III. Essentially normal comp. ops.

1. *Operator theoretic setting*:

- $\mathcal{H}$: A Hilbert space
- $\mathcal{L}(\mathcal{H})$: All bounded linear ops on $\mathcal{H}$.
- $T \in \mathcal{L}(\mathcal{H})$ is
  - *normal* if: $T^*T = TT^*$
  - *essentially normal* if:

    $$[T^*, T] := T^*T - TT^*$$

    is *compact*.

- Examples: "Trivially" ess. normal ...
  - Normal ops.
  - Compact ops.
  - Normal + compact

2. *"Nontrivial" Examples: shifts*

(a) For $x = (x(0), x(1), x(2), \ldots) \in \ell^2$

- *Forward shift*

  $$Sx = (0, x(0), x(1), x(2), \ldots)$$

- *Backward shift* $(= S^*)$

  $$Bx = (x(1), x(2), x(3), \ldots)$$

(b) Prop. S *and* B *are essentially normal*

$$\begin{aligned} S^*S &= I \\ SS^*x &= (0, x(1), x(2), \ldots) \\ [S^*, S]x &= (S^*S - SS^*)x = (x(0), 0, 0, \ldots) \\ [S^*, S] &= \text{rank 1 proj'n (compact)} \end{aligned}$$

(c) S, B *not* "normal plus compact"

(Fredholm index $\neq 0$).

3. *Which C_φ are normal?*

(a) *Example:* Complex dilations.

Fix $a \in \overline{\mathbb{U}}$

Define $\varphi_a(z) := az \qquad (z \in \mathbb{U})$

$\varphi_a(z^n) = a^n z^n$

$[C_{\varphi_a}] = \text{diag}\,\{1, a, a^2, \ldots\}$,

Conclude: C_{φ_a} *normal.* □

(b) *H. J. Schwartz (1969):*

$$C_\varphi \text{ normal} \Rightarrow \varphi = \varphi_a, \quad \exists a \in \overline{\mathbb{U}}.$$

(c) *Normality Lemma*:

T normal $\Rightarrow$

$\cdot\ \ker T = \ker T^*$
$\cdot\ Tf = \lambda f \ \Rightarrow\ T^* f = \overline{\lambda} f$

Proof. Enough to show:

$$\|Tf\| = \|T^* f\| \quad \forall f \in \mathcal{H} \text{ !!}$$

$$\begin{aligned} \|Tf\|^2 &= \langle Tf, Tf\rangle = \langle T^*Tf, f\rangle \\ &= \langle TT^*f, f\rangle \qquad (T \text{ normal}) \\ &= \langle T^*f, T^*f\rangle = \|T^*f\|^2 \end{aligned}$$

□

4. *Pf. of Schwartz's Thm.*

To Show: C_φ *normal* $\Rightarrow$ $\varphi = \varphi_a, \quad \exists a \in \overline{\mathbb{U}}$.

(a) C_φ *normal* $\Rightarrow$ $\varphi(0) = 0$.

Proof. $C_\varphi 1 = 1$

$C_\varphi^* 1 = 1$ (Normality Lemma)

$$\begin{aligned} 0 &= \langle z, 1\rangle = \langle z, C_\varphi^* 1\rangle \\ &= \langle C_\varphi z, 1\rangle = \langle \varphi, 1\rangle \\ &= \varphi(0) \end{aligned}$$

□

(b) $\varphi(z) \equiv \varphi'(0)z$

Proof: $\forall\ f \in H^2$

$$\begin{aligned} \langle f, C_\varphi^* z\rangle &= \langle C_\varphi f, z\rangle = \widehat{f \circ \varphi}(1) \\ &= (f \circ \varphi)'(0) = f'(\underbrace{\varphi(0)}_{=0})\varphi'(0) \\ &= \langle \varphi'(0) f, z\rangle = \langle f, \overline{\varphi'(0)} z\rangle \end{aligned}$$

$$\Rightarrow C_\varphi^* z = \overline{\varphi'(0)}\, z$$

$$\Rightarrow \underbrace{C_\varphi z}_{\varphi(z)} = \varphi'(0)\, z \quad \text{(Normality Lemma)}$$

□

5. *Which C_φ are ess. normal?*

Notation

- HSM ($\mathbb{U}$): All holom. selfmaps $\varphi : \mathbb{U} \to \mathbb{U}$.
- LFT ($\mathbb{U}$): All *linear-fractional* $\varphi \in$ HSM ($\mathbb{U}$)

  $\varphi(z) = \frac{az+b}{cz+d} \qquad (ad - bc \neq 0)$

- Aut ($\mathbb{U}$): All $\varphi \in$ LFT ($\mathbb{U}$) with $\varphi(\mathbb{U}) = \mathbb{U}$.

Nina Zorboska (1999):

(a) | $\varphi \in$ Aut ($\mathbb{U}$) — C_φ ess. normal | $\Rightarrow$ | $\varphi = \varphi_a$, — $\exists\, |a| = 1$ |

"Ess. normal autos are normal"

(b) Question: $\exists$? *nontriv. ess. normal C_φ?*

(c) | $\varphi \in$ HSM ($\mathbb{U}$) — C_φ ess. normal — (nontrivially) | $\Rightarrow$ | φ has NO — f.p. in $\mathbb{U}$. |

6. *Main result*

Thm. (BLNS 2003) *For* $\varphi \in \mathrm{LFT}\,(\mathbb{U})$*:*

$$C_\varphi \textit{ nontriv. ess. normal}$$
$$\iff$$
$$\varphi \textit{ a parabolic non-auto.}$$

(a) Need only consider φ non-auto w/ no f.p. in U. i.e.,

- φ parabolic: one f.p. in $\widehat{\mathbb{C}}$—on $\partial\mathbb{U}$
- φ hyperbolic: two f.p.'s; one $\in \partial\mathbb{U}$, one $\notin \partial\mathbb{U}$

(b) Parabolic LFT's 0f $\mathbb{U}$ — w/ f.p. at 1

- $\varphi_t(z) = \dfrac{(2-t)z+t}{-tz+(2+t)}$ (Re $t \geq 0$)
- On RHP: $\Phi_t(w) = w + t$ (Re $t \geq 0$)
- $\varphi_t = K^{-1} \circ \Phi_t \circ K$ where $\underbrace{K(z) = \dfrac{1+z}{1-z}}_{Cayley\ Transform}$
- φ_t non-automorphic $\iff$ Re $t > 0$
- $\varphi_2(z) = \dfrac{1}{2-z}$ *non*-automorphic

7. Toward pf. of Main Theorem

$$\varphi(z) = \frac{az+b}{cz+d} \quad \in \quad \mathsf{LFT}\,(\mathbb{U})$$

(a) *Cowen's Adjoint Formula* (1988)

$$C_\varphi^* = T_g \, C_\sigma \, T_h^*$$

where

$$\sigma := \rho \circ \varphi^{-1} \circ \rho \qquad \in \mathsf{LFT}\,(\mathbb{U})$$

$$g(z) := \frac{1}{-\bar{b}z + \bar{d}} \qquad \in H^\infty$$

$$h(z) := cz + d \qquad \in H^\infty$$

(b) *The Commutator Formula*

$$\begin{aligned}
[C_\varphi^*, C_\varphi] &:= C_\varphi^* C_\varphi - C_\varphi C_\varphi^* \\
&= T_g \, [C_\sigma, C_\varphi] \, T_h^* + T_g \, C_\sigma \, [T_h^*, C_\varphi] \\
&\quad + (T_g - T_{g\circ\varphi}) \, C_{\sigma\circ\varphi} \, T_h^*
\end{aligned}$$

(c) Corollary. *For*

$\varphi \in \mathsf{LFT}\,(\mathbb{U}) \backslash \mathsf{Aut}\,(\mathbb{U})$
w/ f.p. on $\partial\mathbb{U}$

C_φ *essentially normal* $\iff$ $[C_\sigma, C_\varphi]$ *compact*

8. *So far:*

(a) *For* $\varphi \in \mathrm{LFT}\,(\mathbb{U}) \setminus \mathrm{Aut}\,(\mathbb{U})$ *with f.p. on* $\partial\mathbb{U}$*:*

$$C_\varphi \text{ ess. normal } \iff [C_\varphi, C_\sigma]$$

(b) Suppose φ parabolic non-auto.

To show: C_φ ess. normal

(c) Enough to show: $[C_\varphi, C_\sigma]$ compact.

φ has one fixed pt. in $\widehat{\mathbb{C}}$—on $\partial\mathbb{U}$

Same for $\sigma := \rho \circ \varphi^{-1} \circ \rho$

σ parabolic with same FP as φ.

$\sigma \circ \varphi = \varphi \circ \sigma$

$[C_\sigma, C_\varphi] = 0$ $\square$

(d) So far:

- $\varphi \in \mathrm{Aut}\,(\mathbb{U})$ (not rot'n) $\Rightarrow C_\varphi$ NOT ess normal
- C_φ ess normal $\Rightarrow \varphi$ has NO f.p. in $\mathbb{U}$

  $\Rightarrow \varphi$ HAS f.p. on $\partial\mathbb{U}$
  (Denjoy-Wolff, 1920's)
- φ parabolic non-auto $\Rightarrow C_\varphi$ ess normal

9. To show:

$\varphi \in$ LFT $(\mathbb{U})$ *Non-auto* *Hyperbolic*	$\Rightarrow$	C_φ *NOT* *essentially* *normal*

(a) $\varphi \circ \sigma$ and $\sigma \circ \varphi$ parabolic w/ same f.p.

(b) $\varphi \circ \sigma = \varphi_t$ and $\sigma \circ \varphi = \varphi_s$, $\exists\ s, t \in$ RHP.

(c) Eigenvalues of C_{φ_s}, C_{φ_t} :

$$f_\lambda(z) \;=\; \exp\left\{-\lambda\frac{1+z}{1-z}\right\} \in H^\infty \qquad \forall\lambda > 0$$

$$C_{\varphi_t}(f_\lambda) = e^{\lambda t} f_\lambda$$

$$[C_\varphi, C_\sigma] f_\lambda = (C_{\varphi_t} - C_{\varphi_s}) f_\lambda = (e^{-\lambda t} - e^{-\lambda s}) f_\lambda$$

(d) $\text{spectrum}\,([C_\varphi, C_\sigma]) \supset \underbrace{\{e^{-\lambda t} - e^{-\lambda s} : \lambda > 0\}}_{curve\ in\ \mathbb{C}\ (s \neq t)}$

(e) Conclude:

- $[C_\varphi, C_\sigma]$ not compact (Riesz Theory !!)
- C_φ not essentially normal. $\square$

10. Beyond Linear-Fractional

$\varphi, \psi : \mathbb{U} \to \mathbb{U}$, holo in nbd of $\overline{\mathbb{U}}$

(a) Defns.

- *Horocyclic disc* (at $\eta \in \partial\mathbb{U}$)
- Defn. φ, ψ have *same 2nd order data* at $\zeta \in \partial\mathbb{U}$ if: values, 1st derivs, 2nd derivs agree at ζ.
- Defn. φ is *horocyclic at* $\eta \in \partial\mathbb{U}$ if
  $\varphi(\mathbb{U}) \subset$ horocyc. disc at η.

(b) Thm. (BLNS 2003 / SS 1990):

- φ, ψ horocyclic at $\eta \in \partial\mathbb{U}$
- $\varphi^{-1}(\eta) = \psi^{-1}(\eta) = \zeta \in \partial\mathbb{U}$
- Same 2d order bdry data at ζ

$\Rightarrow$

$C_\varphi - C_\psi$ compact on H^2 !!

(c) Some (nontriv) ess normal C_φ's:

- $\varphi(z) = \frac{-z+1}{z-3}$ $\qquad$ $\Phi(w) = w + 1$
- $\varphi(z) = \frac{z^2+3}{z^2-4z+7}$ $\qquad$ $\Phi(w) = w + 1 + \frac{1}{w+1}$
- $\sigma(z) = \frac{15z-21-2\sqrt{3-2z}}{25z-33}$ $\qquad$ $\varphi(z) = \frac{9z^2-30z+25}{13z^2-42z+33}$, $\quad \Phi(w) = w + 6 + \frac{1}{w+3}$

Selected References

1. P. S. Bourdon, D. Levi, S. K. Narayan, J. H. Shapiro, *Which linear-fractional composition operators are essentially normal?* J. Math. Anal. App. 280 (2003), 30–55.

2. Howard J. Schwartz, *Composition Operators on H^p, Thesis, Univ. of Toledo (Ohio) 1969*

3. J. H. Shapiro, Carl Sundberg, *Isolation amongst the composition operators,* Pacific J. Math 145 (1990), 117–151.

4. Nina Zorboska, *Closed range essentially normal composition operators are normal,* Acta Sci. Math. (Szeged) 65 (1999), 287–292.