Whatever happened to ℓ^p for $p < 1$?

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April 29, 2017

The aim of this note is to (partially) remedy the scandalous omission, from beginning courses in functional analysis, of the sequence space ℓ^p for $0 < p < 1$.

1 *Introduction*

EVERY COURSE in functional analysis introduces the space ℓ^p for indices $p \geq 1$. For $p < \infty$ this is the collection of real (or complex, if you wish) sequences $f = (f(n))_1^\infty$ for which

$$\|f\|_p^p := \sum_{n=1}^{\infty} |f(n)|^p < \infty, \tag{1}$$

and for $p = \infty$ it is the space of sequences f that are *bounded*:

$$\|f\|_\infty := \sup_{n \in \mathbb{N}} |f(n)| < \infty.$$

We learn in our functional analysis course that $\|\cdot\|_p$ is, for $1 \leq p \leq \infty$, a *norm* that makes ℓ^p into a *Banach space* (meaning: The metric induced by this norm is *complete*).

However, definition (1) also makes sense for $0 < p < 1$, again defining a vector space of sequences, but about which functional analysis courses say little to nothing. Why? Perhaps because for $0 < p < 1$ the functional $\|\cdot\|_p$ is not sub-additive (exercise), so no longer a norm.

On the other hand, for $p < 1$ the *p-th power* of $\|\cdot\|_p$ *is* subadditive (thanks to the easily-proved numerical inequality $(a+b)^p \leq a^p + b^p$, valid for all $a.b \geq 0$). Thus ℓ^p is, indeed, a vector space over $\mathbb{R}$, and the distance function

$$d(f,g) = \|f-g\|_p^p \qquad (f,g \in \ell^p) \tag{2}$$

is a metric on it. Moreover, thanks to the same argument that worked for $1 \leq p < \infty$, the metric space ℓ^p is *complete* for $0 < p < 1$. The problem now is that:

For $p < 1$ the functional $\|\cdot\|_p^p$ is no longer homogeneous.

It is instead, "p-homogeneous:"

$$\|af\|_p^p = |a|^p \|f\|_p^p \qquad (f \in \ell^p, a \in \mathbb{R}).$$

Thus ℓ^p is, for $p < 1$, not a Banach space; let's call its norm a "p-norm," and the space itself a "p-Banach space." In fact, let's attach these names to any subadditive p-homogeneous functional on a real (or complex) vector space, and to the vector space itself, should it be complete in the induced metric.

The same proofs that work for Banach spaces show that for $p < 1$ their p-Banach cousins possess the fundamental category theorems of functional analysis: the Uniform Boundedness Principle, Open Mapping Theorem, and the Closed Graph Theorem. However for p-norms when $p < 1$ the usual proof of the Hahn-Banach Extension Theorem[1] fails miserably at the first step (make a norm-preserving extension by one dimension–try it!). This does not mean, however, that the theorem itself has to fail; indeed, there are p-normed Banach spaces for which it obviously holds. For example, if X is a Banach space in the norm $\|\cdot\|$, then for $p < 1$ the p-norm $\|\cdot\|_p^p$ makes X into a p-Banach space whose topology is the same as the original one, hence for which the extension property guaranteed by the Hahn-Banach Theorem[2] is true.

[1] *Every continuous linear functional on a closed subspace has a continuous linear extension to the whole space*. The theorem also guarantees that such an extension can be found that has the same norm as the original, but we won't emphasize this here.

[2] Henceforth: the "Hahn-Banach Extension Property."

In a similar vein if $p < 1$ and $n \in \mathbb{N}$, we can impose a p-norm on $\mathbb{R}^n$ by Eqn. (1), where now the sum on the right-hand side just extends from 1 to n. The result is a p-Banach space that has the same topology as the usual euclidean one, and therefore possesses the Hahn-Banach Extension Property. But what about ℓ^p for $p < 1$? Is it, too, a Banach space in disguise? Does it have the Hahn-Banach Extension Property? We'll find out before long!

Exercise: Sketch the unit balls induced on $\mathbb{R}^2$ by the Euclidean norm, the "1-norm" and the "$\frac{1}{2}$-norm."

2 *The dual space of ℓ^p*

THE HAHN-BANACH THEOREM bestows many benefits on normed linear spaces, the first of which is that each such space has enough bounded linear functionals to separate points. The spaces ℓ^p, even for $p < 1$ also have this property. Indeed, for each $n \in \mathbb{N}$ the "n-th evaluation functional" Λ_n defined on ℓ^p by

$$\Lambda_n(f) = f(n) \qquad (f \in \ell^p)$$

is easily seen to be continuous on ℓ^p, and the collection of all these functionals clearly separates points.

Here "boundedness" for a linear functional means "bounded on the unit ball." Just as for Banach spaces, a linear functional on a p-normed space ($p < 1$) is continuous if and only if it is bounded in this sense.

As the index p gets smaller, so do the spaces ℓ^p, and their "norms" get larger. In particular, if $g \in \ell^\infty$ then the linear functional

$$\Lambda_g(f) = \sum_{n=1^\infty} f(n)g(n) \qquad (f \in \ell^p)$$

is defined, for each $0 \le p \le 1$ on ℓ^p, with

$$|\Lambda_g(f)| \le \|f\|_p \|g\|_\infty.$$

Indeed, if $f \in \ell^p$ with $\|f\|_p \le 1$, then $|f(n)| \le 1$ for each $n \in \mathbb{N}$, so if $q > p$ then we have pointwise: $|f|^q \le |f|^p$, hence $f \in \ell^q$ with $\|f\|_q \le \|f\|_p$. The same is true in general, thanks to the homogeneity of the functionals $\|\cdot\|_p$.

for each $f \in \ell^p$. Thus ℓ^p, for $p < 1$, has at least as many bounded linear functionals as does ℓ^1. In fact, as the exercise at the right shows, it has *exactly the same* bounded linear functionals as ℓ^1. Is this evidence of a Hahn-Banach Extension Property?

Exercise: If $0 < p \le 1$ and Λ is a bounded linear functional on ℓ^p, then $\Lambda = \Lambda_g$ for some $g \in \ell^\infty$. Moreover, $\|\Lambda\| = \|g\|_\infty$.

3 *A cautionary tale: L^p for $0 < p < 1$*

ALONG WITH the sequence space ℓ^p we have the Lebesgue spaces $L^p = L^p([0,1])$, the space of (a.e.-equivalence classes of) measurable functions f on the unit interval which, for $p < \infty$, have absolute values are p-th power integrable with respect to Lebesgue measure:

$$\|f\|_p^p := \int_0^1 |f(x)|^p \, dx \, .$$

Unlike their sequence-space relatives which increase in size with p, these Lebesgue spaces *decrease* as p gets larger (and their "norms" *increase*). Thus for $0 < p < 1$ the space L^p is *larger* than L^1, in fact, *strictly* larger (for example, the function $f(x) = 1/x$ belongs to L^p for each $p < 1$, but not to L^1.) So for $p < 1$ there's no hope of imitating the argument that created bounded linear functionals on ℓ^p: we can't integrate every L^p-function against a fixed bounded measurable function.

IT GETS WORSE! Fix $f \in L^p$, and $n \in \mathbb{N}$, and convince yourself that the unit interval can be divided into n contiguous subintervals $I_1, I_2, \dots I_n$, such that

$$\int_{I_j} |f(x)|^p \, dx = \frac{1}{n} \qquad (1 \le j \le n).$$

Now for $1 \le j \le n$ set g_j equal to nf on I_j and zero elsewhere. Thus $f = (1/n) \sum_{j=1}^n g_j$, so f is a convex combination of the functions $g_1, g_2, \dots g_n$, all of which belong to L^p. Moreover:

$$\|g_j\|_p^p = \int_{I_j} |nf(x)|^p \, dx = \frac{n^p}{n} = \frac{1}{n^{1-p}} \, ,$$

thus for each $n \in \mathbb{N}$ the L^p-vector f belongs to the convex hull of the ball of radius $1/n^{1-p}$ centered at the origin. Now $1/n^{1-p} \to 0$ as $n \to \infty$, so:

Theorem 3.1. *The convex hull of any ball in L^p centered at the origin is the entire space.*

Since translation by any fixed vector is an affine isomorphism of L^p, the same is true for every ball, and since translation is also a homeomorphism, the same is true for every nonempty open set.

Since a linear functional that is bounded on a set is bounded (with the same supremum) on its convex hull:

Corollary 3.2. *If $p < 1$, then the only bounded linear functional on L^p is the zero-functional!*

M. M. Day, Bull. Amer. Math. Soc. 69 (1962), 638–640.

In particular, if $p < 1$ the p-normed space L^p is *not* a Banach space in disguise: it possesses no vestige of the Hahn-Banach theorem!

4 *What about ℓ^p?*

Still outstanding for us are these questions about ℓ^p when $p < 1$:

Q1 Is it really a Banach space in disguise (recall the example of $\mathbb{R}^2$ in its p-norm).

Q2 Does it possess the Hahn-Banach Extension Property?

Regarding these questions, recall from §2 that even for $p < 1$ the spaces ℓ^p have lots of linear functionals, and we know exactly what they are! However these spaces also possess a surprising "universal mapping property" that's well known for $p = 1$, and whose proof works just as well for $p < 1$.

Theorem 4.1. *If $0 \leq p \leq 1$ and X is any separable p-normed space, then there is a continuous linear transformation taking ℓ^p* onto *X.*

S. Banach and S. Mazur, Studia Math. 4 (1933), 100–112.

Before proving this result, let's explore its consequences for questions Q1 and Q2 above. Fix $p < 1$ and, in the above theorem, take $X = L^p$ and $T\colon \ell^p \to L^p$ the surjective continuous linear transformation promised therein. I claim that the kernel (i.e, the null space) M of this transformation is of considerable interest. Note that it's a closed subspace of ℓ^p (since T is continuous), and that it's not all of ℓ^p (since T is not the zero-operator).

Now suppose Λ is a continuous linear functional on ℓ^p that takes value 0 at every point of M, i.e., $M \subset \ker \Lambda$. Then we can factor Λ through L^p as $\Lambda = \tilde{\Lambda} \circ T$, where $\tilde{\Lambda}$ is a linear functional on L^p. In fact:

The point here is that the definition $\tilde{\Lambda}(Tf) := \Lambda(f)$, for $f \in \ell^p$, is a "well-made" definition, since $T(\ell^p) = L^p$ and $Tf_1 = Tf_2$ implies $f_1 - f_2 \in \ker T \subset \ker \Lambda$, hence $\Lambda(f_1) = \Lambda(f_2)$. Thus the definition of $\tilde{\Lambda}(Tf)$ depends only on Tf, and not on f.

$\tilde{\Lambda}$ is continuous on L^p!

Indeed, fix U, an open subset of $\mathbb{R}$. Then by continuity, $V := \Lambda^{-1}(U)$ is an open subset of ℓ^p. Since $\Lambda = \Lambda \,\tilde{\circ}\, T$ we have $V = T^{-1}(\Lambda^{-\tilde{1}}(U))$. But T, being a linear mapping of one p-Banach space *onto* another one is an open mapping.[3] Thus $\tilde{\Lambda}^{-1}(U) = T(V)$ is open in L^p, hence $\tilde{\Lambda}$ is continuous on L^p, so by Corollary 3.2, $\tilde{\Lambda} \equiv 0$ on L^p, and therefore $\Lambda \equiv 0$ on ℓ^p.

We've shown that the closed subspace $M = \ker T$ of ℓ^p has the remarkable property that if a bounded linear functional on ℓ^p vanishes on it, then that functional must be the zero-functional on ℓ^p. Put differently:

[3] Recall that the "category theorems," in particular the Open Mapping Theorem, of Banach space theory go through for p-Banach spaces with the same proofs as for ordinary Banach spaces.

Theorem 4.2. *If $p < 1$ there is a proper, closed subspace M of ℓ^p that cannot be separated, by a continuous linear functional, from any point of its complement.*

J. H. Shapiro, Israel J. Math. 7 (1969), 369–380.

Consequences: "NO" to questions Q1 and Q2. For $p < 1$ the spaces ℓ^p do not have the Hahn-Banach Extension Property; in particular they're not "Banach spaces in disguise."

Before declaring victory, however, need to pick up a loose end.

Proof of Theorem 4.1. Fix $p < 1$ and denote by e_n the n-th *standard unit vector* in ℓ^p, i.e., the sequence having 1 in its n-th place and zeros elsewhere. Then each $f \in \ell^p$ has the representation $f = \sum_{n=1}^{\infty} f(n)e_n$, where the series converges[4] to f in the metric of the space. The space L^p is separable, hence so is its closed unit ball, so we may fix a sequence (F_n) of vectors in L^p that is dense in that unit ball. Define $Te_n = F_n$ for $n \in \mathbb{N}$, and extend linearly to a map taking the linear span of $(e_n)_1^{\infty}$ into L^p.

[4] Meaning: "the sequence of partial sums converges".

Claim: T extends to a continuous linear transformation of ℓ^p onto L^p.

Proof of Claim: Fix $f \in \ell^p$ and observe that for positive integers $1 \le M \le N$ we have by the sub-additivity and "p-homogeneity" of the p-norm in ℓ^p:

$$\Big\| \sum_{n=M}^{N} f(n)F_n \Big\|_p^p \le \sum_{n=M}^{N} \|f(n)F_n\|_p^p = \sum_{n=M}^{N} |f(n)|^p \underbrace{\|F_n\|_p^p}_{\le 1} \le \sum_{n=M}^{N} |f(n)|^p .$$

Thus the partial sums of the series $\sum_{n=1}^{\infty} f(n)F_n$ form a Cauchy sequence in the metric of L^p, hence by the completeness of that space, the series converges therein to a vector Tf with $\|Tf\|_p^p \le \|f\|_p^p$. Thus the mapping T, originally defined on linear span of the standard unit vectors in ℓ^p, extends by the definition

$$Tf := \sum_{n=1}^{\infty} f(n)F_n \qquad (f \in \ell^p)$$

to a continuous linear mapping taking ℓ^p into L^p.

Remains to prove: $T(\ell^p) = L^p$.

To this end, fix $F \in L^p$. We must find $f \in \ell^p$ for which $Tf = f$. Without loss of generality we may assume $\|F\|_p^p = 1$ (exercise). By the density of the sequence $(f_n)_1^{\infty}$ in the closed unit ball of L^p there exists an index n_1 such that $\|F - F_{n_1}\|_p^p < \frac{1}{2^p}$.

Thus $2(F - F_{n_1})$ belongs to the unit ball of L^p, so there exists an index n_2 such that $\|2(F - F_{n_1}) - F_{n_2}\|_p^p < \frac{1}{2^p}$, i,e., such that

$$\left\| F - \left(F_{n_1} + \tfrac{1}{2}F_{n_2}\right)\right\|_p^p < \left(\tfrac{1}{2^p}\right)^2. \tag{3}$$

Now Eqn. (3) asserts that $2^2[F - (F_{n_1} + \frac{1}{2}F_{n_2})]$ belongs to the unit ball of L^p, so there exists F_{n_3} in that ball whose distance to this vector is $< \frac{1}{2^p}$, hence

$$\left\| F - \left(F_{n_1} + \tfrac{1}{2}F_{n_2} + \tfrac{1}{4}F_{n_3}\right)\right\|_p^p < \left(\tfrac{1}{2^p}\right)^3.$$

Continuing in this manner we create a sequence (n_k) of positive integers such that for each $K \in \mathbb{N}$:

$$\left\| F - \sum_{k=1}^{K} \tfrac{1}{2^k} F_{n_k} \right\| < \left(\tfrac{1}{2^p}\right)^K$$

Thus F is the sum, in the metric of L^p, of the series

$$\sum_{k=1}^{\infty} \tfrac{1}{2^k} F_{n_k} = \sum_{k=1}^{\infty} \tfrac{1}{2^k} T(e_{n_k}).$$

Now the series $\sum_{k=1}^{\infty} \frac{1}{2^k} e_{n_k}$ converges in ℓ^p to a vector f (i.e., to the sequence with 2^{-k} in position n_k and zeros elsewhere), so

$$Tf = T(\sum_{k=1}^{\infty} \tfrac{1}{2^k} e_{n_k}) = \sum_{k=1}^{\infty} \tfrac{1}{2^k} T(e_{n_k}) = \sum_{k=1}^{\infty} \tfrac{1}{2^k} F_{n_k} = F$$

and we're done! □

5 *It's the convexity!*

To SUM UP: For $0 < p < 1$ there are p-normed spaces that possess the Hahn-Banach theorem (e.g., the "fake" ones: $\mathbb{R}^n$ in its p-norm, Banach spaces with their norms raised to the p-th power), and there are others that do not (e.g., ℓ^p and L^p). What distiguishes the "fakes" from the "real" p-normed spaces?

For the answer, it helps to extend our reach a bit. All the spaces we've considered have complete metrics that induce a topology that "respects" the vector operations, in that vector addition and scalar multiplication are jointly continuous. Such "complete, metrizable, topological vector spaces" are called *F-spaces* ("F" in honor of the French mathematician Maurice Fréchet).

For example, "joint continuity" of addition means that if we have two sequences of vectors, both convergent in the metric of the space, say $v_n \to v$ and $w_n \to w$, then $v_n + w_n \to v + w$.

An F-space, and more generally a topological vector space, is said to be *locally convex* if at the origin (and therefore, by the continuity of translations, at *every* point) the topology has a base of convex sets.

Clearly every normed linear space is locally convex as are our examples of "fake p-normed spaces." It's well known that *every locally convex (Hausdorff) topological vector space* has the Hahn-Banach Extension Property. For F-spaces it's known that this property *characterizes* local convexity: *If an F-space is not locally convex, then it doesn't have the Hahn-Banach Extension Property!*

N. J. Kalton, Proc. Edinburgh Math. Soc. (2) 19 (1974) 151–167.

In order to give our F-spaces a fighting chance to have further Hahn-Banach properties, let's assume that they have enough continuous linear functionals to separate points, i.e., that they have *separating duals*. It turns out that: *If a p-normed space with separating dual is not locally convex, then it has a proper, closed, "weakly dense" subspace*, like the subspace of Theorem 4.2 that we discovered in ℓ^p .

N. J. Kalton, Glasgow Math. J. 19 (1978) 103–108.

References

1. S. Banach and S. Mazur, *Zur Theorie de linearen Dimension,* Studia Math. 4 (1933), 100–112.
2. M. M. Day, *The spaces L^p with* $0 < p < 1$, Bull. Amer. Math. Soc. 69 (1962), 638–640.
3. N. J. Kalton, *Basic sequences in F-spaces and their applications*, Proc. Edinburgh Math. Soc. (2) 19 (1974) 151–167.
4. N. J. Kalton, *Quotients of F-spaces*, Glasgow Math. J. 19 (1978) 103–108.
5. J. H. Shapiro, *Examples of proper, closed, weakly dense subspaces in nonlocally convex F-spaces*, Israel J. Math. 7 (1969), 369–380.