = \sqrt{\frac{\pi}{\lambda}}$. Since $\sin(\lambda x^2) \geq 0$ on the interval $[0, \delta]$ we have

$$\begin{aligned}
\sqrt{\lambda}\left|\int_0^\delta e^{i\lambda x^2}\, dx\right| \;&\geq\; \sqrt{\lambda}\int_0^\delta \sin(\lambda x^2)\, dx \;=\; \int_0^{\sqrt{\pi}} \sin(x^2)\, dx \\
&>\; \int_0^\infty \sin(x^2)\, dx \;=\; \frac{\sqrt{\pi}}{2\sqrt{2}}
\end{aligned}$$

[2] This is the less important part of the theorem. We could, of course, fall back upon Lemma 2.1, which would yield a better asymptotic estimate whose precise statement would, however, involve minimum of $|\alpha'|$, which we are trying to avoid here.

where the second inequality comes from the fact that the integral following it—a classical Fresnel integral—can be expressed as a convergent alternating series having as leading term the integral preceding it. Complex methods provide evaluation of the final integral; see [1, §2.3, Exercise 13, page 117], for example.

We're interested in what happens when λ is large, so we may assume that $\delta < a$. Then since $\alpha'(x) \equiv 2x \geq 2\delta$ for x in $[\delta, a]$, Lemma 2.1 tells us that

$$\left|\int_\delta^a e^{i\lambda x^2}\,dx\right| \leq \frac{2}{(2\delta)\lambda} = \frac{1}{\sqrt{\pi\lambda}}\,.$$

Putting together these two estimates we find that for all $\lambda > 0$:

$$\left|\int_0^a e^{i\lambda x^2}\,dx\right| \geq \left|\int_0^\delta e^{i\lambda x^2}\,dx\right| - \left|\int_\delta^a e^{i\lambda x^2}\,dx\right| \geq \frac{c}{\sqrt{\lambda}}$$

where

$$c = \left(\frac{\sqrt{\pi}}{2\sqrt{2}} - \frac{1}{\sqrt{\pi}}\right) = \frac{\pi - 2\sqrt{2}}{2\sqrt{2\pi}} > 0.$$

Conclusions. (a) The exponent "1/2" implicit in the upper bound provided by Lemma 2.3 cannot, in general, be increased.

(b) Near a stationary point the oscillation of $e^{i\lambda\alpha(x)}$ slows down enough to significantly inhibit the decay of the resulting integral over any interval containing that stationary point.

Figure 1 below illustrates, for this example, just what's going. Here $I(\lambda) = \int_0^3 \exp(i\lambda x^2)\,dx$ is plotted for integer values of λ between 10 and 300 (the dots), and these values are compared with two constant multiples of $1/\sqrt{\lambda}$ (the solid curves).

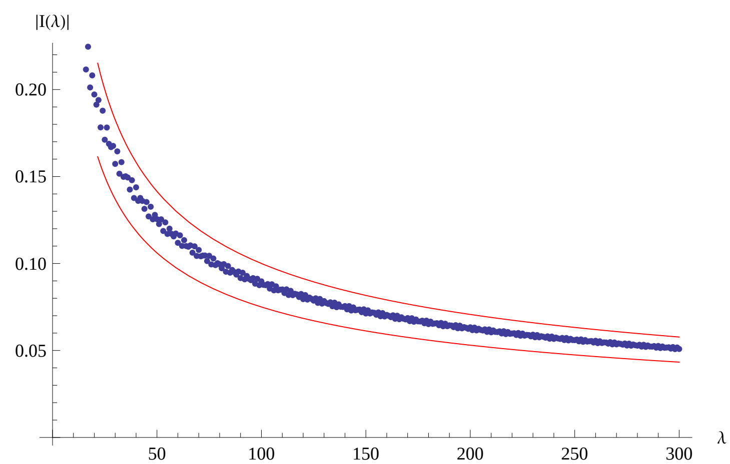

FIGURE 1. *Bounds for* $I(\lambda) = \int_0^3 e^{i\lambda x^2} dx$. *Upper curve is* $y = \frac{1}{\sqrt{\lambda}}$, *lower one is* $y = \frac{0.75}{\sqrt{\lambda}}$.

Note that in this graphic it appears that $|I(\lambda)|$ is asymptotic to a constant multiple of $1/\sqrt{\lambda}$. This is not an accident; for more details see [6, Chapter VIII, §1.3].

The argument that deduced Lemma 2.3 from Lemma 2.1 is the "$k = 2$" stage of an induction that produces a similar result where $\alpha \in C^{(k)}([a,b])$ and $\alpha^{(k)} \geq m > 0$ on $[a,b]$ (see [6, p. 333] for the details). Here's a summary, incorporating this generalization, of what we know so far:

Theorem 2.5. Van der Corput's Lemma(s): *For a positive integer k: suppose $\alpha \in C^{(k)}([a,b])$ and $|\alpha^{(k)}| \geq m > 0$ on $[a,b]$. Then there exists a constant c_k independent of λ and α such that:*

(a) *If $k \geq 2$, then* $\left|\int_a^b e^{i\lambda\alpha(x)}\,dx\right| \leq \frac{c_k}{(m\lambda)^{1/k}} \qquad (\lambda > 0).$

(b) *The conclusion of part (a) is also true for $k = 1$, provided that α' is, in addition, monotonic on $[a,b]$. Here $c_1 = 2$ is the best possible constant.*

Remark. We observed in Corollary 2.2, that the conclusion of part (b) of Theorem 2.5 only required that its hypotheses be satisfied on the open interval (a,b). The same argument establishes this extension for part (a) as well.

The weight reappears. So far the work of this section has dealt with oscillatory integrals having weight $w \equiv 1$. It's surprisingly easy to pass from this special case to that of general weights that are, say, continuously differentiable. For such a weight w we'll write

$$\|w\|_{1,1} := |w(b)| + \int_a^b |w'(x)|\,dx\ .$$

Theorem 2.6. *Suppose $\alpha \in C^2([a,b])$ and $w \in C^1([a,b])$.*

(a) If α' is monotonic and of magnitude $\geq m > 0$ on $[a,b]$, then for all $\lambda > 0$:

$$\left|\int_a^b e^{i\lambda\alpha(x)} w(x)\,dx\right| \leq \left(\frac{2\|w\|_{1,1}}{m}\right)\frac{1}{\lambda}\ .$$

(b) If $|\alpha''| \geq m > 0$ on $[a,b]$ then for all sufficiently large λ:

$$\left|\int_a^b e^{i\lambda\alpha(x)} w(x)\,dx\right| \leq \left(\frac{6\|w\|_{1,1}}{\sqrt{m}}\right)\frac{1}{\sqrt{\lambda}}\ .$$

Proof. The same argument works for both parts, so let's just prove (b). Fix λ, and for $x \in [a,b]$ let $F(x) := \int_a^x e^{i\lambda\alpha(t)}\,dt$. By Lemma 2.3, for λ sufficiently large:

$$|F(x)| \leq 6/\sqrt{m\lambda} \qquad (a \leq x \leq b). \tag{9}$$

An integration by parts yields

$$\int_a^b e^{i\lambda\alpha(x)} w(x)\,dx \;=\; \int_a^b w(x)dF(x) \;=\; F(b)w(b) \;-\; \int_a^b F(x)w'(x)\,dx\ .$$

The desired result follows from this equation upon placing absolute values around both sides, then, on the right, crashing these absolute values through the difference and integral, and using inequality (9) on both $|F(b)|$ and on the term $|F(x)|$ in the integrand. □

Remarks. (a) Once again, the conclusions of Theorem 2.6 remain true if the hypotheses are satisfied on the open interval (a, b).

(b) There is, of course, a weighted analogue of part (a) of Theorem 2.5; I leave its formulation and proof to the reader.

Examples. Let's consider the argument $\alpha(x) = \sin x$ for two different intervals.

(a) *The non-stationary case:* $[a, b] = [0, \pi/4]$. Here $\alpha'(x) = \cos x$ is decreasing, with minimum value $\sqrt{2}/2$, so by Lemma 2.1 we know that

$$|I(\lambda)| \;=\; \left|\int_0^{\pi/4} e^{i\lambda \sin x}\,dx\right| \;\le\; \frac{2\sqrt{2}}{\lambda}$$

for all $\lambda > 0$. Figure 2 below illustrates the result of numerically computing $|I(\lambda)|$ (the lower group of dots) for λ on the horizontal axis running through positive integers from 10 to 1000 in integer steps, and compares this with corresponding values of $2\sqrt{2}/\lambda$ (the upper curve).

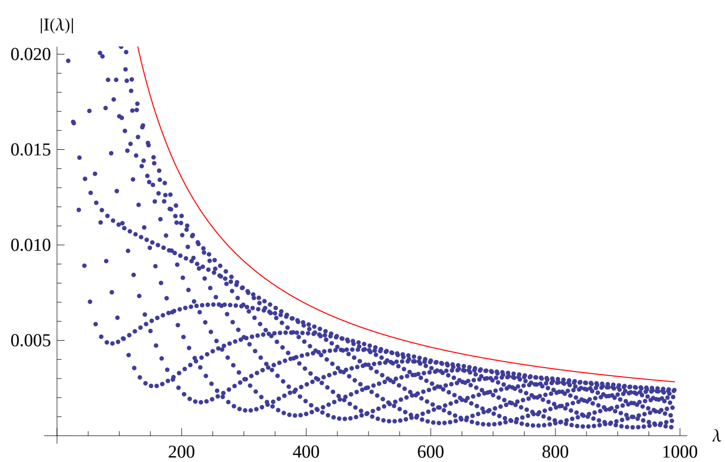

FIGURE 2. *Nonstationary:* $I(\lambda) = \displaystyle\int_0^{\pi/4} e^{i\lambda \sin x} dx$, *upper curve is* $y = \frac{2\sqrt{2}}{\lambda}$.

(b) *The stationary case:* $[a, b] = [\pi/4, 3\pi/4]$. Here $\alpha(x) = \sin x$ has derivative vanishing in the center of the interval, and second derivative of magnitude $\ge \sqrt{2}/2$ on the interval of integration. Figure 3 below shows the values of $|I(\lambda)|$ for the same values of λ as the previous one, but the upper curve now shows the values of $3/\sqrt{\lambda}$, as predicted by Lemma 2.3.

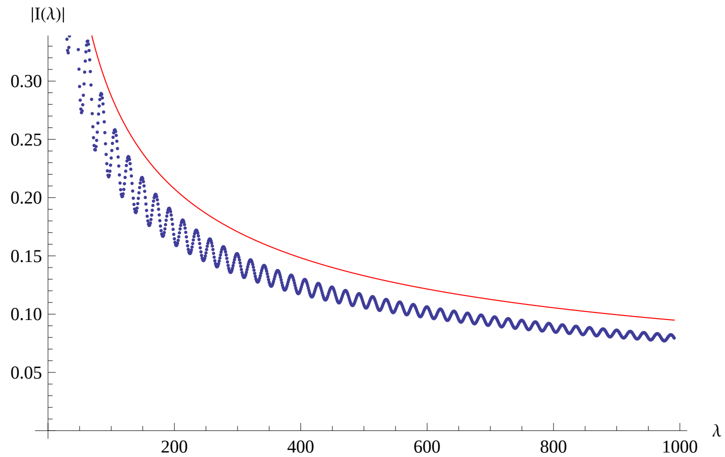

FIGURE 3. *Stationary:* $I(\lambda) = \int_{\pi/4}^{3\pi/4} e^{i\lambda \sin x} dx$*, upper curve is* $y = \frac{3}{\sqrt{\lambda}}$.

Note once more how the appearance of a stationary point for the phase function increases the size of the resulting integrals, and how—as we observed in the discussion following Corollary 2.4—it appears that $|I(\lambda)|$ is asymptotic to a constant multiple of $1/\sqrt{\lambda}$.

(c) *Bessel Functions* (cf. [6, p. 338]): Here $[a, b] = [\pi, \pi]$, $\alpha(x) = \sin x$ (as in the previous two examples), and for $m \in \mathbb{Z}$, $w(x) = (2\pi)^{-1} \exp(-imx)$. The resulting integral is the Bessel function of integral order m:

$$J_m(\lambda) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\lambda \sin x} e^{-imx} dx \,.$$

Note that $\alpha'(x) \equiv \cos x$ has zeros $\pm\pi/2$, at which points α'' has magnitude 1.

Let $J_1 = [-3\pi/4, -\pi/4]$, $J_2 = [\pi/4, 3\pi/4]$, and $J_3 = [-\pi, \pi] \backslash (J_1 \cup J_2)$. Then $J_m(\lambda)$ splits into three integrals. Since $|\alpha''| \geq \sqrt{2}/2$ on both J_1 and J_2, Theorem 2.6(b) guarantees that for $k = 1, 2$:

$$\left| \int_{J_k} e^{i\lambda \sin x} w(x)\, dx \right| = \mathrm{O}(\frac{1}{\sqrt{\lambda}}) \quad \text{as} \quad \lambda \to \infty.$$

Now J_3 is the disjoint union of four intervals, on each of which α' is monotonic and has magnitude $\geq \sqrt{2}/2$. Thus by Theorem 2.6(a) we see that

$$\left| \int_{J_3} e^{i\lambda \sin x} w(x)\, dx \right| = \mathrm{O}(\frac{1}{\lambda}) \quad \text{as} \quad \lambda \to \infty.$$

Conclusion: For each integer m: $J_m(\lambda) = \mathrm{O}(\frac{1}{\sqrt{\lambda}})$ as $\lambda \to \infty$.

(d) *Composition matrices.* Suppose φ is a holomorphic function on the open unit disc $\mathbb{U}$, with $\varphi(\mathbb{U}) \subset \mathbb{U}$. We call such a map φ a "holomorphic selfmap of $\mathbb{U}$." For each such mapping φ the transformation $C_\varphi : f \to f \circ \varphi$ is a linear "composition operator" on $\mathrm{Hol}(\mathbb{U})$, the space of all functions holomorphic on $\mathbb{U}$. With respect to the monomial "basis" $(z^n)_0^\infty$ of $\mathrm{Hol}(\mathbb{U})$, this composition operator has a matrix $[C_\varphi]$ whose (j,k) entry is $\widehat{\varphi^k}(j)$.

Open Problem [2, page 8]: *If φ is not a rotation, do the diagonals of $[C_\varphi]$ converge to zero?*

If φ *is* a rotation, say $\varphi(z) = \omega z$ for some fixed unimodular number ω, then $[C_\varphi]$ is the matrix with main diagonal (ω^n) and all other entries zero. Thus the main diagonal does not converge to zero (in fact, unless $\omega = 1$ it does not converge at all). It is known that for all other self-maps φ, if a diagonal of $[C_\varphi]$ converges, then it must converge to zero (see [2, Theorem 2.2] or [5, §4, pp. 528-9] for more details).

Here I want to consider the asymptotic behavior of the main diagonal of $[C_\varphi]$ where φ is the *unit singular function:*

$$\varphi(z) = \exp\left\{\frac{z+1}{z-1}\right\}.$$

This mapping takes $\mathbb{U}$ onto $\mathbb{U}\backslash\{0\}$, covering each point of the image infinitely often (in fact, φ is a *covering map.*)

It is known that, for this particular φ, the diagonals of $[C_\varphi]$ all converge to zero (see [2, Corollary 2.4]); the question I want to explore here is: *How fast do the diagonals converge to zero?* For simplicity I'll restrict the discussion to the main diagonal, although it can easily be extended to handle all diagonals.

Returning for the moment to general holomorphic selfmaps φ, note that $d_n(\varphi)$, the n-th element of the main diagonal of $[C_\varphi]$ is the n-th Maclaurin coefficient of φ^n, which is the n-th Fourier coefficient of the restriction of φ^n to the unit circle, i.e.,

$$d_n(\varphi) := \widehat{\varphi^n}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(e^{it})^n e^{-int}\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} (\varphi(e^{it})e^{-it})^n\,dt \tag{10}$$

Now once again suppose φ is our unit singular function. Since

$$\frac{1+e^{it}}{1-e^{it}} = -i\cot\frac{t}{2}$$

we see from (10) that

$$d_n(\varphi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\alpha(t)}\,dt \qquad (n = 0, 1, 3, \ldots), \tag{11}$$

with $\alpha(t) := -(\cot\frac{t}{2} + t)$. Thus it appears that van der Corput's Lemma(s) should be able to handle give us good information about the speed at which $d_n(\varphi)$ converges to zero.

For this we need only consider the integral in (11) over the interval $[0,\pi]$ (since α is an odd function, the corresponding integral over $[-\pi, 0]$ is just the complex conjugate of this one). Now $\alpha(t) \to \infty$ as $t \to 0+$, so we restrict our attention to the *open* interval $(0, \pi)$. There the derivative $\alpha'(t)$ is strictly increasing, and equal to zero at $t = \pi/2$. Furthermore, α'' is strictly decreasing on $[0, \pi]$, and equal to 1 at $t = \pi/2$. Thus we can proceed as in the Bessel function example, breaking the open interval $(0, \pi)$ into two pieces: the interval $[\pi/4, 3\pi/4]$, and everything else. Since α'' is bounded away from zero on the first piece, the integral over that interval is, by Theorem 2.5(b), bounded by a multiple (independent of n) of $1/\sqrt{n}$. "Everything else" consists of two disjoint intervals on which α' is monotonic with magnitude bounded away from zero. Thus the integral over this is, by Corollary 2.2 (also incorporated into the remark following Theorem 2.5), bounded by a multiple of $1/\sqrt{n}$.

Conclusion: $d_n(\varphi) = \mathrm{O}(1/\sqrt{n})$ as $n \to \infty$.

The same estimate holds for the other diagonals of $[C_\varphi]$; one need only employ the weighted versions of van der Corput's lemmas.

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