I. Intertwining

1. *Setting*:

- $\mathbb{U} = \{|z| < 1\}$

- $\varphi : \mathbb{U} \to \mathbb{U}$ holomorphic.

- $C_\varphi : \text{Hol}(\mathbb{U}) \to \text{Hol}(\mathbb{U})$ defined by:

$$C_\varphi f = f \circ \varphi.$$

(a) *J. E. Littlewood (1925)*:

$$C_\varphi : H^2 \to H^2 \qquad \text{bounded lin. op.}$$

(b) *Challenge*:

Properties of φ (Function theory)	$\longleftrightarrow$	Properties of C_φ (Operator theory)

2. *Research directions (selected):*

(A) C_φ *compact (on H^2) for which φ?*

- *Easy:* $\varphi(z) \equiv z \Rightarrow C_\varphi$ not compact.

  $\|\varphi\|_\infty < 1 \Rightarrow C_\varphi$ compact.

- *Less easy:*

  $$\varphi(z) = 1 - \sqrt{1-z} \quad \Rightarrow \quad C_\varphi \text{ compact.}$$

  $$\varphi(z) = \tfrac{1+z}{2} \quad \Rightarrow \quad C_\varphi \text{ not compact}$$

- *Partial answer:* For φ univalent:

$$C_\varphi \text{ compact} \iff \lim_{|z|\to 1-} \frac{1-|\varphi(z)|}{1-|z|} = \infty$$

$\iff$ φ "conformal" at no boundary pt.

(Julia-Caratheodory Theorem (1920's))

3. *"Selected directions"* *(cont'd.)*

(B) *Spectra–eigenvalues:* $f \circ \varphi = \lambda f$

(Schroeder's equation, 1870's)

- *Königs, 1880's:* $\varphi(0) = 0, \ \varphi'(0) \neq 0$

  $\Rightarrow$

  (1) $\{\text{Eigenvalues of } C_\varphi\} = \{\varphi'(0)^n\}_{n=0}^{\infty}$

  (2) Each eigenvalue has multiplicity one.

  (3) Eigenvector for $\varphi'(0)^n$ "is" σ^n
  where $\sigma \circ \varphi = \varphi'(0)\,\sigma$
  (the "Königs function").

- *Caughran & Schwartz (1970's):*

  C_φ compact $\Rightarrow$:

  (1) φ has fixed point in $\mathbb{U}$.

  (2) spectrum of $\varphi = \{\text{"Königs eigenvalues"}\} \cup \{0\}$

  (3) $\sigma^n \in H^2$ $(n = 0, 1, 2, ...)$, i.e. $\sigma \in H^p \ \forall \, p < \infty$

- *Question: Properties of* σ $\longleftrightarrow$ *Cpctness of* C_φ.

· Smith, Stegenga & JHS · Poggi-Corradini	(1990's)

4. *Comp. ops. and Toeplitz ops.*

(a) *Defn.* For $g \in H^\infty$ define $T_g : H^2 \to H^2$ by

$$T_g f = g \cdot f$$

(the *analytic Toeplitz op.* with symbol g).

(b) *$C_\varphi - T_g$ connection:*

$$C_\varphi T_g = T_{g\circ\varphi} C_\varphi$$

(C_φ *intertwines* T_g and $T_{g\circ\varphi}$)

(c) *Defn:* For Hilbert space ops. A, B, $X \neq 0$:

X *intertwines* A and B if $XA = BX$.

Notation: $A \propto B$ or $A \propto_X B$

(d) *Example:* $T_g \propto_\varphi T_{g\circ\varphi}$

(e) *Defn:* h *subordinate to* g means

$$\exists\ \varphi : \ h = g \circ \varphi$$

Notation: $h \prec g$ or $h \prec_\varphi g$

(f) *Prop.* $T_g \propto_\varphi T_h \iff h \prec_\varphi g$

5. *Intertwining & Subordination*

(a) *Prop.* $T_g \; \propto_{C_\varphi} \; T_h \iff h \; \prec_\varphi \; g$

Proof. ($\Leftarrow$: already proved)

$\Rightarrow$: Suppose $\quad C_\varphi T_g = T_h C_\varphi$

i.e. $\quad T_{g\circ\varphi} C_\varphi \; = \; T_h C_\varphi$

To show: $\quad g \circ \varphi = h$

(b) *Qn.* $T_g \propto T_h$? $\Rightarrow$? $h \prec g$

i.e. Analyt toep ops intertwined

?$\Rightarrow$?

Intertwined by a comp. op?

(c) *Ans.* NOT ALWAYS:

$$T_{z^2} \propto T_z \quad \text{but} \quad z \not\prec z^2$$

(d) *Thm.* $\quad g$ inner $\quad \Rightarrow \quad T_g \propto_{C_g^*} T_z$

$g(0) = 0$

6. *"Non-subord." Intertwining*

(a) *Thm.* g inner $\Rightarrow T_g \propto_{C_g^*} T_z$
$g(0) = 0$

(b) Deddens (1972)

(c) Proof of Thm.

- *To Show:* $C_g^* T_g = T_z C_g^*$

i.e. $\forall\, f \in H^2$:

$$(*) \qquad \langle C_g^* T_z f,\, f\rangle = \langle T_z C_g^* f,\, f\rangle$$

$$\text{LHS}(*) = \langle g \cdot f,\, f \circ \varphi\rangle$$

$$\begin{aligned}
\text{RHS}(*) &= \langle C_g^* f,\, T_z^* f\rangle = \langle f,\, C_g T_z^* f\rangle \\
&= \langle f,\, C_g \frac{f(z) - f(0)}{z}\rangle = \langle f,\, \frac{f(g(z)) - f(0)}{g(z)}\rangle \\
&= \langle f,\, \overline{g}\,(f \circ g - f(0))\rangle = \langle g \cdot f,\, f \circ g - f(0)\rangle \\
&= \langle g \cdot f,\, f \circ g\rangle - \overline{f(0)}\,\langle g \cdot f,\, 1\rangle
\end{aligned}$$

(d) (Bourdon–JHS 2007): For φ *univalent:*
"Intertwining $\Rightarrow$ sbdn"

7. *Proof of Littlewood's Thm*

Given: $\varphi : \mathbb{U} \to \mathbb{U}$ holomorphic.

To Show: $C_\varphi : H^2 \to H^2$ bounded operator.

(a) Crucial case: $\varphi(0) = 0$.

Enough to assume: f polynomial (degree n).

$$\begin{aligned} f(z) &= \hat{f}(0) + \hat{f}(1)\, z + \hat{f}(2)\, z^2 + \cdots \hat{f}(n) z^n \\ &= \hat{f}(0) + z \cdot T_z^* f(z) \\ C_\varphi f &= \hat{f}(0) + \varphi\, C_\varphi(T_z^*\, f) \\ \|C_\varphi f\|^2 &= |\hat{f}(0)|^2 + \|\varphi\, C_\varphi(T_z^*\, f)\|^2 \quad \text{(!!)} \\ &\leq |\hat{f}(0)|^2 + \|C_\varphi(T_z^*\, f)\|^2 \end{aligned}$$

Repeat, with $T_z^* f$ in place of f:

$$\begin{aligned} \|C_\varphi T_z^* f\|^2 &\leq |\widehat{T_z^* f}(0)|^2 + \|C_\varphi (T_z^*)^2 f\|^2 \\ &\leq |\hat{f}(1)|^2 + \|C_\varphi (T_z^*)^2 f\|^2 \\ \|C_\varphi f\|^2 &\leq |\hat{f}(0)|^2 + |\hat{f}(1)|^2 + \|C_\varphi (T_z^*)^2 f\|^2 \\ &\vdots \\ &\leq |\hat{f}(0)|^2 + |\hat{f}(1)|^2 + \cdots + |\hat{f}(n)|^2 \\ &= \|f\|^2 \end{aligned}$$

8. *Littlewood—endgame*

(a) So far: $\varphi(0) = 0$

$$\Rightarrow$$

$$\|C_\varphi f\| \;\leq\; \|f\| \qquad \forall f \in H^2$$

(b) To Show: $\varphi(0) = p \neq 0$

$$\Rightarrow$$

$$\|C_\varphi f\| \;\leq\; \text{const. } \|f\| \quad \forall\, f \in H^2$$

(c) Proof:

$$\alpha_p(z) \;=\; \frac{p - z}{1 - \overline{p}\, z} \;\in \mathrm{Aut}(\mathbb{U}) \quad \text{“self-inverse”}$$

$$\psi \;:=\; \alpha_p \circ \varphi$$

$$\psi(0) \;=\; 0 \qquad \& \qquad \varphi = \alpha_p \circ \psi$$

$$C_\varphi \;=\; C_\psi\, C_{\alpha_p}$$

$$\|C_\varphi\| \;\leq\; \|C_\psi\|\; \|C_{\alpha_p}\| \;=\; \sqrt{\frac{1 + |p|^2}{1 - |p|^2}} \qquad \square$$

9. *When does intertwining $\Rightarrow$ sbdn. ?*

So far:

(a) $h \prec g \Rightarrow T_g \propto T_h$

$(h = g \circ \varphi \quad \Rightarrow \quad C_\varphi T_g = T_h C_\varphi)$

(b) $T_g \propto T_h \not\Rightarrow h \prec h$

$(C_{z^2} \propto C_z$ but $z \not\prec z^2)$

<u>Thm.</u> (PB & JHS, 2007). For φ *univalent:*

$$T_g \propto T_h \quad \Rightarrow \quad h \prec g \qquad (\text{so } T_g \propto_{C_\varphi} T_h)$$

<u>Reproducing Kernel</u> for $a \in \mathbb{U}$:

$$K_a(z) := \frac{1}{1 - \overline{a}z} = \sum_{n=0}^{\infty} \overline{a}^n z^n$$

$$\langle f, K_a \rangle = \sum_{n=0}^{\infty} \hat{f}(n)\, a^n = f(a) \quad \forall\, f \in H^2$$

$$\langle f, T_g^* K_a \rangle = \langle T_g f, K_a \rangle = g(a) f(a) = \langle f, \overline{g(a)}\, K_a \rangle$$

<u>Prop.</u> $T_g^* K_a = \overline{g(a)}\, K_a \quad \forall\, a \in \mathbb{U}$.

10. *Intertwining ?⇒? sbdn. (cont'd)*

Thm. For φ *univalent:*

$$T_g \propto T_h \quad \Rightarrow \quad h \prec g \qquad (\text{so } T_g \propto_{C_\varphi} T_h)$$

Prop. $$T_g^* K_a = \overline{g(a)}\, K_a \quad \forall\, a \in \mathbb{U}.$$

Proof:

$$X\,T_g = T_h\,X \qquad (X \neq 0)$$

$$T_g^*\,X^* = X^*\,T_h^*$$

$$T_g^*\,(X^* K_a) = X^*\,T_h^*\,K_a = \overline{h(a)}\,(X^* K_a)$$

Conclude: $X^* K_a \neq 0 \Rightarrow X^* K_a$ an eigvector of T_g^* for eigvalue $\overline{h(a)}$

Fact: g univalent $\Rightarrow$ $\{\text{Eigval's of } T_g^*\} = \overline{g(\mathbb{U})}$

Consequence:

$X^* K_a \neq 0 \Rightarrow h(a) \in g(\mathbb{U})$

$h(\mathbb{U}) \subset g(\mathbb{U})$

$h = g \circ \varphi$ where $\varphi = g^{-1} \circ h$. □

11. *Further results*

(a) So far: $h \prec g \Rightarrow T_g \propto T_h$

Converse true for g *univalent.*

(b) Converse true when:

- g a *covering map,*

or when

- g *singly covers* an open subset of $h(\mathbb{U})$.

(c) Recall: $T_{z^2} \propto_{C^*_{z^2}} T_z$

What is $C^*_{z^2}$?

<u>Answer</u> (proof next time):

$C^*_{z^2} f(z) = \frac{1}{2}[f(\sqrt{z}) + f(-\sqrt{z})]$!!

Selected References

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