II. *The Maximal Function and the Dirichlet Problem*

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In the previous lecture [5] we deduced the Lebesgue Differentiation Theorem from the Hardy-Littlewood Maximal Theorem. Here we'll connect the Maximal Theorem with one of the classical problems of partial differential equations: the *Dirichlet Problem*.

1 *The Dirichlet Problem*

Our setting is the euclidean plane $\mathbb{R}^2$. Given a domain[1] Ω in $\mathbb{R}^2$, and "boundary data" expressed by a real-valued function f on $\partial\Omega$, the most general formulation of the *Dirichlet Problem* asks for a real-valued function u defined on Ω, and at each point of Ω has continuous second partial derivatives, satisfies *Laplace's equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\,, \tag{1}$$

and is—in some sense—equal to f on $\partial\Omega$, the boundary of Ω.

Functions u satisfying Laplace's equation on Ω are said to be *harmonic* on Ω. If we regard $\mathbb{R}^2$ to be the complex plane $\mathbb{C}$, then it follows from the Cauchy-Riemann equations[2] that every complex-valued function that is analytic on Ω has its real and imaginary parts harmonic on Ω. If Ω is *simply connected*[3] then the converse is true: every real-valued function that is harmonic on Ω is the real (resp. imaginary) part of a function that is analytic on Ω.

Most commonly one assumes the domain Ω to be *bounded*, and $\partial\Omega$ to be not too bad, e.g., piecewise smooth. In cases like this, the Dirichlet problem will have a unique solution whenever the boundary data is assigned by a function f that is continuous on $\partial\Omega$.[4]

By contrast, here we'll be considering Ω to be the *upper half-plane* of $\mathbb{R}^2$, i.e, the domain

$$\mathrm{UHP} = \{(x,y) \in \mathbb{R}^2 \colon y > 0\},$$

in which case non-uniqueness is possible. For example, the function $(x,y) \to y$ is clearly harmonic on $\mathbb{R}^2$, and $\equiv 0$ on $\mathbb{R}$ (now viewed as $\{(x,0) \colon x \in \mathbb{R}\}$). Thus given any solution u of the Dirichlet problem with boundary data f, the function $(x,y) \to u(x,y) + y$ is also a solution.

[1] A *domain* is a connected open set.

In the real world, one might imagine Ω to be cut out of a flat metal sheet, to which a time-invariant temperature $f(p)$ is applied at each point $p \in \partial\Omega$. Then the resulting steady-state temperature $u(x,y)$ at the point $(x,y) \in \Omega$ satisfies Laplace's equation for the boundary temperature distribution f.

[2] See, e.g., [4, pp. 16-17].

[3] Here "Ω simply connected" means that "every point surrounded by Ω lies in Ω." More rigorously: "The complement of Ω in the extended complex plane $\mathbb{C} \cup \{\infty\}$ is connected." *Examples*: open discs are simply connected, open annuli are not. See, e.g., [4], pp. 125-128 for more on these matters.

[4] In particular, this is true when Ω is the open unit disc—a companion to our half-plane situation. See, e.g. the (freely downloadable) paper [2] for a nice discussion.

2 The Poisson Integral

All our work on the Dirichlet problem for the upper half-plane springs from the unassuming function $\varphi\colon \mathbb{R} \to \mathbb{R}$ defined by:

$$\varphi(x) := \frac{1}{\pi}\frac{1}{1+x^2} \qquad (x \in \mathbb{R}), \tag{2}$$

where the factor $\frac{1}{\pi}$ is inserted to make the integral of φ over the real line equal to 1. The *Poisson kernel for level* $y > 0$ is the function P_y defined on $\mathbb{R}$ by

$$P_y(x) := \frac{1}{y}\varphi\left(\frac{x}{y}\right) = \frac{1}{\pi}\frac{y}{x^2+y^2} \qquad (x,y) \in \text{UHP}. \tag{3}$$

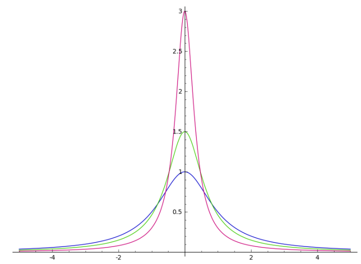

Figure 1: Graph of Poisson kernel $P_y(x) = \frac{1}{y}\varphi(x/y)$ for three different values of $y > 0$ (smaller values give "peakier" graphs).

Each Poisson kernel P_y is clearly non-negative, bounded, continuous on $\mathbb{R}$, and $\to 0$ as $x \to \pm\infty$. By a simple change of variable to the $y = 1$ case, $\int_{\mathbb{R}} P_y(x)\,dx = 1$ for each $y > 0$.

Proposition 2.1. *For $f \in L^1(\mathbb{R})$, the "Poisson integral"*

$$P[f](x,y) := \int_{\mathbb{R}} P_y(x-t)f(t)\,dt \qquad (x \in \mathbb{R})$$

exists for every $x \in \mathbb{R}$, and $P[f]$ is harmonic on UHP.

Proof. Let's identify the point $(x,y) \in \mathbb{R}^2$ with the complex number $z = x + iy$. Upon observing that $P_y(x) = i/(\pi z)$ for each $z \in \mathbb{C}\backslash\{0\}$, we see that:

$$\pi P[f](z) := \int_{\mathbb{R}} \text{Re}\,\frac{i}{z-t}\,f(t)\,dt = \text{Re}\int_{\mathbb{R}} \frac{i}{z-t}\,f(t)\,dt$$

with the last equality coming from real-valuedness of f and the definition of integral of a complex-valued function as the integral of its real and imaginary parts. Thus to show that $P[f]$ is harmonic it will be enough to show that the function F defined on UHP by

$$F(z) := \int_{\mathbb{R}} \frac{f(t)}{z-t}\,dt \qquad (z \in \text{UHP})$$

The existence of the integral is not in doubt, since f is integrable and integrand's denominator is bounded below by the distance from the (fixed) point $z \in \text{UHP}$ to the real line.

is analytic on UHP, i.e., that its complex derivative exists there.

To this end, fix points z and z_0 in UHP and compute:

$$\frac{F(z)-F(z_0)}{z-z_0} = -\int_{\mathbb{R}} \frac{f(t)}{(z-t)(z_0-t)}\,dt\,.$$

In the integral on the right, the integrand's denominator is bounded away from zero (with bound depending only on z and z_0), and as $z \to z_0$ the integrand converges pointwise (almost everywhere) to $f(t)/(z_0-t)^2$.

Thus by the Dominated Convergence Theorem, F is differentiable at z_0, with

$$F'(z_0) = -\int_{\mathbb{R}} \frac{f(t)}{(z-t)^2}\,dt\,.$$

What we've really done here is justify an interchange of differentiation (with respect to z) with integration.

Consequence: $P[f]$ is the real part of the iF, a function analytic on the upper half-plane, hence $P[f]$ is harmonic there. □

In the next few sections we'll explore ways in which $u = P[f]$ solves the Dirichlet problem on UHP with "boundary data" f.

3 *Uniform Convergence to Boundary Data*

SUPPOSE THAT our boundary data $f \in L^1(\mathbb{R})$ is both bounded and uniformly continuous on $\mathbb{R}$. For example, f could be continuous on $\mathbb{R}$, with compact support. We'll show that under these hypotheses the Poisson integral $u = P[f]$ solves the Dirichlet problem for UHP in the strongest possible fashion.

Theorem 3.1. *Suppose $f \in L^1(\mathbb{R})$ is bounded and uniformly continuous on $\mathbb{R}$. Then the function $u\colon \overline{\mathrm{UHP}} \to \mathbb{R}$ defined by*

$$u = \begin{cases} P[f] & \text{on} \quad \mathrm{UHP} \\ f & \text{on} \quad \mathbb{R} \end{cases}$$

is harmonic on UHP *and continuous on* $\overline{\mathrm{UHP}}$.

$\overline{\mathrm{UHP}} := \{z = x + iy \in \mathbb{C} \colon y \geq 0\}$, the "closed upper half-plane."

Proof. We'll need the following properties of the Poisson kernel:

(P1) $P_y \geq 0$ for each $y > 0$,

(P2) $\int_{\mathbb{R}} P_y(x)\,dx = 1$ for each $y > 0$, and

(P3) $\lim_{y\to 0+} \int_{|x|>\delta} P_y(x)\,dx = 0$ for each $\delta > 0$.

We noted the first two of these properties in the paragraph preceeding Proposition 2.1 above. For the third, fix positive numbers δ and y; then use the even-ness of P_y and the change-of-variable $x = ty$ to compute:

$$\int_{|x|>\delta} P_y(x)\,dx = 2\int_{x=\delta}^{\infty} \frac{1}{y}\varphi\left(\frac{x}{y}\right)dx = 2\int_{t=\delta/y}^{\infty} \varphi(t)\,dt\,.$$

Property (P3) follows from this and the integrability of φ over $\mathbb{R}$.

We prove the Theorem showing that $\lim_{y\to 0+} P[f](x,y) = f(x)$, where the convergence is *uniform* for $x \in \mathbb{R}$. To this end, fix $\varepsilon > 0$ and use the uniform continuity of f to choose $\delta_0 > 0$ such that

$$|t| < \delta_0 \implies |f(x-t) - f(x)| < \varepsilon/2 \quad (\forall\, x \in \mathbb{R})\,.$$

By condition (P3) above, we can choose $\delta > 0$ so that

$$0 < y < \delta \implies \int_{|t| \geq \delta_0} P_y(t)\,dt < \frac{\varepsilon}{2\|f\|_\infty}\,.$$

$\|f\|_\infty := \sup_{t \in \mathbb{R}} |f(t)|$.

Fix $(x, y) \in$ UHP and note that:

$$\begin{aligned} P[f](x,y) - f(x) &= \int_{\mathbb{R}} P_y(x-t) f(t)\,dt - f(x) \\ &= \int_{\mathbb{R}} f(x-t) P_y(t)\,dt - f(x) && \text{[Change of variable]} \\ &= \int_{\mathbb{R}} [f(x-t) - f(x)]\, P_y(t)\,dt\,. && \text{[Property (P2) of Poisson kernel]} \end{aligned} \tag{4}$$

Now estimate (using in the first line the positivity of P_y):

$$\begin{aligned} |P[f](x,y) - f(x)| &\leq \int_{\mathbb{R}} |f(x-t) - f(x)|\, P_y(t)\,dt \\ &= \int_{|t|<\delta_0} + \int_{|t \geq \delta_0} |f(x-t) - f(x)|\, P_y(t)\,dt \\ &\equiv I_1 + I_2, \end{aligned}$$

where

$$I_1 = \int_{|t|<\delta_0} \underbrace{|f(x-t) - f(x)|}_{<\varepsilon \text{ by choice of } \delta_0} P_y(t)\,dt < \frac{\varepsilon}{2} \underbrace{\int_{|t|<\delta} P_y(t)\,dt}_{\leq 1 \text{ by } (P2)} \leq \frac{\varepsilon}{2}\,.$$

For the second integral we have this estimate for $0 < y < \delta$:

$$\begin{aligned} |I_2| &= \int_{|t| \geq \delta_0} \underbrace{|f(x-t) - f(x)|}_{\leq 2\|f\|_\infty} P_y(t)\,dt \\ &\leq 2\|f\|_\infty \underbrace{\int_{|t| \geq \delta_0} P_y(t)\,dt}_{< \frac{\varepsilon}{2\|f\|_\infty} \text{ by choice of } \delta} \\ &< \frac{\varepsilon}{2} \end{aligned}$$

Thus for each $x \in \mathbb{R}$:

$$0 < y < \delta \implies \big|P[f](x,y) - f(x)\big| \leq I_1 + I_2 < \varepsilon\,,$$

which establishes the uniform convergence of $P[f](x,y)$ to f. □

The result above is "global" in that uniform continuity of f on $\mathbb{R}$ implies uniform convergence of $P[f](\cdot, y)$ to f on $\mathbb{R}$ as $y \to 0+$. The same argument gives the following "local" version:

Corollary 3.2 (of the above proof)**.** *Suppose $f \in L^1(\mathbb{R})$ is continuous at the point $x \in \mathbb{R}$. Then* $\lim_{y \to 0+} P[f](x,y) = f(x)$

4 Mean Convergence to Boundary Data

Now suppose $f \in L^1(\mathbb{R})$ is encumbered by no extra assumptions of continuity or boundedness. Then $P[f]$ is still a harmonic function on UHP; is there some sense in which it has f as "boundary data?" The result of this section says that this is so ... "in the mean."

Theorem 4.1. *If $f \in L^1(\mathbb{R})$, then* $\lim_{y\to 0+} \|P[f](\cdot, y) - f\|_1 = 0.$

$\|g\|_1 := \int_{\mathbb{R}} |g(x)|\, dx$ for $g \in L^1(\mathbb{R})$.

Proof. For $y > 0$ we have, by the computation (4) in the proof of Theorem 3.1:

$$P[f](x,y) - f(x) = \int_{\mathbb{R}} [f(x-t) - f(x)] P_y(t)\, dt$$

from which it follows that for each $y > 0$:

$$\begin{aligned} \|P[f](\cdot,y) - f\|_1 &= \int_{\mathbb{R}} \left|P[f](x,y) - f(x)\right| dx \\ &= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} [f(x-t) - f(x)]\, P_y(t)\, dt \right| dx \\ &\le \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x-t) - f(x)|\, P_y(t)\, dt \right) dx \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x-t) - f(x)|\, dx \right) P_y(t)\, dt\,. \end{aligned}$$

[Fubini's Theorem]

Upon using the notation $f_t(x)$ for $f(x-t)$, we can write the conclusion of this calculation as:

(5) $$\|P[f](\cdot,y) - f\|_1 \le \int_{\mathbb{R}} \|f_t - f\|_1 P_y(t)\, dt \qquad (y > 0).$$

Thanks to the Lemma below, the mapping $t \to \|f - f_t\|_1$ is continuous at the origin, (where it takes the value zero), so by Corollary 3.2, the integral on the right-hand side of (5) converges to zero as $y \to 0+$, hence the same is true of the left-hand side. □

Lemma 4.2. *If $f \in L^1(\mathbb{R})$ then* $\lim_{t\to 0} \|f_t - f\|_1 = 0.$

Proof. The result is easily seen to be true if $f \in C_c(\mathbb{R})$, the collection of continuous functions on $\mathbb{R}$ that have compact support (hint: uniform continuity).

$C_c(\mathbb{R})$ is a dense subspace of $L^1(\mathbb{R})$ (easy consequence of Urysohn's Lemma; see, e.g., [5], sidenote middle of page 2).

Suppose $f \in L^1(\mathbb{R})$, and we are given $\varepsilon > 0$. By the density of $C_c(\mathbb{R})$ in $L^1(\mathbb{R})$, there exists $g \in C_c(\mathbb{R})$ with $\|f - g\|_1 < \varepsilon$. Since the theorem is true for g, there exists $\delta > 0$ such that

$$|t| < \delta \implies \|g - g_t\|_1 < \frac{\varepsilon}{3}\,.$$

Thus for each $|t| < \delta$:

This proof is an elementary example of use of a uniform estimate to transfer a desired result from a dense subset to the whole space. The proof of the Lebesgue Differentiation Theorem in [5], as well as the result of the next section give further examples, more sophisticated in that the uniform estimate is supplied by a maximal theorem.

$$\begin{aligned}
\|f - f_t\| &= \|(f-g) + (g - g_t) + (g_t - f_t)\| \\
&\leq \|f-g\|_1 + \|g-g_t\|_1 + \underbrace{\|g_t - f_t\|_1}_{=\|g-f\|_1} \\
&= 2\underbrace{\|f-g\|_1}_{<\frac{2\varepsilon}{3}} + \underbrace{\|g-g_t\|_1}_{<\frac{\varepsilon}{3}} \\
&< \varepsilon
\end{aligned}$$

Here we note that $g_t - f_t = (g-f)_t$, and that (by a change-of-variable) $\|h_t\|_1 = \|h\|_1$ for every $h \in L^1(\mathbb{R})$.

as desired. □

5 *a.e. Convergence to Boundary Data*

So far we've shown that for each $f \in L^1(\mathbb{R})$, the Poisson integral solves—in some "mean sense"—the Dirichlet problem for UHP for boundary data f. Can we say more about the connection between $P[f]$ and its boundary function f?

Integration theory supplies hope that something more is possible: if a sequence converges in $L^1(\mathbb{R})$, then some subsequence converges almost-everywhere.[5] Thus for each $f \in L^1(\mathbb{R})$ there's a sequence $y_n \searrow 0$ such that $P[f](\cdot, y_n) \to f$ a.e. on $\mathbb{R}$. However, the sequence (y_n) may vary from one f to another, so this result is less-than-satisfactory. What we'd really like is a result reminiscent of the Lebesgue Differentiation Theorem, e.g., $P[f](\cdot, y) \to f$ a.e. as $y \to 0+$. This is what we'll prove here, leaning heavily on the techniques developed in the previous notes [5] for the Lebesgue Differentiation Theorem.

[5] See, e.g., [3], Theorem 3.12, page 68.

Not surprisingly, the desired result will follow from a maximal theorem. For $f \in L^1(\mathbb{R})$ consider the "Poisson Maximal Function," $P^* f$ defined by:

$$P^* f(x) := \sup_{y>0} P[|f|](x,y) \qquad (x \in \mathbb{R}).$$

Our goal is to prove:

Theorem 5.1 (The "Poisson Maximal Theorem"). *There exists a positive constant C such that for each $f \in L^1(\mathbb{R})$ and $\lambda > 0$:*

$$m\{P^* f > \lambda\} \leq \frac{C}{\lambda} \|f\|_1 .$$

"m" denotes Lebesgue measure on $\mathbb{R}$.

We'll get to the proof of this theorem in a moment. But first, the result we're really after.

Corollary 5.2 (a.e. convergence for Poisson integrals). *For each $f \in L^1(\mathbb{R})$:*

$$\lim_{y \to 0+} P[f](x,y) = f(x) \quad \text{for } \text{ a.e } x \in \mathbb{R}.$$

Proof of Corollary. Theorem 3.1 provides the desired result for the dense subset $C_c(\mathbb{R})$ of $L^1(\mathbb{R})$. Thanks to his fact and the "uniform estimate" of Theorem 5.1, the "maximal theorem" proof of the Lebesgue Differentiation Theorem (see, e.g., [5], §3, pp. 4-5) works *mutatis mutandis*[6] to show that

[6] Medieval Latin for: "once the necessary changes have been made."

$$\lim_{y\to 0+} P\big[|f-f(x)|\big](x) = 0 \quad \text{for a.e } x \in \mathbb{R}\,;$$

a result even stronger than the one promised. □

Proof of Theorem. Fix $y > 0$. We'll construct a "wedding-cake" function whose graph lies between the graphs of P_y and $2P_y$. To do this, draw the largest possible closed rectangle (sides parallel to the coordinate axes) that has top edge at height $2P_y(0)$ and which fits between the graphs of P_y and $2P_y$. Now draw another such rectangle, the largest one whose top edge contains the bottom edge of the first rectangle, and which again fits between the two graphs. Continue the process (forever), obtaining a function W_y whose graph looks like the side view of a wedding cake (see Figure 2).

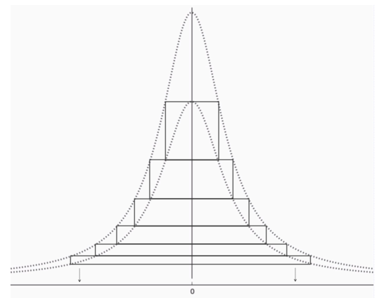

Figure 2: A "wedding-cake" function

For $j \in \mathbb{N}$, denote by B_j the interval obtained by projecting the base of j-th rectangle formed above onto the horizontal axis, and let $H_j = \frac{1}{m(B_j)}\chi_{B_j}$. Thus $\int_{\mathbb{R}} H_j\,dm = 1$ for each $j \in \mathbb{N}$, and it's clear from its defining picture that there is a sequence (α_j) of positive constants such that $W_y = \sum_j \alpha_j H_j$. Moreover: $\sum_j \alpha_j \leq 2.^{(*)}$

χ_E denotes the characteristic function of the set $E \subset \mathbb{R}$ (= 1 on E, and 0 off E).

(*) *Proof*: $2P_y \geq W_y$ and $\int_{\mathbb{R}} P_y\,dm = 1$, so

$$2 \geq \int_{\mathbb{R}} W_y\,dm = \sum_j \alpha_j \int_{\mathbb{R}} H_j\,dm = \sum_j \alpha_j.$$

Fix $f \in L^1(\mathbb{R})$ and $x \in \mathbb{R}$, and calculate:

$$\begin{aligned}
P\big[|f|\big](x,y) &= \int_{\mathbb{R}} |f(x-t)|P_y(t)\,dt \\
&\leq \int_{\mathbb{R}} |f(x-t)|W_y(t)\,dt \\
&= \sum_j \alpha_j \int_{\mathbb{R}} f(x-t)H_j(t) \\
&= \sum_j \alpha_j \frac{1}{m(B_j)} \int_{B_j} |f(x-t)|\,dt
\end{aligned}$$

Now make the substitution $s = x - t$ in the integrals in the last line above, and write $B_j(x)$ for $B_j + x$, the interval of the same size as B_j, but now with center x. The result is

$$P\big[|f|\big](x,y) \leq \sum_j \alpha_j \underbrace{\frac{1}{B_j(x)} \int_{B_j(x)} |f(s)|\,ds}_{\leq Mf(x)}\,.$$

Mf is the *Hardy-Littlewood Maximal Function*: $Mf(x) = \sup_B \int_B |f|\,dm$, where the supremum is over all balls (in this case—intervals) B centered at x (see, e.g., [5, §1]).

Conclusion: For each $(x,y) \in \text{UHP}$ and each $f \in L^1(\mathbb{R})$:

$$P\big[|f|\big](x,y) \leq 2\,Mf(x)\,. \tag{6}$$

Upon taking the supremum of the left-hand side of (6) over $y > 0$ we obtain:

(7) $$P^* f(x) \leq 2\, Mf(x) \qquad \forall\, f \in L^1(\mathbb{R}) \text{ and } x \in \mathbb{R}.$$

Thus, for each $f \in L^1(\mathbb{R})$ and $\lambda > 0$:

$$m\{P^* f > \lambda\} \;\leq\; m\{2\, Mf > \lambda\} \;\leq\; \frac{6}{\lambda}\|f\|_1 .$$

with the final inequality provided by the Hardy-Littlewood Maximal Theorem. □

For the real line, this theorem asserts: $m\{Mf > \lambda\} \leq \frac{3}{\lambda}\|f\|_1$ for each $\lambda > 0$ and $f \in L^1(\mathbb{R})$ (see, e.g., [5, §2]).

6 Final Remarks

6.1 The "wedding-cake construction"

(a) If, in this construction, we replace $2P_y$ by $(1+\varepsilon)P_y$, there results a corresponding improvement on the right-hand side of (7). Since this is true for each $\varepsilon > 0$, the true story is:

$P^* f \leq Mf$ for each $f \in L^1(\mathbb{R})$.

(b) You may have noticed that not many properties of P_y got used in this construction. In fact, the construction works for any "kernel" $y^{-1}\varphi(x/y)$, where φ is non-negative, integrable, even, decreasing for $x > 0$, and normalized so that $\int_{\mathbb{R}} \varphi\, dm = 1$.

Conclusion: The Hardy-Littlewood Maximal Theorem is a lot more general than at first meets the eye!

6.2 Higher dimensions

Let $\mathscr{U}$ denote the upper half-space of $\mathbb{R}^{n+1}$, i.e.,

$$\mathscr{U} := \{(x,y) \in \mathbb{R}^n \times \mathbb{R} \colon y > 0\}.$$

The analogue for $\mathscr{U}$ of the Poisson kernel for UHP is:

$$P_y(x) = \frac{1}{y^{n-1}}\varphi\left(\frac{x}{y}\right) \quad \text{where} \quad \varphi(x) = c_n \frac{1}{(1+|x|^2)^{n/2}} \quad (x \in \mathbb{R}^n, y > 0),$$

with c_n chosen to make $\int_{\mathbb{R}^n} \varphi\, dm = 1$. Then a change-of-variable in $\mathbb{R}^n$ shows—just as for the $n = 1$ case—that $\int_{\mathbb{R}^n} P_y\, dm = 1$ for each $y > 0$; in fact all the properties $P1 - P3$ for our original Poisson kernel[7] continue to hold for these higher dimensional ones.

Now m = Lebesgue measure on $\mathbb{R}^n$.

[7] See the proof of Theorem 3.1 for these.

We can then form the Poisson integral

$$P[f](x,y) := \int_{\mathbb{R}^n} P_y(x-t) f(t)\, dt \qquad ((x,y) \in \mathscr{U})$$

of any $f \in L^1(\mathbb{R}^n)$, verify that $P[f]$ is harmonic on $\mathscr{U}$ (now using an interchange of derivative and integral to verify that it satisfies Laplace's equation on $\mathscr{U}$), and proceed to prove by exactly the same steps as before that $P[f]$ solves the Dirichlet problem for $\mathscr{U}$, with boundary data f on $\mathbb{R}^n$. See [1], Chapter 7, pp. 144-151 for the continuous and "mean" versions of the problem.

The higher dimensional analogue of the Poisson Maximal Theorem (Theorem 5.1) and its resulting a.e. convergence theorem (Corollary 5.1) continue to hold, with essentially the same proofs—now based on the n-dimensional version of the Hardy-Littlewood Maximal Theorem, as presented in [5].

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