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Composition Operators and Schröder's Functional Equation

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Introduction

This article sketches a case history in which the study of composition operators does what it does best: link fundamental concepts of operator theory with the beautiful classical theory of holomorphic selfmaps of the unit disc. Here the operator-theoretic issues begin with the notion of *compactness*, while the associated function theory revolves around *Schröder's functional equation:*

$$f \circ \varphi = \lambda f. \tag{1}$$

In Schröder's equation, φ is the given quantity, a holomorphic selfmap of the unit disc $U = \{z \in \mathbf{C} : |z| < 1\}$, and the goal is to find $\lambda \in \mathbf{C}$ and f holomorphic on U so that (1) is satisfied.

Schröder's equation is, of course, the eigenvalue equation for the composition operator C_φ, defined by $C_\varphi f = f \circ \varphi$, where, at least for now, f is allowed to range through the entire space $H(U)$ of functions holomorphic on U. The study I want to describe begins with a question that has long intrigued me:

> *Can you determine whether or not C_φ is compact on the Hardy space H^2 by studying the growth of solutions of Schröder's equation for φ?*

In the pages that follow I intend to show you why the question is a natural one, both for operator theory and for classical function theory, and to sketch how it has recently been resolved. I use the word "resolved," rather than "solved," advisedly, because there is a surprising twist: you will see that the relevant operator-theoretic concept for this problem turns out *not* to be compactness, but instead, the more general notion of "Rieszness." Even better, you'll learn how the work on Riesz operators, in turn, serves as a base camp for further explorations that lead into the realm of Fredholm theory.

1. Ernst Schröder

Our story begins in the early 1870s with Ernst Schröder's pioneering work on iteration of analytic functions. In trying to understand Newton's method in the complex plane, Schröder arrived at the idea of using iteration to find solutions of equations involving analytic functions. His great insight was to realize that each univalent solution f of (1) established a conformal "conjugation" between the action

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of φ on U and the simpler mapping of multiplication by λ on $f(U)$. Although he investigated (1) for many specific mappings φ, Schröder never succeeded in finding methods that guaranteed solutions for general classes of maps. In fact he more-or-less admitted defeat, settling instead for what one might call the "Opposite Approach," wherein one *begins* with f and λ, and then *defines* $\varphi(z) = f^{-1}(\lambda f(z))$, thereby obtaining ready-made examples of "pre-solved" Schröder equations [**22**, page 302].

Even though Schröder did not develop general theorems about solutions of his equation, he did originate the systematic study of iteration as a means of solving analytic equations, and he was the first to use conjugation as a fundamental tool to understand iteration near an attractive fixed point. These ideas of Schröder occur in every book on complex dynamics, but they are rarely attributed to him. Ironically, the work for which he *does* get credit, his proof of the Schröder-Bernstein theorem of set theory, contains a fundamental mistake. For a readable and informative account of these matters, I recommend Alexander's book [**1**, Chapter 1], and Gamelin's review [**10**] of that book.

2. Gabriel Koenigs

The name most closely associated with Schröder's equation is not Schröder, it is Gabriel Koenigs, who in 1884 published the fundamental existence-uniqueness theory for analytic solutions of Schröder's equation near an attractive fixed point [**12**]. Let us call a holomorphic selfmap of U a *Schröder map* if it fixes a (necessarily unique) point $a \in U$, and if $0 < |\varphi'(a)| < 1$. By the Schwarz lemma, if φ is *any* holomorphic selfmap of U with $a \in U$ its fixed point, then $|\varphi'(a)| \leq 1$. For Schröder maps, the additional hypothesis of strict inequality simply says that φ is not a conformal automorphism of U, while the requirement that $\varphi'(a) \neq 0$ asserts that φ is conformal in some neighborhood of the fixed point a. I'll explain the theorem of Koenigs in two parts.

Koenigs's Theorem, Part I. *If φ is a Schröder map, then the eigenvalues of $C_\varphi : H(U) \to H(U)$ are the non-negative powers of $\varphi'(a)$:*

$$1, \varphi'(a), \varphi'(a)^2, \varphi'(a)^3, \dots \ .$$

Each of these "Koenigs eigenvalues" has multiplicity one, and if σ denotes the eigenfunction for $\varphi'(a)$, then σ^n is the eigenfunction for $\varphi'(a)^n$ $(n = 0, 1, 2, \dots)$.

The terminology "multiplicity one" for an eigenvalue means that the associated eigensubspace has dimension one; i.e., all the eigenfunctions for that eigenvalue are scalar multiples of each other. Thus our reference in the statement above to "the" eigenfunction for $\varphi'(a)^n$ should cause no ambiguity. It's clear that 1 is an eigenvalue of C_φ, with eigenfunction $\equiv 1$, and it's also clear that once you've found an eigenfunction σ for $\varphi'(a)$ then σ^n is an eigenfunction for $\varphi'(a)^n$ for each positive integer n (just raise both sides of the eigenfunction equation $\sigma \circ \varphi = \varphi'(a)\sigma$ to the n-th power).

Thus the real work in proving Koenigs's Theorem, Part I, involves establishing the multiplicity-one property for eigenvalues, and showing that $\varphi'(a)$ is an eigenvalue. In view of its importance, we call $\varphi'(a)$ the *principal eigenvalue* of C_φ (or simply "of φ"), and σ the *principal eigenfunction.* (In some papers σ is called the *Koenigs eigenfunction.*)

Koenigs constructed σ as a limit of normalized iterates of φ:

Koenigs's Theorem, Part II. *If φ is a Schröder map with fixed point $a = 0$, then the principal eigenfunction for φ is given by the limit*

$$\sigma(z) = \lim_{n\to\infty} \frac{\varphi_n(z)}{\varphi'(0)^n}\ , \tag{2}$$

which converges uniformly on compact subsets of U.

A little argument involving the Schwarz Lemma shows that the hypotheses $\varphi(0) = 0$ and $|\varphi'(0)| < 1$ imply that $\varphi_n(z) \to 0$ uniformly on compact subsets of U. This much was known to Schröder, and was, in fact, the basis of his study of equation-solving by iteration. So what Koenigs showed is that one can precisely balance this convergence to zero by normalizing with successive powers of $\varphi'(0)$.

By substituting $\varphi(z)$ for z on both sides of (2) you see right away that the function σ defined by (2) satisfies Schröder's equation with $\lambda = \varphi'(0)$. From this and the fact that $\varphi'(0) \neq 1$, it's easy to see that $\sigma(0) = 0$ and that σ must be nonconstant. Finally, it follows from the Chain Rule that each function $\varphi_n/\varphi'(a)^n$ has derivative 1 at the origin, so the same is true of σ as defined by (2). Unless we say otherwise, we'll always reserve the symbol σ, and the term "principal eigenfunction" for this particular $\varphi'(a)$-eigenfunction of C_φ. (Don't forget: by Koenigs's "multiplicity-one" result, all the other $\varphi'(a)$-eigenfunctions are constant multiples of this one.)

In case φ is also *univalent*, then the same is true of each of the functions $\varphi_n/\varphi'(0)^n$, so by the Argument Principle, the limit function σ, because it is nonconstant, must also be univalent. For the record:

Corollary. *If φ is a univalent Schröder map, then its principal eigenfunction is also univalent.*

You can find a more detailed account of these matters in [**26**, Chapter 6].

3. Compactness and the principal eigenfunction

The connection between Schröder's equation and the compactness problem for composition operators on Hardy spaces was first studied by J. G. Caughran and H. J. Schwartz. Their paper [**7**], which appeared in 1975, just a little more than one century after Schröder's original papers on iteration, established this theorem:

The Caughran-Schwartz Theorem. *Suppose φ is a holomorphic self-map of U for which C_φ is compact on H^2. Then φ has a fixed point $a \in U$. If in addition $\varphi'(a) \neq 0$ then φ is a Schröder map, and:*

(*) *The principal eigenfunction σ of φ lies in $\cap_{p<\infty} H^p$.*

We see here the first instance of a recurrent theme: even though we are only studying operators that act on the Hardy space H^2, which you may think of as the space of functions holomorphic on U with square-summable Taylor series coefficients (in the expansion about the origin), the other Hardy spaces H^p creep in anyhow. For the definition of these spaces, see the first chapter of [**6**]. For a penetrating analysis of the Hardy spaces, consult: [**20**, Chapter 17], [**9**], or [**13**].

The existence of the fixed point $a \in U$, is a delicate matter that is intimately connected with the dynamics of φ, i.e., with the Denjoy-Wolff theorem (cf. [**26**, Chapter 5]). Since we are already assuming that $\varphi'(a) \neq 0$, to show that φ is a

Schröder map we need only show $|\varphi'(a)| < 1$. For this, note that since C_φ is compact, φ cannot be a conformal automorphism of U (automorphisms induce isomorphisms of H^2 onto itself, and isomorphisms of infinite dimensional Banach spaces are never compact), hence by the conformally invariant version of the Schwarz Lemma, $|\varphi'(a)| < 1$. Thus φ is a Schröder map, and so has a principal eigenfunction σ.

The proof of (*) is an interesting synthesis of matrix theory and functional analysis. By making a conformal similarity (which induces an operator similarity) we can take the fixed point a to be 0. Then, even if C_φ is not compact, the fact that $\varphi(0) = 0$ forces the matrix of C_φ with respect to the orthonormal monomial basis $\{z^n\}_0^\infty$ to be lower triangular, with the sequence $\{\varphi'(0)^n\}_0^\infty$ of Koenigs eigenvalues on the diagonal. So the adjoint of C_φ has an *upper triangular* matrix with the complex conjugates of these numbers on *its* diagonal. These complex conjugates are therefore eigenvalues of C_φ^*, and hence the original diagonal sequence at least lies in the spectrum of C_φ. But if C_φ is in addition *compact*, then by the Riesz theory of compact operators, every nonzero spectral point is an eigenvalue; in particular each Koenigs eigenvalue is an "H^2-eigenvalue" (see [**26**, §6.2] for more details).

Fix a positive integer n. Since $\varphi'(a)^n$ is an eigenvalue of $C_\varphi : H^2 \to H^2$, there must be a corresponding eigenfunction in H^2. Now Koenigs Theorem tells us that σ^n is an eigenfunction for $C_\varphi : H(U) \to H(U)$, corresponding to the eigenvalue $\varphi'(a)^n$. But, being unique (up to multiplication by a constant), σ^n has to be an eigenfunction for $C_\varphi : H^2 \to H^2$. In other words: $\sigma^n \in H^2$. Thus $\sigma \in H^{2n}$ for each positive integer n, which yields the desired result. □

Now the compactness problem for composition operators on Hardy spaces has, in a certain sense, been solved. It was shown early on that if C_φ is compact on some space H^p, then it is compact on all of them ($p < \infty$), and that compactness for C_φ means that in some sense "the values of φ can not be too close to the unit circle too often" (see [**24**]). In a paper that appeared in 1987 I was able to describe exactly what this "closeness" means in terms of the distribution of values of φ [**25**]. In the case that φ is univalent, some beautiful results of Julia and Carathéodory turn this description into something particularly simple:

The "angular derivative criterion." *Suppose φ is univalent. Then C_φ is* not *compact on H^2 if and only if there is a point ζ on the unit circle such that:*

(a) *φ has a nontangential limit of modulus one at ζ, and*
(b) *φ' has a finite nontangential limit at ζ.*

The conclusion is often phrased: "φ has an angular derivative at ζ", hence the name of the theorem. See [**26**, Chapter 4] for an exposition of this result (but be warned: in that book the phrase "angular derivative criterion" denotes the *negation* of the one above). The point of much that follows is that there is a fascinating interaction between this angular derivative criterion for compactness and the behavior of solutions to Schröder's equation.

4. A "Caughran-Schwartz Converse?"

In this section we'll assume that φ is a *univalent* Schröder map with fixed point $a \in U$. Since univalence guarantees that $\varphi'(a) \neq 0$, what we are really assuming, in addition to univalence, is that φ is not a "conformal rotation" about the fixed point a.

As usual we denote by σ the principal eigenfunction of φ (guaranteed by Koenigs's theorem), so σ is univalent, $\sigma(a) = 0$, and σ satisfies this version of Schröder's equation:

$$\sigma \circ \varphi = \varphi'(a)\sigma, \tag{3}$$

which we can rewrite as

$$\varphi(z) = \sigma^{-1}(\varphi'(a)\sigma(z)) \qquad (z \in U). \tag{4}$$

This takes us back to the ideas of Schröder wherein σ establishes a conformal conjugacy between the action of φ on U and the much simpler action of the multiplication map $w \to \varphi'(a)w$ on the simply connected domain $\sigma(U)$. (Note that (3) insures that this multiplication map takes $\sigma(U)$ into itself.) Thus Schröder's equation (3) transfers all the subtleties of the map φ to the geometry of $\sigma(U)$. We think of the map of multiplication by $\varphi'(a)$, acting on $\sigma(U)$ as a *geometric model* for the action of φ on U, and set ourselves the task of untangling the relationship between the model and the original map.

For example, the work of Caughran and Schwartz makes this connection between geometric models and compactness of composition operators:

$$C_\varphi \textit{ compact on } H^2 \quad \Rightarrow \quad \sigma(U) \textit{ is "small."}$$

Now in this case the meaning of "small" is: $\sigma \in \cap_{p<\infty} H^p$. This implies that σ is "almost in H^∞," in fact the standard estimate on the growth of H^p-functions shows that in this case $|\sigma(z)| = O(1 - |z|)^{-\epsilon}$ for every $\epsilon > 0$ (see, for example, [**9**, page 36]). When σ is univalent, this notion of smallness has a geometric consequence: *$\sigma(U)$ cannot contain an angular sector!*

This is true by a standard subordination argument, which I now present from the composition-operator standpoint. First you have to check that the standard univalent map $\sigma_1(z) = (1 + z)/(1 - z)$, which takes the unit disc onto the right half-plane, is not in H^1 (easily done; note that it is, however, in H^p for each $p < 1$). Then it follows readily that for $0 < \gamma < 2$ the map $\sigma_\gamma = \sigma_1^\gamma$ takes U univalently onto the sector $S_\gamma = \{|\arg(w)| < \gamma\pi/2\}$, and is not in $H^{1/\gamma}$. Now if σ is univalent, and $\sigma(U)$ contains some sector S_γ $(0 < \gamma < 2)$, then $\sigma_\gamma = \sigma \circ \psi$ where $\psi = \sigma^{-1} \circ \sigma_\gamma$ is a holomorphic selfmap of U. In other words, $\sigma_\gamma = C_\psi(\sigma)$, so, because σ_γ is not in $H^{1/\gamma}$, the Hardy-space boundedness of C_ψ forces the same conclusion for σ itself. □

It is natural, therefore, to speculate about the possibility of a converse to the Caughran-Schwartz theorem. The importance of this question traces back to Schröder's Opposite Approach, wherein one generates Schröder maps of U by the following procedure:

- Fix a complex number λ with $0 < |\lambda| < 1$.
- Fix a simply connected plane domain G that contains the origin has the property: $w \in G \Rightarrow \lambda w \in G$.
- Let σ be any univalent mapping of U onto G (such maps exist by the Riemann Mapping Theorem).
- Define $\varphi : U \to U$ by $\varphi(z) = \sigma^{-1}(\lambda\sigma(z))$.

One checks easily that the map φ so defined is a Schröder map with fixed point $a = \sigma^{-1}\{0\}$, principal eigenvalue $\varphi'(a) = \lambda$, and principal eigenfunction σ.

Of course, one can choose the map σ taking U onto G in many different ways, with each choice resulting in a different disc map φ. However this causes no problem: if σ and $\tilde{\sigma}$ are two such choices, then $\tilde{\sigma} = \sigma \circ \alpha$, where α is a conformal automorphism of the disc. Since the composition operator C_α is an isomorphism on every H^p space, both σ and $\tilde{\sigma}$ have the same H^p-membership. Moreover you can check easily that α establishes a conformal conjugacy between the resulting φ's, hence the corresponding composition operators induced on H^2, while different, will nonetheless be similar.

The importance of Koenigs's theorem is that it shows that *every* univalent Schröder map φ arises from this procedure! Now if we could prove (for our univalent case) a converse to the Caughran-Schwartz theorem, of the form "$\sigma(U)$ small $\Rightarrow C_\varphi$ compact," then we could determine from the geometry of $\sigma(U)$ just when a Schröder map induces a compact composition operator. Equally, we could adopt the Opposite Approach, and create compact composition operators just by drawing pictures of appropriate simply connected domains and invoking the Riemann Mapping Theorem to call the Koenigs eigenfunction σ into being!

Here are five examples that help clarify the issues involved.

Example 1. Suppose $\|\varphi\|_\infty < 1$, i.e., that the closure of the image $\varphi(U)$ does not contact the unit circle. Then φ fails part (a) of the angular derivative criterion at each point of ∂U, hence C_φ is compact on H^2. In this case the principal eigenfunction σ is *very* small—it is *bounded.* To see this, let M denote the maximum of $|\sigma(w)|$ as w runs through the disc of radius $\|\varphi\|_\infty$ centered at the origin. Then by (3) we have for each $z \in U$,

$$|\sigma(z)| = \frac{|\sigma(\varphi(z))|}{|\varphi'(a)|} \leq \frac{M}{|\varphi'(a)|} ,$$

hence $\|\sigma\|_\infty \leq |\varphi'(a)|^{-1}M < \infty$, as promised.

For the next three examples we adopt the Opposite Approach, starting with the map σ (univalent on U with $\sigma(0) = 0$), and defining φ by means of the equation

$$\varphi(z) = \sigma^{-1}(\sigma(z)/2) \qquad (z \in U). \tag{5}$$

The map φ so defined is a univalent Schröder map that fixes the origin, has eigenvalue $\lambda = \varphi'(0) = 1/2$, and principal eigenfunction σ. In some of these examples we relax the requirement that σ' take value 1 at the interior fixed point of φ.

Example 2: *The strip map.* Let $\sigma(z) = \log \frac{1+z}{1-z}$, a conformal mapping of U onto the strip $\{w \in \mathbf{C} : |\mathrm{Im}\, w| < \pi/2\}$. It is not difficult to check that σ belongs to every H^p space ($p < \infty$), so, even though it is unbounded, σ is still "small" in the sense of the Caughran-Schwartz theorem; in particular, note that $\sigma(U)$ contains no angular sector.

A little map chasing shows that φ, as defined by (5), takes U onto a lens-shaped subregion whose boundary contacts the unit circle only at the points ± 1 and makes an angle of 45 degrees with the unit circle at these points (see [**26**, page 27]). It is easy to check that φ fails the angular derivative criterion at every point of ∂U, hence C_φ is compact on H^2.

How large must $\sigma(U)$ be in order to insure that C_φ is *not* compact? The next two examples turn out to be, in some sense, typical.

Example 3: *A half-plane map.* Let $\sigma(z) = 2z/(1-z)$, a conformal mapping taking U onto the half-plane $\{w \in \mathbf{C} : \mathrm{Re}\, w > -1\}$. So $\sigma(U)$ is "very large". Now (5) defines a linear-fractional mapping taking U onto a subdisc that is tangent to the unit circle at the fixed point 1. This map does have an angular derivative at 1, so the associated composition operator is not compact on H^2. It is not difficult to check directly that, in this case, $\sigma \notin H^1$.

Example 4: *A sector-map.* Continuing in the spirit of the last example, suppose σ is any univalent function on U that fixes the origin, and whose image contains a *sector*, say of total angular opening $\pi\gamma$. Then, as we saw a couple of pages ago, σ does not belong to $H^{1/\gamma}$, and therefore by the Caughran-Schwartz theorem the disc-map φ defined by (5) induces a non-compact composition operator on H^2.

So far our examples support the idea that there might indeed be a converse to the Caughran-Schwartz theorem. However the next example shows that something more subtle is involved.

Example 5: *Reality sets in.* Let $\varphi(z) = (1-z)/2$. Then φ is univalent, $\varphi(-1) = 1$, and φ has an angular derivative at -1 (it's analytic there). So C_φ is not compact on H^2. On the other hand, $\varphi(\varphi(z)) = (1+z)/4$ maps U into the disc $\{|z| < 1/2\}$, so $C_\varphi^2 = C_{\varphi\circ\varphi}$ is compact.

Now φ has a fixed point at 1/3, and $\varphi'(1/3) = -1/2$. Thus φ is a Schröder map and so it has a principal eigenfunction σ. Now $\varphi_2 = \varphi \circ \varphi$ is also a Schröder map with principal eigenfunction σ: it fixes the point 1/3, has derivative $1/4 \neq 0$ there, and $\sigma \circ \varphi_2 = (1/4)\sigma$. Since $\|\varphi_2\|_\infty < 1$ we see from Example 1 above that σ must be bounded.

LESSON: *There are Schröder maps φ, with bounded principal eigenfunction, such that C_φ fails to be compact.*

This example shows that, up to now, we have been neglecting a fundamental property of the Schröder-Koenigs approach to composition operators: the principal eigenfunction for a Schröder map φ is also the principal eigenfunction for each of its iterates! Thus any property of a composition operator that arises from the behavior of its principal eigenfunction is really a property of the sequence of powers of that operator. From this point of view, Example 5 seems to be suggesting that in our search for a converse to the Caughran-Schwartz theorem we should drop our demand that the operator itself be compact, and try settling instead for compactness of some power (equivalently: of all sufficiently large powers) of that operator.

5. Power-compactness

Let us call an operator *power-compact* if one of its positive powers is compact. A little more care in executing the Caughran-Schwartz argument shows that its conclusion holds under the weaker assumption of power-compactness for C_φ. This is, in fact, what Caughran and Schwartz proved in [**7**].

Now the (noncompact) operator of Example 5 is power-compact, and this raises the hope that there might be a converse to the "power-compact" version of the Caughran-Schwartz theorem. The next example, which played a crucial role in [**27**], shows that the real truth lies still deeper.

The bulge-map. Let G be the domain formed by taking the union of the strip $\{0 < \operatorname{Im} w < 1\}$ and the unit disc. So G is a strip with a bulge. Let σ be a conformal mapping of U onto G that fixes the origin (Riemann Mapping Theorem). Define φ on U by (5). Now for each positive integer n the mapping of multiplication-by-$(1/2)^n$ leaves (a large) part of the boundary of G on the boundary, hence φ_n leaves arcs of the boundary of U on the boundary. It follows from the reflection principle that φ_n is analytic over such arcs, and therefore the associated composition operator C_φ^n is (by the angular derivative criterion) not compact. However a standard subordination argument shows that σ belongs to every H^p space for $0 < p < \infty$.

LESSON: *There exist Schröder maps φ, with principal eigenfunction in $\cap_{p<\infty} H^p$, such that C_φ is not even power-compact.*

Thus, if there is to be any hope of obtaining a Caughran-Schwartz converse for power-compact operators, something extra must be assumed. Here is a hypothesis that eliminates the bulge-map.

Definition. Let us call a simply connected plane domain G *strictly starlike* if $0 \in G$ and $t\overline{G} \subset G$ for all $0 < t < 1$, where $\overline{G}$ denotes the closure of G in $\mathbf{C}$.

Under the hypothesis of strict starlikeness, Wayne Smith, David Stegenga, and I were able to prove the following converse to the "power-compact" version of the Caughran-Schwartz Theorem ([**27**], see also [**26**, Chapter 9]):

A Caughran-Schwartz Converse. *Suppose φ is a univalent Schröder map with principal eigenfunction σ. If $\sigma(U)$ is strictly starlike, then:*

$$\sigma \in \cap_{p<\infty} H^p \quad \Rightarrow \quad C_\varphi \text{ power-compact.}$$

Moreover, if $\varphi'(a) > 0$, then, in the conclusion above, "power-compact" can be replaced by "compact."

Example. The region G inside the parabola $y^2 = x + 1$ contains the origin and is taken into itself by the mapping of multiplication by $1/2$. Let σ be any conformal mapping of U onto G that fixes the origin. Since G lies in sectors of arbitrarily small angular opening, subordination shows that $\sigma \in \cap_{p<\infty} H^p$. Thus, if we define φ on U by $\varphi(z) = \sigma^{-1}(\sigma(z)/2)$, then φ is a Schröder map and, because G is strictly starlike, our "Caughran-Schwartz Converse" shows that C_φ is compact on H^2.

Consider the following three conditions which may or may not hold for a univalent Schröder map φ and its principal eigenfunction σ:

(a) C_φ is power-compact,
(b) $\sigma \in \cap_{p<\infty} H^p$,
(c) $\sigma(U)$ contains no sector.

We've already noted that Caughran and Schwartz proved (a)→(b), and have indicated in Example 4 of the previous section why (b) → (c) is *always* true. For the proof of our Caughran-Schwartz converse, Stegenga, Smith, and I were able to show that that (c) → (a) under the additional assumption of strict starlikeness for $\sigma(U)$. In the course of our work we needed to introduce the notion of a *twisted sector*—a type of domain you should think of as being just like an ordinary angular sector except that the axis is a curve instead of a straight line (see [**27**] or [**26**, Chapter 9] for the details). We showed that, for *any* univalent map σ (not necessarily a principal eigenfunction), the smallness condition (b) implies:

(c′) $\sigma(U)$ contains no *twisted* sector,

and that under any of several nonequivalent geometric assumptions about φ or σ, (c′) implies that C_φ is power-compact.

To this point, the story about the Caughran-Schwartz theorem and the search for a converse can be summarized like this:

(a)→(b): The original result of Caughran and Schwartz [**7**].
(b)→(c′): Proved in [**27**] for every univalent function σ.
(c′)→(a): Proved in [**27**] under additional hypotheses on $\sigma(U)$ or φ.

Not long after the appearance of [**27**] in preprint form, Pietro Poggi-Corradini entered the fray, and in his dissertation [**14**] removed the extra assumptions needed on φ or σ in [**27**], thus obtaining the implication (c′) → (b) *for any principal eigenfunction* σ (see also [**15**]). Thus we have what one might call:

The Twisted Sector Theorem. *Suppose σ is a univalent map, holomorphic on the unit disc, with $\sigma(0) = 0$, and suppose there exists $0 \neq \lambda \in U$ such that $\lambda\sigma(U) \subset \sigma(U)$. Then:*

$$\sigma \in \cap_{p<\infty} H^p \iff \sigma(U) \text{ contains no twisted sector.}$$

Since in this section we are describing situations where the principal eigenfunction σ establishes a congruence between the action of the Schröder map φ on the unit disc U and the action of a simple multiplication map on $\sigma(U)$, you might expect that any results that emerge would have a lot to do with conformal invariants of $\sigma(U)$. This is precisely what happens: in [**27**] the invariant is *hyperbolic distance*, while in [**15**] it is *extremal length*. The proofs of these results illustrate perfectly how the study of composition operators can enrich both operator theory (by providing new examples) and classical function theory (by providing new problems).

6. Riesz composition operators

The final assault on the problem of finding a converse to the univalent version of the Caughran-Schwartz theorem was sparked by Poggi-Corradini, who asked in [**15**, §5] if some operator-theoretic concept might be added to the conclusion of the Twisted Sector Theorem. This question connected with one put to me by Michael Neumann after I'd talked at Mississippi State University on the results of [**15**] and [**27**]. Neumann wanted to know if there were any composition operators on H^2 that were *Riesz*, but not power-compact.

Now a *Riesz operator* is one that has essential spectrum $\{0\}$. More precisely, if T is a bounded operator on a Banach space, define its *essential norm* $\|T\|_e$ to be its distance, in the operator norm, from the set of compact operators. Then T is a Riesz operator if and only if $\lim_{n\to\infty} \|T^n\|_e^{1/n} = 0$. In particular, every nilpotent operator is Riesz. Since the set of compact operators on a Banach space is closed in the space of all operators (norm topology), an operator is compact if and only if its essential norm is zero. Thus the class of Riesz operators contains every power-compact operator, and, as we will soon see, it contains even more.

Riesz operators were introduced, not by Riesz, but by Ruston [**21**, 1954]. The name comes from their spectral properties: Riesz operators are the ones that have all the spectral properties that the Riesz theory guarantees for compact operators. In particular:

- The origin is always in the spectrum.
- Every non-zero spectral point is an eigenvalue of finite multiplicity.
- If there are infinitely many eigenvalues, then they form a sequence that tends to zero.

You can find all this explained in Dowson's book [**8**, Chapter 3], and developed in a very general setting in my soon-to-be-finished monograph [**28**].

If you go back to the argument that established the Caughran-Schwartz theorem, you'll see that the compactness of C_φ shows up in two places: it forces φ to fix a point of U, and it insures that all the nonzero spectral points have to be eigenvalues. As I just mentioned, the latter property is true more generally of Riesz operators, and in [**3**] Paul Bourdon and I were able to prove that Riesz composition operators must also satisfy the former (the fixed-point property). So there is the following "Riesz" version of the Caughran-Schwartz theorem:

The "Riesz" Caughran-Schwartz Theorem. *Suppose φ is a holomorphic self-map of U for which C_φ is a Riesz operator on H^2. Then φ has a fixed point $a \in U$. If in addition $\varphi'(a) \neq 0$ then φ is a Schröder map, and:*

(*) *The principal eigenfunction σ of φ lies in $\cap_{p<\infty} H^p$.*

Now in [**25**] the solution to the compactness problem for composition operators on H^2 was derived from a more precise result—a formula for the essential norm of a composition operator. For univalent inducing functions φ this formula reduces to:

$$\|C_\varphi\|_e = \left[\min_{\zeta \in \partial U} |\varphi'(\zeta)|\right]^{-1/2},$$

where $\varphi'(\zeta)$ denotes the angular derivative of φ at the boundary point ζ. If this angular derivative does not exist at ζ, then the convention is to set $|\varphi'(\zeta)| = \infty$. This renders the modulus of the angular derivative lower semicontinuous, which is why the "inf" you might expect to see in the above formula is a "min" (see [**3**, §3] for the details).

Using this formula, Bourdon and I showed in [**3**] that the bulge-map of the last section, although it's not power-compact, is nonetheless a Riesz operator—thus answering Neumann's question. Our method gave the same result for some more complicated counterexamples that occurred in [**27**], and this convinced us that the operator-theoretic concept that Poggi-Corradini wanted to fit into the Twisted Sector Theorem was: "C_φ is Riesz."

We communicated our conjecture to Poggi-Corradini, who quickly established it ([**16**], [**17**]), thus completing the following result, which (in my dreams) I think of as the "Theorem of Bourdon, Caughran, Koenigs, Poggi-Corradini, Schröder, Schwartz, Shapiro, Smith, and Stegenga," but which (in the light of day) I call:

The Univalent Riesz Composition Operator Theorem. *If φ is a univalent Schröder map with principal eigenfunction σ, then the following conditions are equivalent:*

(a) *C_φ is a Riesz operator on H^2.*
(b) *$\sigma \in \cap_{p<\infty} H^p$.*
(c) *$\sigma(U)$ contains no twisted sector.*

As in the study of compactness, our emphasis on the space H^2 is not a restriction. The concept of Rieszness makes sense for Banach spaces, and even for p-Banach spaces like the Hardy space H^p with $0 < p < 1$ (for this generality, see [**28**]). In [**3**] Bourdon and I were able to show that a composition operator is Riesz on H^p if and only if it is Riesz on H^2, so the H^2 case really tells the whole story about Rieszness on Hardy spaces.

For univalently induced composition operators there is also an angular derivative condition equivalent to Rieszness. It's due to Poggi-Corradini [**18**], and should be added to the three previous equivalences that make up the Univalent Riesz Composition Operator Theorem:

(d) *For each positive integer n, the iterate φ_n does not have an angular derivative at any of its boundary fixed points.*

In this statement we call a point $\zeta \in \partial U$ a *boundary fixed point* of φ_n if φ_n has radial limit ζ at ζ (by Fatou's theorem, φ_n has *some* radial limit at almost every point of ∂U).

The story doesn't end here, however. In the next section you'll see how it continues beyond the realm of Riesz operators, and into Fredholm theory.

7. Schröder's equation and Fredholm theory

The work I've just described deals with ways of determining when the principal eigenfunction of a Schröder map belongs to every H^p space ($0 < p < \infty$). How, then, do you determine when this principal eigenfunction belongs to some H^p spaces, but not to others? To make matters precise, define the *Hardy number* of a holomorphic function σ on U to be:

$$h(\sigma) = \sup\{p > 0 : \sigma \in H^p\}.$$

Thus $\sigma \in H^p$ for all $p < h(\sigma)$, and $\sigma \notin H^p$ for all $p > h(\sigma)$. For general holomorphic functions the case $p = h(\sigma)$ can go either way, but we will see shortly that this doesn't happen if σ is a principal eigenfunction!

Now part of the Univalent Riesz Composition Operator Theorem of the last section can be rephrased, at least for univalent Schröder maps φ, as follows:

$$C_\varphi \textit{ is Riesz } \iff h(\sigma) = \infty.$$

This raises the question of how to characterize the case $h(\sigma) < \infty$ in terms of operator theory.

In [**2**] Bourdon and I resorted to Fredholm theory to get a handle on this problem—we related the Hardy number of a principal eigenfunction to the essential spectral radius of its parent composition operator. A *Fredholm operator* T on a Banach space X is one that is "invertible modulo compact operators" in the sense that there is a bounded operator S such that both $TS - I$ and $ST - I$ are compact. You should think of Fredholm operators as being "almost invertible." Thus the notion of "Fredholm" breaks up the spectrum of any operator into two pieces:

- the *essential spectrum*, those $\lambda \in \mathbf{C}$ for which $T - \lambda I$ is so noninvertible that it's not even Fredholm, and
- the *inessential spectrum*, those $\lambda \in \mathbf{C}$ for which $T - \lambda I$, while not invertible, at least has the decency to be Fredholm.

You can interpret the essential spectrum of T to be the spectrum of its coset, modulo the compact operators, in the *Calkin Algebra*—the quotient of the algebra of all bounded operators on X modulo the closed ideal of compact operators. Since the Calkin algebra is a Banach algebra, the essential spectrum of T is nonempty and compact. The *essential spectral radius* of T is defined to be

$$r_e(T) = \sup\{|\lambda| : \lambda \in \text{ essential spectrum of } T\}.$$

Recall that earlier we defined an operator T to be *Riesz* if $r_e(T) = 0$, and commented that this means $\|T^n\|_e^{1/n} \to 0$. You can see now that the latter characterization follows from the former by applying the spectral radius formula to the Calkin algebra.

Now Fredholm theory tells us that, for any Banach space operator, the spectral points in the unbounded component of the complement of the essential spectrum (if there are any) have to be eigenvalues of finite multiplicity. Moreover, if there are infinitely many of these, then they form a sequence that clusters only on the essential spectrum (see [**28**] for the details). Riesz operators constitute the extreme case of this phenomenon—they have essential spectrum $\{0\}$. Thus the spectral properties they share with compact operators (each non-zero spectral point is an eigenvalue of finite multiplicity, and if there are infinitely many of these, they form a sequence tending to zero) follow from this more general consequence of Fredholm theory.

Perhaps more to the point, suppose φ is a Schröder map with fixed point $a \in U$, and σ is its principal eigenfunction. Suppose that for some positive integer n the Koenigs eigenvalue $\varphi'(a)^n$ has modulus $> r_e(C_\varphi)$ (the essential spectral radius of $C_\varphi : H^2 \to H^2$). Then $\varphi'(a)^n$ lies both in the spectrum of C_φ (as we showed in our proof of the original Caughran-Schwartz Theorem, see §3), and in the unbounded component of the complement of the essential spectrum, hence it must be an H^2-eigenvalue. Now comes a familiar argument: by the "multiplicity-one" property of Koenigs eigenvalues, the eigenfunction of $C_\varphi : H^2 \to H^2$ corresponding to the eigenvalue $\varphi'(a)^n$ must be the eigenfunction of $C_\varphi : H(U) \to H(U)$ for that eigenvalue, and this, by Koenigs's Theorem, must be σ^n. Thus σ^n must belong to H^2, i.e., $\sigma \in H^{2n}$. Summarizing: *for every positive even integer p,*

$$|\varphi'(a)| > r_e(C_\varphi)^{2/p} \quad \Rightarrow \quad \sigma \in H^p. \tag{6}$$

In [**2**] Bourdon and I proved the inequality (6) *for every* $0 < p < \infty$. Our idea was to apply the Koenigs-Fredholm argument given above with $n = 1$, but with C_φ viewed as a bounded operator on H^p. Remarkably, the required Fredholm theory goes through unchanged even when $0 < p < 1$ (this is all worked out in [**28**]), and this shows that if $|\varphi'(a)| > r_{e,p}(C_\varphi)$ (the essential spectral radius of $C_\varphi : H^p \to H^p$) then $\sigma \in H^p$. Then we finished our result by working hard to show that $r_{e,p}(C_\varphi) = r_e(C_\varphi)^{2/p}$ for each $0 < p < \infty$. Phrasing our result in terms of the Hardy number:

Theorem. *If φ is a Schröder map with fixed point $a \in U$, then*

$$r_e(C_\varphi) \geq |\varphi'(a)|^{h(\sigma)/2} \tag{7}$$

(see [**2**, Section 3,"Main Theorem"]).

We proved the converse of (6) for Schröder maps φ that are analytic on the closed unit disc (meaning: analytic in an open set that contains the closed unit disc), thus generalizing earlier work of Carl Cowen [**4**] and Herbert Kamowitz [**11**]. Thus, for this case, $\sigma \in H^p$ if and only if $|\varphi'(a)| > r_e(C_\varphi)^{2/p}$. Along with the theorem above, this yields:

Theorem [**2**, Theorem 4.7]**.** *If φ is a Schröder map that is analytic in the* closed *unit disc, and has fixed point $a \in U$, and if σ is its principal eigenfunction, then*

$$r_e(C_\varphi) = |\varphi'(a)|^{h(\sigma)/2} \tag{8}$$

and

$$p = h(\sigma) \quad \Rightarrow \quad \sigma \notin H^p. \tag{9}$$

There is also the problem of finding an effective way to calculate the essential spectral radius of $C_\varphi : H^2 \to H^2$. If φ is analytic on the closed unit disc, and the closure of $\varphi(U)$ contacts the unit circle only at boundary fixed points of φ, then letting S denote the (necessarily finite) collection of such fixed points, it follows from Cowen's work ([**4**, Corollary 2.5], see also [**2**, Theorem 4.1]) that

$$r_e(C_\varphi) = \max\{\varphi'(\zeta)^{-1/2} : \zeta \in S\} \tag{10}$$

(it turns out that the derivatives in question are necessarily positive). The situation is illustrated by the following example, which modifies one that you can find in [**2**].

Example. Let S denote the set of sixteenth roots of unity, and consider the map

$$\varphi(z) = \frac{z}{2 - z^{16}} \qquad (z \in \mathbf{C}). \tag{11}$$

Clearly φ fixes the origin, and a little consideration shows that it also fixes each point of S, and maps the rest of the closed unit disc into the open unit disc. Now $\varphi'(0) = 1/2$, so φ is a Schröder map, and $\varphi' \equiv 17$ on S. Thus from (10),

$$r_e(C_\varphi) = 1/\sqrt{17} = .242\ldots < 1/4,$$

while from (8),

$$h(\sigma) = \frac{\log 17}{\log 2} = 4.08\ldots\ .$$

So $\sigma \in H^p \iff p < \frac{\log 17}{\log 2}$, and in particular $\sigma^2 \in H^2$, but $\sigma^3 \notin H^2$. Thus the Koenigs eigenvalues 1, 1/2, and 1/4 are actual H^2-eigenvalues, but the other points of the Koenigs sequence: 1/8, 1/16,..., while still spectral points, are not H^2-eigenvalues.

The significance of the fact that the principal eigenfunction σ does not belong to H^p for *p equal to* the Hardy number emerges if you consider instead the map $\varphi(z) = z/(1 - z^{15})$. Now the same calculations show that $r_e(C_\varphi) = 1/4$, and $h(\sigma) = 4$. Thus in this case $\sigma^2 \notin H^2$, so 1/4 is not an H^2-eigenvalue.

In [**17**] Poggi-Corradini proves (8) for *univalent* Schröder maps φ, under no extra boundary regularity conditions, and in [**18**]—still under the assumption of univalence—he obtains a formula for $h(\sigma)$ in terms of angular derivatives of iterates of φ at their boundary fixed points, thus significantly generalizing (10) for the univalent case.

These results tie in strongly with work of Carl Cowen and Barbara MacCluer, who recently showed that the spectrum of a univalently induced composition operator on H^2 consists of the Koenigs eigenvalues along with a closed disc Δ centered at the origin [**5**] (this closed disc may degenerate to just the origin itself, as is the case when C_φ is a Riesz operator). A nice exposition of this result appears in [**6**, §7.6]. Cowen and MacCluer also show that Δ is the essential spectrum of $C_\varphi : H^2 \to H^2$, so we have the following picture of the spectrum of an H^2-composition operator induced by a univalent Schröder map:

- The spectrum contains the sequence $\{\varphi'(a)^n\}_0^\infty$ of Koenigs eigenvalues.
- It also contains the essential spectral disc Δ, which may degenerate to the singleton $\{0\}$. This degenerate situation occurs if and only if C_φ is a Riesz operator.
- Whenever a Koenigs eigenvalue $\varphi'(a)^n$ lies outside Δ, we have $\sigma \in H^{2n}$, and whenever $\varphi'(a)^n \in \Delta$, we have $\sigma \notin H^{2n}$.

Returning to the example worked out above, we see that for $\varphi(z) = z/(1-z^{16})$, the spectrum of C_φ is the closed disc of radius $1/\sqrt{17}$ centered at the origin, none of whose points are eigenvalues, along with the eigenvalues 1, 1/2, and 1/4.

8. Epilogue: The non-univalent case

Recall from the preceding section: Bourdon and I proved that

$$r_e(C_\varphi) \geq |\varphi'(a)|^{h(\sigma)/2}$$

for any Schröder map φ with fixed point $a \in U$, and that the opposite inequality holds when φ is either analytic on the *closed* unit disc (Bourdon and I), or *univalent* (Poggi-Corradini).

Just recently (see [**19**]) Poggi-Corradini succeeded in obtaining the opposite inequality without any extra assumptions on φ, so we have the following complete result:

Theorem *(*[**2**, §3] *and* [**19**]*). If φ is a Schröder map with fixed point $a \in U$, then*

$$r_e(C_\varphi) = |\varphi'(a)|^{h(\sigma)/2}. \tag{12}$$

In addition, Poggi-Corradini established the "critical exponent result" in complete generality:

$$p = h(\sigma) \quad \Rightarrow \quad \sigma \notin H^p.$$

One case of (12) deserves special mention. Recall that a holomorphic selfmap φ of U is said to be *inner* if its radial limit function has modulus one at almost every point of the unit circle. In [**2**] Bourdon and I showed that if φ is inner, then $h(\sigma) = 0$ in a very strong sense: σ does not even belong to the Nevanlinna class! We also observed that $r_e(C_\varphi) = 1$ for any inner function. In [**19**] Poggi-Corradini shows that if φ is a Schröder map that is *not* inner then σ belongs to some H^p space, i.e., $h(\sigma) > 0$, so $r_e(C_\varphi) < 1$. This provides an amusing characterization of inner Schröder maps:

A Schröder map φ is inner if and only if $r_e(C_\varphi) = 1$.

The requirement that φ be a Schröder map is essential (no pun intended!) for the "$\Leftarrow$" direction of this result. It follows from [**2**, Lemma 5.3] that if φ is any holomorphic selfmap of U, inner or not, with Denjoy-Wolff point $\omega \in \partial U$ and angular derivative equal to 1 at ω, then C_φ has essential spectral radius 1.

In a more serious vein, (12) removes the hypothesis of univalence from some of the equivalent conditions that characterize Riesz composition operators. It shows that for any Schröder map of U, whether univalent or not, $r_e(C_\varphi) = 0$ if and only if $\sigma \in \cap_{p<\infty} H^p$. This result, along with the work previously outlined in §5 establishes the following:

The General Riesz Composition Operator Theorem. *For any Schröder map φ of U with fixed point $a \in U$, the following are equivalent:*

(a) *C_φ is a Riesz operator on H^2.*
(b) *$\sigma \in \cap_{p<\infty} H^p$.*
(c) *The spectrum of C_φ is $\{0\} \cup \{\varphi'(a)^n\}_0^\infty$.*

(In [**7**] Caughran and Schwartz obtained (c) for power-compact maps φ.)

There is, of course, still work to be done. Right now it's difficult to compute the essential spectral radius of a composition operator induced by an arbitrary

Schröder map φ. The angular derivative, which does the trick if φ is either univalent or analytic across ∂U, doesn't handle the general case. Also, the connection between the Hardy number of σ and the geometry of $\sigma(U)$ becomes murky when φ is not univalent. In the univalent case Poggi-Corradini described this connection in terms of a number he called the "hyperbolic stricture" of $\sigma(U)$, a function-theoretic quantity that plays the role for twisted sectors that the angular opening plays for ordinary ones [**17**, §2]. It is not clear, however, what the analogue of this should be in general.

Nevertheless, I hope you'll agree that we know a lot more about the connection between composition operators and Schröder's equation than we did a few years ago, and I hope this little survey has given you a feeling for how the study of composition operators can make life better for everyone. I find this particular story pleasing, in that the initial focus on compact operators and Schröder's equation led naturally into work on conformal invariants, and this in turn moved the investigation away from compactness, and toward the more general, and often underappreciated, operator-theoretic concept of "Rieszness." Not only did the class of Riesz operators turn out to be the proper setting for the original work, they provided the jumping-off point for the introduction of Fredholm theory, which in turn led to even more precise results.

I think this is a wonderful example of how different branches of mathematics—in this case classical function theory and operator theory—can reinforce each other and make everyone a winner!

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References

[1] D. S. Alexander, *A History of Complex Dynamics from Schröder to Fatou and Julia*, Vieweg 1994.

[2] P. S. Bourdon and J. H. Shapiro, *Mean growth of Koenigs eigenfunctions*, Journal Amer. Math. Soc., to appear.

[3] P. S. Bourdon and J. H. Shapiro, *Riesz composition operators*, Pacific J. Math, to appear.

[4] C. C. Cowen, *Composition Operators on* H^2, J. Operator Th. 9 (1983), 77–106.

[5] C. C. Cowen and B. D. MacCluer, *Spectra of some composition operators,* J. Functional Analysis 125 (1994), 223–251.

[6] C. C. Cowen and B. D. MacCluer, *Composition Operators on Spaces of Analytic Functions,* CRC Press 1995.

[7] J. G. Caughran and H. J. Schwartz, *Spectra of compact composition operators*, Proc. Amer. Math. Soc. 51 (1975), 127–130.

[8] H. Dowson, *Spectral Theory of Linear Operators*, Academic Press 1978.

[9] P. L. Duren, *Theory of* H^p *Spaces*, Academic Press, 1970.

[10] T. W. Gamelin, *Review of* [**1**], Historia Math. 23 (1966), 74–84.

[11] H. Kamowitz, *The spectrum of composition operators on H^p*, J. Functional Analysis 18 (1975), 132–150.
[12] G. Koenigs, *Recherches sur les intégrales de certaines équationes functionelles*, Annales Ecole Normale Superior (3) 1 (1884), Supplément, 3–41.
[13] P. Koosis, *Introduction to H^p spaces*, London Math. Society Lecture Notes #40, Cambridge University Press, 1980.
[14] P. Poggi-Corradini, *Geometric Models, Iteration, and Composition Operators*, Thesis, University of Washington, Seattle 1996.
[15] P. Poggi-Corradini, *Hardy spaces and twisted sectors for geometric models*, Trans. Amer. Math. Soc. 348 (1996), 2503–2518.
[16] P. Poggi-Corradini, *Riesz composition operators and their Koenigs eigenvalues*, preprint.
[17] P. Poggi-Corradini, *The Hardy class of geometric models and the essential spectral radius of composition operators*, J. Functional Analysis, to appear.
[18] P. Poggi-Corradini, *Angular derivatives at boundary fixed points for self-maps of the disk*. Proc. Amer. Math. Soc., to appear.
[19] P. Poggi-Corradini, *The Hardy class of Koenigs maps*, preprint.
[20] W. Rudin, *Real and Complex Analysis, 3rd edition*, McGraw-Hill, 1987.
[21] A. F. Ruston, *Operators with a Fredholm theory,* J. London Math. Soc. 29 (1954), 318–326.
[22] E. Schröder, *Ueber unendlich viele Algorithmen zur Auflösung der Gleichungen*, Math. Ann. 2 (1870), 317–365.
[23] E. Schröder, *Ueber iterite Functionen*, Math. Ann. 3 (1871), 296–322.
[24] J. H. Shapiro and P. D. Taylor, *Compact, nuclear, and Hilbert-Schmidt composition operators on H^p*, Indiana Univ. Math. J. 23 (1973), 471–496.
[25] J. H. Shapiro, *The essential norm of a composition operator*, Annals of Math. 125 (1987), 375–404.
[26] J. H. Shapiro, *Composition Operators and Classical Function Theory*, Springer-Verlag, 1993.
[27] J. H. Shapiro, W. Smith, and D. A. Stegenga, *Geometric models and compactness of composition operators*, J. Functional Analysis 127 (1995), 21–62.
[28] J. H. Shapiro, *Essential Fredholm Theory*, in preparation.

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