IV. Toeplitzness of Comp. Ops.

1. *Setting / Operators*

(a) $\mathbb{U} = \{|z| < 1\} \subset \mathbb{C}$

$\partial\mathbb{U} = \{|z| = 1\}$

(b) $H^2 : \left\{ f(z) = \sum_0^\infty \widehat{f}(n)\, z^n : \sum |\widehat{f}(n)|^2 < \infty \right\}$

$$\{f \in L^2(\partial\mathbb{U}) : \widehat{f}(n) = 0 \ \forall n < 0\}$$

(c) $P : L^2(\partial\mathbb{U}) \to H^2$ Orthog (Riesz) proj'n.

(d) *Toeplitz ops:*

For $b \in L^\infty(\partial\mathbb{U})$,

$T_b f = P(bf) \qquad (f \in H^2)$

$T_b : H^2 \to H^2$ *bounded lin op*

(e) Two Toeplitz Examples: $b(z) \equiv z$

$T_b =$ *forward shift* $:= S$

$T_{\bar{b}} =$ *backward shift* $= S^*$

2. *Which C_φ are Toeplitz ops?*

Obvious: $C_z = I = T_1$

Claim: No others !!

(a) *Matrix of a Toep. op:*

$$b \sim \sum_{-\infty}^{\infty} \hat{b}(n) e^{in\theta} \in L^\infty(\partial\mathbb{U})$$

$$[T_b] = \begin{bmatrix} \hat{b}(0) & \hat{b}(-1) & \hat{b}(-2) & \hat{b}(-3) & \cdots \\ \hat{b}(1) & \hat{b}(0) & \hat{b}(-1) & \hat{b}(-2) & \cdots \\ \hat{b}(2) & \hat{b}(1) & \hat{b}(0) & \hat{b}(-1) & \cdots \\ \hat{b}(3) & \hat{b}(2) & \hat{b}(1) & \hat{b}(0) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

Constant on Diagonals !

$$S^* \, T_b \, S = T_b$$

3. *Which C_φ are Toeplitz?–cont'd*

$$[C_\varphi] = \begin{bmatrix} 1 & \hat{\varphi}(0) & \widehat{\varphi^2}(0) & \widehat{\varphi^3}(0) & \cdots \\ 0 & \hat{\varphi}(1) & \widehat{\varphi^2}(1) & \widehat{\varphi^3}(1) & \cdots \\ 0 & \hat{\varphi}(2) & \widehat{\varphi^2}(2) & \widehat{\varphi^3}(2) & \cdots \\ 0 & \hat{\varphi}(3) & \widehat{\varphi^2}(3) & \widehat{\varphi^3}(3) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

- Main diag: $\hat{\varphi}(1) = 1$
- Sub-diag's: $\hat{\varphi}(n) = 0 \quad \forall\ n > 1$
- 1st Super-diag: $\hat{\varphi}(0) = \widehat{\varphi^2}(1) = 2\hat{\varphi}(0)\hat{\varphi}(1)$

Conclude: C_φ Toeplitz $\Leftrightarrow \varphi(z) \equiv z \quad (C_\varphi = I)$

4. *Asymptotic Toeplitz Operators*

(Barría & Halmos, 1982)

Defn. $T \in \mathcal{L}(H^2)$ "Asymptotically Toeplitz"

means:

"$\{S^{*n}\,T\,S^n\}$ strongly convergent."

(a) "Asymp Toep." $\Rightarrow$ "Diagonals converge"

$$
\begin{aligned}
< S^{*n} T S^n z^\alpha, z^\beta > &= < T S^n z^\alpha, S^n z^\beta > \\
&= < T z^{\alpha+n}, z^{\beta+n} > \\
&= [T]_{\alpha+n,\beta+n}
\end{aligned}
$$

(b) "Compact" $\Rightarrow$ "Asymp. Toep."

5. *Which C_φ are Asymp. Toeplitz?*

(Fedor Nazarov & JHS 2006)

(a) *Thm. 1* $|\varphi| < 1$ a.e. on $\partial\mathbb{U}$

$$\Rightarrow$$

C_φ Asymp. Toep.

(b) *Example.* $C_{\frac{1+z}{2}}$ is asymp. Toep.

nontrivially (not compact).

(c) *Proof of Thm. 1:* Fix $f \in H^2$.

$$\|S^{*n} C_\varphi S^n f\|^2 \leq \|\varphi^n (f \circ \varphi)\|^2$$

$$= \int_{\partial\mathbb{U}} |\varphi|^{2n} |f \circ \varphi|^2 \, dm$$

$\rightarrow$ 0 by LDCT $\square$

6. *Converse of Thm. 1?*

$\varphi(z) \not\equiv z$ from now on !!

Thm. 2 (Partial Converse)

$$\left.\begin{array}{c} C_\varphi \text{ asymp. Toep.} \\ \\ \varphi(0) = 0 \end{array}\right\} \Rightarrow \quad |\varphi| < 1 \text{ a.e. on } \partial\mathbb{U}$$

Consequence. φ inner, $\varphi(0) = 0$

$$\Rightarrow$$

C_φ not asymp. Toeplitz on H^2.

Rmk. True even if $\varphi(0) \neq 0$!!

Importance of "$\varphi(0) = 0$":

$\exists$ C_φ asymp. Toep.

with

$m\{|\varphi| = 1\} > 0$

7. ***"Partial Converse" pf.*** $\varphi(0) = 0$.

Assume: $m\{|\varphi| = 1\} > 0$.

To Show: C_φ *not* Asymp. Toeplitz.

(a) C_φ asymp. Toep. $\Rightarrow$ $S^{*n} C_\varphi S^n \to \underline{\underline{0}}$ strongly (later).

(b) $\varphi(0) = 0 \Rightarrow \psi(z) := \varphi(z)/z$, holo on $\mathbb{U}$.

$$\begin{aligned} \|S^{*n}\, C_\varphi\, S^n\, 1\|^2 &= \int_{\partial\mathbb{U}} |S^{*n} \varphi^n|^2\, dm \\ &= \int_{\partial\mathbb{U}} |S^{*n}\, z^n\, \psi^n|^2\, dm \\ &= \int_{\partial\mathbb{U}} |\psi|^{2n}\, dm \\ &\geq m\{|\varphi| = 1\} > 0 \end{aligned}$$

Summary:

- $|\varphi| < 1$ a.e. on $\partial\mathbb{U} \Rightarrow C_\varphi$ asymp Toep.
- [C_φ asymp Toep & $\varphi(0) = 0$] $\Rightarrow$ $|\varphi| < 1$ a.e. on $\partial\mathbb{U}$.
- C_φ asymp Toep $\not\Rightarrow$ $|\varphi| < 1$ a.e. on $\partial\mathbb{U}$.
- C_φ asymp Toep $\iff$??

8. *"Mean" Asymp. Toeplitzness*

Thm 3. $\forall\ \varphi, \quad \forall \alpha > 0$:

$$C_\varphi \text{ is } \quad \text{"}(C,\alpha)\text{–asymp. Toeplitz"}$$

In fact, $\forall\ f \in H^2$:

$$\varphi(z) \not\equiv z \quad \Rightarrow \quad \|S^{*n} C_\varphi S^n f\| \overset{(C,\alpha)}{\longrightarrow} 0.$$

- $s_n \overset{(C,1)}{\longrightarrow} s$ means $\displaystyle \frac{1}{n+1}\sum_{j=0}^{n} s_j \to s$.

- $s_n \overset{(C,\alpha)}{\longrightarrow} s$ means $\displaystyle \sum_{j=0}^{n} c_{nj}^{(\alpha)} s_j \to s$.

- $C_\alpha \neq C_1^\alpha$, *but* $C_\alpha \approx C_1^\alpha \quad (\alpha = 2, 3, \ldots)$.

- $(C,\alpha) \searrow$ as $\alpha \searrow$

9. *Matrix Convergence Methods*

Defn. "$A = [a_{ij}]_{i,j=0}^{\infty}$ is a *regular matrix*"

means

$$\lim_n s_n = s \quad \Rightarrow \quad \lim_n \sum_j a_{n,j}\, s_j = s$$

Classical Thm. A regular iff

(i) $\lim_n a_{n,j} = 0 \quad \forall\, j$ (col's $\to$ 0)

(ii) $\sup_n \sum_j |a_{n,j}| < \infty$ ("absol. row sums" bndd)

(iii) $\lim_n \sum_j a_{n,j} = 1$ (row sums $\to$ 1)

Defn. Call A V-regular if, in addition,

(iv) $\lim_n \sum_j |a_{n,j} - a_{n,j+1}| = 0$ (row varn's $\to$ 0)

Rmk. C_α *is V-regular* $\forall\, \alpha > 0$.

10. *A-(asymp) Toeplitzness*

Defn. "T is A-*Toeplitz*" means

$$\left(\sum_j a_{n,j}\ S^{*n}\ T\ S^n\right)_{n=0}^{\infty}$$

strongly convergent on H^2.

Thm 4. A V-regular $\Rightarrow$ every C_φ A-Toeplitz

Proof outline.

- $S^{*n}\ C_\varphi\ S^n \quad = \quad T_{(\overline{z}\varphi)^n}\ C_\varphi \quad = \quad T_{\psi^n}\ C_\varphi$

- $\sum_j a_{n,j}\ S^{*n} T S^n \ = \ T_{\Psi_n} C_\varphi$ where $\Psi_n = \sum_j a_{n,j}\ \psi^j$

11. *A-(asymp) Toeplitzness* (cont'd)

Thm 4. A V-reg. $\Rightarrow$ each C_φ is A-Toep.

Proof outline (so far).

- $S^{*n}\, C_\varphi\, S^n \quad = \quad T_{(\bar{z}\varphi)^n}\, C_\varphi \quad = \quad T_{\psi^n}\, C_\varphi$
- $\sum_j a_{n,j}\, S^{*n} T S^n \;=\; T_{\Psi_n} C_\varphi, \qquad \Psi_n = \sum_j a_{n,j}\, \psi^j$

Therefore for $f \in H^2$:

- $\|\sum_j a_{n,j}\, S^{*n} T S^n f\| = \|T_{\Psi_n} C_\varphi f\| \le \|\Psi_n C_\varphi f\|$

So *enough to show:*

- $\|\Psi_n f\| \to 0 \quad \forall\; f \in H^2$
- $\Psi_n \;=\; \dfrac{1}{(1-\psi)} \left(\sum_j (a_{n,j} - a_{n,j-1})\; \psi^n + a_{n,0} \right)$

12. *A-(asymp) Toeplitzness* (cont'd)

Thm 4. A V-reg. $\Rightarrow$ every C_φ is A-Toep.

Enough to show ($z \in \partial\mathbb{U}$):

- $\|\Psi_n f\| \to 0 \quad \forall\ f \in H^2$,

  $$\Psi_n := \sum_j a_{n,j}\ \psi^j \ = \ \frac{1}{(1-\psi)} \left(\sum_j (a_{n,j} - a_{n,j-1})\ \psi^n \ + \ a_{n,0} \right)$$

  $\psi(z) := \ \bar{z}\varphi(z)$

- $|\Psi_n| \ = \ |\sum_j a_{n,j}\ \psi^j| \le \sum_j |a_{n,j}| \le M$

- $|\Psi_n| \ \le \ \dfrac{1}{|1-\psi|} \left(\sum_j (|a_{n,j} - a_{n,j-1}|)\ \psi^n + |a_{n,0}| \right)$

- $\Psi_n \to 0$ "boundedly a.e." on $\partial\mathbb{U}$

- $\|\Psi_n f\|^2 \ = \ \int_{\partial\mathbb{U}} |\Psi_n|^2 |f|^2 \to 0$ $\square$

13. *Summary* (cont'd)

Have seen:

- For $\varphi(0) = 0$:
  C_φ asymp. Toep. $\iff$ $m\{|\varphi| = 1\} = 0$.
- For $\varphi(0) \neq 0$:

$\Leftarrow$ true
$\Rightarrow$ false

- C_φ A-asymp Toep. $\forall$ V-reg. matrix A

Cor. Every comp. op. matrix has A-convergent diagonals.

Qn. 1. Does every comp. op. matrix have *convergent* diagonals?

Equiv: Is every comp. operator on H^2 "weakly asymptotically Toeplitz?"

Qn. 2. Which comp. ops. on H^2 are asymptotically Toeplitz?

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