7. Cyclic vectors for V

Finally, we prove something!

$$\boxed{\begin{array}{c} \underline{\text{THM}}.\ f \in L^2 \text{ is } V\text{-cyclic} \\ \Longleftrightarrow \\ 0 \in \operatorname{spt} f \end{array}}$$

Proof (assuming Titchmarsh at last step)

Suppose $0 \in \operatorname{spt} f$

TO SHOW: $\operatorname{span}\{f, Vf, V^2f, \dots\}$ dense in L^2.

Have already noted "f cyclic $\Rightarrow 0 \in \operatorname{spt} f$" so converse is what's at stake.

Fix $g \in L^2$, $\perp \{f, Vf, V^2f, \dots\}$

ENOUGH TO SHOW: $g = 0$ a.e.

For this, recall:

In a previous lecture we showed that $\lambda I - V$ invertible on L^2 $\forall \lambda \in \mathbb{C}\setminus\{0\}$

and . . .

$$R_V(\lambda) \stackrel{\text{def}}{=} (\lambda I - V)^{-1} = \sum_{n=0}^{\infty} \frac{V^n}{\lambda^{n+1}}$$

where series on right converges in operator norm.

(since $\|V^n\| \le \frac{1}{(n-1)!}$, $n = 1, 2, \dots$)

Thus for any $f \in L^2$ & $0 \le x \le 1$:

$$R_V(\lambda) f(x) = \sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}} V^n f(x)$$

$$= \frac{1}{\lambda} f(x) + \sum_{n=1}^{\infty} \frac{1}{\lambda^{n+1}} \int_{t=0}^{x} \frac{(x-t)^{n-1}}{(n-1)!} f(t)\, dt$$

$$= \frac{1}{\lambda} f(x) + \frac{1}{\lambda^2} \int_{t=0}^{x} \underbrace{\left[\sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{x-t}{\lambda} \right)^n \right]}_{= e^{\frac{x-t}{\lambda}}} f(t)\, dt$$

so

$$\boxed{R_V(\lambda) f = \frac{1}{\lambda} f + \frac{1}{\lambda^2} e_\lambda * f} \qquad \forall f \in L^2$$

FOR OUR f (w/ $0 \in \operatorname{spt} f$ & $\bar{g} \perp \operatorname{Orb}_V(f)$), $\forall \lambda \in \mathbb{C} \setminus \{0\}$

operator norm conv. of series

$$0 = \sum_{n=0}^{\infty} \frac{\langle V^n f, \bar{g} \rangle}{\lambda^{n+1}} = \left\langle \sum_{0}^{\infty} \frac{V^n f}{\lambda^{n+1}}, \bar{g} \right\rangle$$

$\therefore \forall \lambda \in \mathbb{C} \setminus \{0\}$:

$$0 = \langle R_V(\lambda) f, \bar{g} \rangle$$

$$= \int_0^1 R_V(\lambda) f(x)\, g(x)\, dx$$

$$= \underbrace{\frac{1}{\lambda}\int_0^1 f(x)\, g(x)\, dx}_{=0 \text{ since } \bar{g} \perp f} + \frac{1}{\lambda^2}\int_0^1 \underbrace{e_\lambda * f}_{= e_\lambda * f \text{, recall}}(x)\, g(x)\, dx$$

so

$$0 = \int_0^1 f * e_\lambda(x)\, g(x)\, dx$$

$$= \int_{x=0}^1 \left(\int_{t=0}^x f(x-t)\, e^{t/\lambda}\, dt \right) g(x)\, dx$$

$$= \int_{t=0}^1 \underbrace{\left(\int_{x=t}^1 f(x-t)\, g(x)\, dx \right)}_{\stackrel{\text{def}}{=} F(t)} e^{\frac{t}{\lambda}}\, dt$$

$\therefore$

$$\underbrace{(*)\quad 0 = \int_{t=0}^1 F(t)\, e^{tz}\, dt \quad \forall z \stackrel{d}{=} \frac{1}{\lambda} \in \mathbb{C} \setminus \{0\}}$$

Diff both sides of $(*)$ repeatedly w/r z
& set $z=0$ (i.e. let $z \to 0$)

Result: $\forall n = 0, 1, 2, \ldots$;

$$\int_{t=0}^{1} t^n F(t)\,dt = 0$$

now $F \in L^2$ (actually $\in C([0,1])$)
so by Weierstrass
$F \equiv 0$ on $[0,1]$, i.e., for $0 \le t \le 1$:

$$0 = \int_{x=t}^{1} f(x-t)\, g(x)\, dx$$

substitute $x = 1-u$

$$= \int_{u=0}^{1-t} f(1-t-u)\, \underbrace{g(1-u)}_{\overset{\text{def}}{=} \tilde{g}(u)}\, du$$

$$= f * \tilde{g}\,(1-t)$$

Thus $f * \tilde{g} \equiv 0$ on $[0,1]$
so by TITCHMARSH

FINALLY !!

$\tilde{g} \equiv 0$ a.e. (since $0 \in \operatorname{supp} f$)
so $g \equiv 0$ a.e. on $[0,1]$ $\blacksquare$

8

Proof of the Volterra-Invariant Subspace Thm

Suppose $\mathcal{M}$ is a closed subspace of L^2 invariant for V: $V(\mathcal{M}) \subset \mathcal{M}$.

Want to show: $\exists\ 0 \le a \le 1$

$$\ni \quad \mathcal{M} = \mathcal{M}_a \quad (= L^2([a,1]))$$

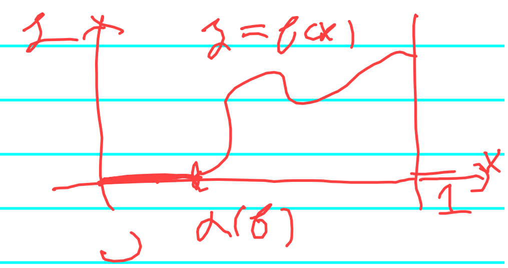
note: $\mathcal{M}_a = \begin{cases} \{0\} & \text{if } a = 1 \\ L^2 & \text{if } a = 0 \end{cases}$

Fix $f \in \mathcal{M}$. Let

$\alpha(f) \overset{d}{=} \inf \operatorname{spt} f$

Since $\operatorname{spt} f$ closed, $\alpha(f) \in \operatorname{spt} f$. Thus

$f \equiv 0$ a.e. on $[0, \alpha(f)]$ — may $= 0$



$f \in \mathcal{M}$

a) Spse $f \in \mathcal{M}$, $\alpha(f) = 0$.

Then $0 \in \operatorname{spt} f$ so by the Cyclicity Thm for V,

$$\mathcal{M} \supset \overline{\operatorname{span}}\{V^n f\}_0^\infty = L^2$$

$f \in \mathcal{M}$
$\Rightarrow V^n f \in \mathcal{M} \quad \forall n$
$\Rightarrow \operatorname{span}\{V^n f\} \subset \mathcal{M}$
$\Rightarrow \overline{\operatorname{span}}\{V^n f\} \subset \mathcal{M}$

$\therefore \quad \mathcal{M} = L^2$

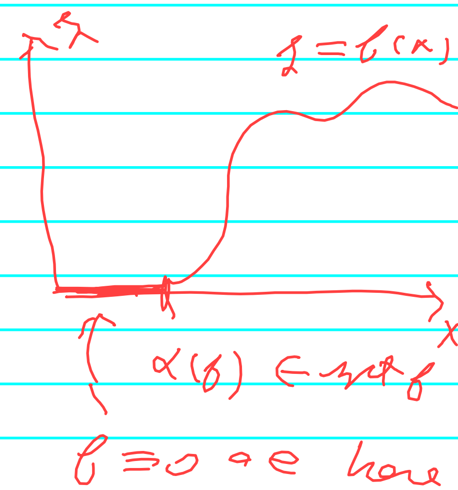
$\longrightarrow$

(b) Suppose $f \in \mathcal{M}$, $\alpha(f) > 0$

Thus $f \in \mathcal{M}_{\alpha(f)}$ & $\alpha(f) \in$ spt f,

so by Volterra cyclic Th ["suitably translated"],

f is cyclic for $V|_{\mathcal{M}_{\alpha(f)}}$



i.e. $\mathcal{M} \supset \overline{\text{span}}\{V^n f\}_0^\infty = \mathcal{M}_{\alpha(f)}$ ← as in part (a)

$\therefore \mathcal{M} \supset \bigcup_{f \in \mathcal{M}} \mathcal{M}_{\alpha(f)}$

Let $\alpha = \inf\{\alpha(f) : f \in \mathcal{M}\}$

<u>Exercise</u> $\bigcup_{f \in \mathcal{M}} \mathcal{M}_{\alpha(f)}$ dense in $\mathcal{M}_\alpha$

$\therefore \mathcal{M} \supset \mathcal{M}_\alpha$

But $f \in \mathcal{M} \Rightarrow f \equiv 0$ a.e. on $[0, \alpha]$

so $\mathcal{M}_\alpha \supset \mathcal{M}$

$\therefore \mathcal{M}_\alpha = \mathcal{M}$ $\blacksquare$

<u>TO DO</u>: Prove Titchmarsh Conv'l Th →

9) PROTOTYPE TITCHMARSH THM

Suppose $f \in C^1([0,1])$ with $f(0) \neq 0$.

If $g \in L^2$ & $f * g \equiv 0$ on $[0,1]$,

then $g \equiv 0$ a.e. on $[0,1]$.

Proof. We're given that

$$0 = (f*g)(x) = \int_{t=0}^{x} f(x-t)g(t)\,dt \quad (0 \leq x \leq 1)$$

Differentiate both sides of this eqn, using Leibnitz's rule on the right. Obtain for $0 \leq x \leq 1$:

$$0 = f(0)g(x) + \underbrace{\int_{t=0}^{x} f'(x-t)\,g(t)\,dt}_{\text{call this } T_{f'}g(x)}$$

$\therefore$ the convol. op $T_{f'}$ has $-f(0) \neq 0$ as an eigenvalue (w/ g as corresp. eigenvector)

$\longrightarrow$

Previously we proved that if
$K \in C([0,1]\times[0,1])$ w/

$$\|K\|_\infty \stackrel{d}{=} \max_{0\le x,\xi\le 1} |K(x,\xi)|$$

and $T_K : L^2 \to L^2$ is defined by

$$T_K f(x) = \int_{t=0}^{x} K(x,\xi)\, f(\xi)\, d\xi$$

then T_K is a bounded lin. op. on L^2, and $\forall\, n \in \mathbb{N}$ & $f \in L^2$

$$\|T_K^n f\| \le \frac{\|K\|_\infty^n}{\sqrt{2n-1}\,(n-1)!} \|f\| \quad (*)$$

We'll review the proof at end of these notes. Right now note that we've shown $\forall\, f \in C^1([0,1])$ w/ $f(0) \ne 0$

$$\text{if } g \in L^2 \ \&\ f*g \equiv 0 \text{ a.e. on } [0,1]$$

$$\text{then} \quad T_{f'}g \stackrel{d}{=} f'*g = -f(0)g$$

Thus

$$T_{f'}^n g = (-f(0))^n g \quad (**)$$

$\longrightarrow$

but from ineq (*) on previous page,

$$\|T_{f'}^n g\| \le \frac{\|f'\|_\infty^n}{\sqrt{2n-1}\,(n-1)!}\|g\| \quad (**)$$

From (**) & (***) we see for $n \in \mathbb{N}$:

$$\underbrace{|f'(0)|^n}_{\neq 0 \text{ by hypothesis}} \|g\| \le \underbrace{\frac{\|f'\|_\infty^n}{\sqrt{2n-1}\,(n-1)!}}_{\to 0 \text{ as } n\to\infty} \|g\|$$

Thus $\|g\| = 0$, i.e. $g = 0$ a.e. on $[0,1]$

$\blacksquare$

10 Loose Ends

Here's the proof of the estimate needed for $\|T_{f'}g\|$. This is the case $K(x,y) = f(x-y)$ of the next result

$\longrightarrow$

THM. Suppose $K \in C([0,1]\times[0,1])$ and $\|K\| \stackrel{\Delta}{=} \max_{0 \le x, \xi \le 1} |K(x,\xi)|$

Define $T_K g(x) = \int_{\xi=0}^{x} K(x,\xi)\, g(\xi)\, d\xi$

Then $\|T_K^n g\| \le \dfrac{\|K\|_\infty^n}{\sqrt{2n-1}\,(n-1)!} \|g\|$

$\forall\, g \in L^2,\ n \in \mathbb{N}$

Pf. For $g \in L^2$, $0 \le x \le 1$, $n \in \mathbb{N}$:

$$|T_K^n g(x)| = \left| \int_{t_1=0}^{x} K(x,t_1) \int_{t_2=0}^{t_1} K(t_1,t_2) \cdots \right.$$

$$\left. \cdots \int_{t_n=0}^{t_{n-1}} K(t_{n-1},t_n)\, g(t_n)\, dt_n \cdots dt_1 \right|$$

$$\le \|K\|_\infty^n \int_{t_1=0}^{x} \int_{t_2=0}^{t_1} \cdots \int_{t_n=0}^{t_{n-1}} |g(t_n)| \cdot \underbrace{}_{dt_n \cdots dt_1}$$

$$= \|K\|_\infty^n \left(V^n |g|\right)(x)$$

$$\le \frac{\|K\|_\infty^n}{\sqrt{2n-1}\,(n-1)!} \|g\| \longrightarrow$$

Here we use PROP. For $g \in L^2$ & $n \in \mathbb{N}$, $x \in [0,1]$:
$|V^n g(x)| \le \dfrac{\|g\|}{\sqrt{2n-1}\,(n-1)!}$
proof at end of these notes.

$$\therefore \quad \|T_K^n g\|^2 \le \int_0^1 \frac{1}{(2n-1)(n-1)!^2} |g(x)|^2 \, dx$$

$$= \frac{1}{(2n-1)(n-1)!^2} \|g\|^2 \quad \blacksquare$$

<u>Proof of "Prop"</u> on $|V^n g(x)|$

$$|V^n g(x)| = \left| \int_{t=0}^{x} \frac{(x-t)^{n-1}}{(n-1)!} g(t) \, dt \right|$$

Crash thru integral w/ abs values & use Cauchy-Schwartz ineq

$$\le \left(\int_{t=0}^{x} \frac{(x-t)^{2n-2}}{(n-1)!^2} \, dt \right)^{1/2} \left(\int_{t=0}^{x} |g(t)| \, dt \right)^{1/2}$$

$$= \sqrt{\frac{1}{(2n-1)(n-1)!^2}} \left(\int_{t=0}^{1} |g(t)|^2 \, dt \right)^{1/2}$$

$$= \frac{1}{\sqrt{2n-1}\,(n-1)!} \|g\| \quad \blacksquare$$

<u>REMAINS TO PROVE</u>:

The Titchmarsh Convol. Thm.

— For a later lecture! —