Invariant subspaces of the Volterra operator

Notes on lectures given at PSU Analysis Seminar, Jan 15 & 22, 2016

Text

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Lecture Notes #3

NVARIANT SUBSPACES OF HE VOLTERRA OPERATOR

Analysis Seminar
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. Setting.

$L^2 = L^2([0,1])$

$= \text{all } f^{\text{mble}} : [0,1] \to \mathbb{C}$

$\ni \|f\|^2 \stackrel{d}{=} \int_0^1 |f(x)|^2\,dx < \infty$

a Hilbert space w/ inner prod.

$\langle f, g\rangle \stackrel{d}{=} \int_0^1 f(x)\overline{g(x)}\,dx$

also may refer to
$L^1 = L^1([0,1])$
&
$C = C([0,1])$

.The volterra operator $\underline{\underline{V}}$

For $f \in L^2$, $0 \le x \le 1$

$$Vf(x) \stackrel{d}{=} \int_{t=0}^{x} f(t)\,dt$$

Could as well define f on L^1

Clearly $V(L^2) \subset C$
In fact, $V(L^1) \subset C$

ROP. V a bounded op. on L^2.

<u>Proof of Prop.</u> Fix $f \in L^2$

For $0 \le x \le 1$

$$|Vf(x)| \le \int_{t=0}^{x} |f(t)|\,dt$$

$$= \int_{t=0}^{1} \chi_{[0,x]}(t)\,|f(t)|\,dt$$

$\chi_{[0,x]}$ is the char. fcn of $[0,x]$,

$$= \begin{cases} 1 & \text{on } [0,x] \\ 0 & \text{off } [0,x] \end{cases}$$

$$\le \left(\int_0^1 \underbrace{\chi_{[0,x]}(t)^2}_{\chi_{[0,x]}(t)}\,dt\right)^{1/2} \cdot \left(\int_0^1 |f(t)|^2\,dt\right)^{1/2}$$

$$= \left(\int_0^1 \chi_{[0,x]}(t)\,dt\right)^{1/2} \|f\|$$

$$= \left(\int_0^x dt\right)^{1/2} \|f\|$$

$$= \sqrt{x}\, \|f\|$$

$$\therefore \|Vf\|^2 = \int_0^1 |Vf(x)|^2\,dx \le \left(\int_0^1 x\,dx\right) \|f\|^2$$

$$= \tfrac{1}{2} \|f\|^2$$

<u>CONCLUDE:</u> $Vf \in L^2$, $\|Vf\| \le \frac{1}{\sqrt{2}} \|f\|$

SOR (of PB)

On L^2, $\|V\| \le \frac{1}{\sqrt{2}} \approx .707\ldots$

In fact: $\|V\| = \frac{2}{\pi} \approx .636\ldots$ but that's for later!!

Invariant subspaces of V

Def. For operator T on Hilbert space $\mathcal{H}$, to say a subspace $\mathcal{M}$ of $\mathcal{H}$ is T-invariant means:

- $\mathcal{M}$ is closed in $\mathcal{H}$

&

- $T(\mathcal{M}) \subset \mathcal{M}$

To say $\mathcal{M}$ is nontrivial means $\mathcal{M} \neq \{0\}, \mathcal{H}$

Open Question: Does every bndd op on separable ∞ dim'l Hilbert space have a nontriv. invar. subspace?

Known: NO for some Banach spaces, e.g. ℓ^1

QN What are the non-trivial invar. subspaces of Volterra op V ?

Obvious invar subspaces

For $0 \le a \le 1$ let

$$\mathcal{M}_a \stackrel{d}{=} \{f \in L^2 : f \equiv 0 \text{ a.e. on } [0,a]\}$$

$\mathcal{M}_0 = L^2$

$\mathcal{M}_1 = \{0\}$

all other $\mathcal{M}_a$'s are "nontrivial"

THEOREM. These are the only ones!

i.e., $\mathcal{M}$ closed invar subsp for $V \Rightarrow \exists\ 0 \le a \le 1 \ni$

$$\mathcal{M} = \mathcal{M}_a$$

Thm due to Agmon (1949)

Conj. by Gelfand in 1938

Q. Why are we interested (in invar subspaces)?

← C. Jordan Canon. forms!

Only a few operators for which invar subspaces are completely classified

EXAMPLE — compare w/ V

S = Forward shift on ℓ^2

$S((a_0, a_1, a_2, \cdots)) = (0, a_0, a_1, a_2, \cdots)$

Obvious invariant subspaces for S:

For $n = 1, 2, \dots$.

$$M_n = \{(\underset{\substack{\uparrow \\ 0^{th}\text{ place}}}{0}, 0, \dots, \underset{\substack{\uparrow \\ n^{th}\text{ place}}}{0}, a_{n+1}, a_{n+2}, \dots) \in \ell^2\}$$

But: There are MANY others!! For later talk!

5. CYCLICITY

$\mathcal{H}$ - a Hilbert space — e.g., L^2, ℓ^2

T - a bndd operator on $\mathcal{H}$ — e.g., V on L^2, S on ℓ^2

$x \in \mathcal{H}$

Defn. To say "x is cyclic for T" (or "x is T-cyclic") means:

$$\operatorname{span} \overbrace{\{x, Tx, T^2x, \dots\}}^{\mathrm{Orb}_T(x),\ \text{the } T\text{-orbit of } x}$$

dense in $\mathcal{H}$.

Examples.

a) $\mathbb{1}$ = constant fcn 1 on [0,1] is cyclic for V on L^2

Pf. $V^n \mathbb{1}(x) = \frac{x^n}{n!}$ $(n = 0, 1, 2, \cdots)$

$\therefore$ span $\mathrm{Orb}_V(\mathbb{1})$ = {all polynomials}, dense in L^2. $\boxtimes$

Polynomials $\overset{\text{uniformly}}{\wedge}$ dense in $C([0,1])$ by Weierstrass. $C([0,1])$ dense in L^2.

b) $e_0 = (1, 0, 0, \cdots) \in \ell^2$ cyclic for S

Pf. $S^n e_0 = e_n = (\underset{0^{\text{th}}\text{ coord}}{0}, 0, \ldots, 0, \underset{n^{\text{th}}\text{ coord}}{1}, 0, 0, \cdots)$ for $n = 0, 1, 2 \cdots$.

$(e_n)_{n=0}^{\infty}$ is an orthonormal basis for ℓ^2, so

$\mathrm{Orb}_S(e_0) = \{e_0, e_1, e_2, \cdots\}$ spans dense subspace of ℓ^2. $\boxtimes$

Why care about cyclicity?

(a) Invariant subspaces

$M = \overline{\text{span}}\ \text{orb}_T(x)$ is an invariant subspace for T.

assume $x \neq 0$ so $M \neq \{0\}$

$M \neq \mathcal{H} \iff x$ not T-cyclic.

b) Approximation Thms.

x cyc for T means each $y \in \mathcal{H}$ a limit (in $\mathcal{H}$-norm) of linear comb's of vectors in $\text{Orb}_T(x)$.

QUESTION. Which $f \in L^2$ are cyclic for V?

Here $0 < a \leq 1$

Easy: $f \in M_a \overset{\text{def}}{=} \{f \in L^2 : f \equiv 0 \text{ ae on } [0,a]\}$

$\implies f$ not cyclic for V

i.e., $\boxed{f \text{ cyclic for } V \implies 0 \in \operatorname{spt} f}$

"spt f" is the support of f, i.e., the set of $x \in [0,1] \ni$ for any nbd U of x, f not a.e. $= 0$ on U.

Our immediate goal is:

THM. $f \in L^2$ & $0 \in \operatorname{spt} f \implies f$ is cyclic for V.

The proof is not trivial. It requires the

TITCHMARSH CONVOLUTION THM (1926)

5. CONVOLUTION in L^2

Recall from previous lectures:

Prop. If $n = 1, 2, \cdots$; $f \in L^2$, $0 \leq x \leq 1$, then
$$V^n f(x) = \int_{t=0}^{x} \frac{(x-t)^{n-1}}{(n-1)!} f(t)\,dt$$

Defn. For $f, g \in L^2$, $0 \le x \le 1$, the <u>convolution</u> of f & g is

$$f * g(x) = \int_{t=0}^{x} f(t-x)\, g(t)\, dt$$

<u>EXAMPLE</u>. $V^n f = g_{n-1} * f$, where $g_n(x) = \dfrac{x^n}{n!}$

<u>EXAMPLE</u> $Vf = \mathbb{1} * f$

<u>PROP</u>. For $f, g \in L^2$

(a) $f*g(x)$ exists $\forall x \in [0,1]$

(b) $f*g \in C([0,1])$ & $\|f*g\| \le \|f\|\,\|g\|$ ← L^2 norms

(c) $f*g = g*f$

(d) $(L^2, *)$ is a <u>Banach</u> <u>algebra</u>

<u>HOWEVER</u> $(L^2, *)$ is <u>not</u> an integral domain!

<u>EXAMPLE</u>: If $f = \chi_{[\frac{1}{2}, 1]}$ then $f * f \equiv 0$

CRUCIAL TO ALL THAT FOLLOWS IS:

$\boxed{\begin{array}{l}\textbf{THM} \text{ (The Titchmarsh Convol Th. 1926)} \\ \text{Suppose } f, g \in L^2 \text{ and} \\ f * g \equiv 0. \\ \text{If } 0 \in \operatorname{spt} f, \text{ then } g \equiv 0 \text{ a.e. on } [0,1]\end{array}}$

Restatement. For $f \in L^2$ let $T_f : L^2 \to L^2$ be the convolution op

$$T_f g = f * g \quad (g \in L^2)$$

Then T_f a bndd op on L^2 w/ $\|T_f\| \le \|f\|$.

The Titchmarsh Th. asserts that

$$\boxed{0 \in \operatorname{spt} f \Rightarrow T_f \text{ is } 1\text{-}1}$$

Converse also true: Exercise