Composition Operators
&
Classical Function Theory

Joel H. Shapiro

Portland State University

CMFT 2013

Composition operators on H^2

Setting:

- $\mathbb{U} = \{|z| < 1\} \subset \mathbb{C}$,
- $\varphi : \mathbb{U} \to \mathbb{U}$ holomorphic

Composition operators $\mathrm{H}(\mathbb{U}) \to \mathrm{H}(\mathbb{U})$:

$$C_\varphi f = f \circ \varphi$$

The Hardy Space H^2:

All $f \in \mathrm{H}(\mathbb{U})$ with $\|f\|^2 := \sum_{n=0}^{\infty} |\hat{f}(n)|^2 < \infty$

Littlewood's Theorem (1920's):

$$C_\varphi : H^2 \to H^2, \text{ bounded lin. op.}$$

Three Natural Questions

Compactness?

“Size” of $C_\varphi(\text{Ball}\, H^2)$ vs. ”Shape” of $\varphi(\mathbb{U})$

Spectra?

$C_\varphi f = \lambda f$ (eigen-vector/-value eqn for C_φ)

$f \circ \varphi = \lambda f$ (Schröder’s eqn, 1871)

Dynamics?

$C_\varphi^n = C_{\varphi_n}$ $(\varphi_n = \overbrace{\varphi \circ \varphi \circ \cdots \circ \varphi}^{n \text{ times}})$

Qn: *Which C_φ are “hypercyclic”* (i.e. have a dense orbit) ?

Eigen-values/-vectors

Schröder's Equation:

$$f \circ \varphi = \lambda f \quad \text{i.e.} \quad C_\varphi f = \lambda f \qquad (f \in \mathrm{H}(\mathbb{U}))$$

Koenigs' Theorem (1884). For φ not a rotation:

$$\varphi(0) = 0,\ \varphi'(0) \neq 0 \quad \Longrightarrow$$

- $\exists \sigma \in \mathrm{H}(\mathbb{U})$ with $C_\varphi \sigma = \varphi'(0)\sigma$.
- $C_\varphi(\sigma^n) = \varphi'(0)^n \sigma^n \quad (n \in \mathbb{N})$.
- Eigenvalue $\varphi'(0)^n$ has multiplicity one.

i.e. *Eigen-pairs for* C_φ *on* $\mathrm{H}(\mathbb{U})$ *are*

$$(\varphi'(0)^n, \sigma^n), \qquad n = 0, 1, 2, \ \ldots$$

The Principal Eigenfunction σ of C_φ

Koenigs: $\varphi(0) = 0$ & $\varphi'(0) \neq 0$

$\Longrightarrow$

- $\exists \sigma \in \mathrm{H}(\mathbb{U})$ with $C_\varphi \sigma = \varphi'(0)\sigma$.
- $C_\varphi(\sigma^n) = \varphi'(0)^n \sigma^n \quad (n = 0, 1, \ldots)$.
- Eigen-pairs for C_φ on $\mathrm{H}(\mathbb{U})$: $\{\varphi'(0)^n, \sigma^n\}_0^\infty$.

Now consider $C_\varphi \colon H^2 \to H^2$

- Matrix of C_φ is Lower-Δ with diagonal $\{\varphi'(0)^n\}_0^\infty$
- $\therefore$ Matrix of C_φ^* is Upper-Δ with diagonal $\{\overline{\varphi'(0)}^n\}_0^\infty$
- $\therefore$ $\{\overline{\varphi'(0)}^n\}_0^\infty \subset$ eigval's (C_φ^*) so $\{\varphi'(0)^n\}_0^\infty \subset$ spectrum (C_φ)

Koenigs & H^2

C_φ on H^2: $\varphi(0) = 0$, $\varphi'(0) \neq 0$

- Matrix of C_φ^* is upper-Δ with diagonal $\{\overline{\varphi'(0)}^n\}_0^\infty$
- $\therefore\ \{\overline{\varphi'(0)}^n\}_0^\infty \subset$ eigenvalues (C_φ^*)
- $\therefore\ \{\varphi'(0)^n\}_0^\infty \subset$ spectrum (C_φ)

If, in addition: *C_φ compact on H^2:*

- $\{\varphi'(0)^n\}_0^\infty \ \subset\ $ eigenvalues (C_φ) (Riesz Theory)
- $\therefore\ \sigma^n \in H^2 \qquad (n = 0, 1, 2, \ldots)$ (Koenigs' Thm. !!).

Summary: *if* $\exists\, p \in \mathbb{U}$ *with:* $\varphi(p) = p$ & $\varphi'(p) \neq 0$, then

$$C_\varphi \text{ compact on } H^2 \implies \sigma \text{ "almost bounded"}$$

The Caughran-Schwartz Theorem (∼1970)

Theorem. *C_φ compact on H^2* $\Rightarrow$ $\exists\,(!)\, p \in \mathbb{U}$ *with* $\varphi(p) = p$.

Corollary. *C_φ compact on H^2 & $\varphi'(p) \neq 0$* $\Longrightarrow$

- $\sigma^n \in H^2$ for all $n \in \mathbb{N}$, i.e. σ "almost bounded."
- spectrum$\,(C_\varphi) = \{\varphi'(p)^n\}_0^\infty \cup \{0\}$.

Proof of CS Thm—uses the remarkable theorems of:

Denjoy-Wolff & Julia-Carathéodory (1920's–30's).

The Corollary—raises the question:

For which φ is C_φ compact?

Which C_φ are compact on H^2 ?

Easy necessary Cond'n:

C_φ compact $\implies$ $|\varphi| < 1$ a.e. on $\partial\mathbb{U}$.

Examples

- $\varphi(z) \equiv z \implies C_\varphi = I$. Not compact.
- $\|\varphi\|_\infty < 1 \implies C_\varphi$ is compact
- $\varphi(\mathbb{U}) \subset$ inscribed polygon $\implies C_\varphi$ is compact.
- $\varphi(z) = (1+z)/2$; C_φ <u>not</u> compact.

General "Principle"

C_φ compact iff "$\varphi(z)$ not to close to $\partial\mathbb{U}$ too often."

Which C_φ are compact on H^2 ?

Nevanlinna Counting Function (1925). For $w \in \mathbb{U}$:

$$N_\varphi(w) := \sum_{z\in\varphi^{-1}\{w\}} \log\frac{1}{|z|} \qquad \left(\approx \sum_{z\in\varphi^{-1}\{w\}} (1-|z|)\right)$$

Theorem (Littlewood—1920's):

$$N_\varphi(w) = \mathrm{O}\left(\log\frac{1}{|w|}\right) \quad \text{as} \quad |\mathrm{w}| \to 1- .$$

Theorem (JHS 1987):

$$C_\varphi \text{ compact } \iff N_\varphi(w) = \mathrm{o}\left(\log\frac{1}{|w|}\right) \quad \text{as} \quad |\mathrm{w}| \to 1-$$

Which C_φ are compact on H^2?

Theorem. C_φ compact iff

$$\sum_{z\in\varphi^{-1}\{w\}} (1-|z|) \;=\; o\,(1-|w|) \quad \text{as} \quad |\mathrm{w}| \to 1- \quad .$$

For *φ univalent:* set $w = \varphi(z)$ above to get

$$C_\varphi \text{ compact} \iff \lim_{|z|\to 1-} \frac{1-|z|}{1-|\varphi(z)|} \;=\; 0$$

$$\iff \lim_{|z|\to 1-} \frac{1-|\varphi(z)|}{1-|z|} \;=\; \infty$$

The Angular Derivative

Last result. *For φ univalent:*

$$C_\varphi \text{ compact on } H^2 \iff \lim_{|z|\to 1-} \frac{1-|\varphi(z)|}{1-|z|} = \infty$$

Julia-Carathéodory Theorem.

$$\zeta \in \partial\mathbb{U} \;\; \& \quad \liminf_{z\to\zeta} \frac{1-|\varphi(z)|}{1-|z|} < \infty \quad \Longrightarrow$$

- $\displaystyle\lim_{z\to\zeta\angle} \varphi(z)$ exists, $:= \varphi(\zeta) \in \partial\mathbb{U}$
- $\displaystyle\lim_{z\to\zeta\angle} \frac{\varphi(\zeta)-\varphi(z)}{\zeta - z}$ exists, $:= \varphi'(\zeta)$ ("$\angle$ deriv. of φ at ζ")

Corollary. *For φ univalent: C_φ compact on H^2*

$$\iff$$

φ has $\angle$ derivative at no point of $\partial\mathbb{U}$.

The Four Examples—again (φ univalent)

- $\varphi(z) \equiv z \implies C_\varphi = I$. Not compact:

  $<$-deriv. exists at ALL points of $\partial\mathbb{U}$.

- $\|\varphi\|_\infty < 1 \implies C_\varphi$ is compact:

  $<$-deriv. exists at NO point of $\partial\mathbb{U}$.

- $\varphi(z) \subset$ inscribed polygon $\implies C_\varphi$ is compact:

  $<$-deriv. at NO point of $\partial\mathbb{U}$.

- $\varphi(z) = (1+z)/2$; C_φ not compact:

  $<$ derivative exists at $z = 1$.

Proof of the Caughran-Schwartz Theorem

The Theorem

$$C_\varphi \textit{ compact} \implies \varphi \textit{ has a fixed point in } \mathbb{U}.$$

The Denjoy-Wolff Theorem (1920's)

φ has no fixed pt. in $\mathbb{U} \implies \exists! \omega \in \partial\mathbb{U}$ such that $\varphi_n \to \omega$.

Moreover:

- $\lim\limits_{z \to \omega \angle} \varphi(z) = \omega$, and
- $\angle$ derivative of φ exists at ω.

Previously: (for φ univalent)

φ *has* $\angle$ *deriv. at some pt.* $\in \partial\mathbb{U} \implies C_\varphi$ *not compact.*

Conclude: φ *has no FP in* $\mathbb{U} \implies C_\varphi$ *not compact.* $\square$

Some spectra

φ	Spectrum of C_φ
C_φ compact	$\{\text{Sequence} \to 0\} \cup \{0\}$
$\varphi(z) = 1/(2-z)$	Closed interval: $[0,1]$
φ a parabolic non-auto	Spiral from 1 to 0
φ a parabolic auto	Unit Circle
$\varphi(z) = (1+z)/2$	Closed disc: $\{\lvert\lambda\rvert \le \sqrt{2}\}$
φ a hyperbolic auto	Annulus: $\{\lvert\varphi'(\omega)\rvert \le \lvert\lambda\rvert \le \lvert\varphi'(\omega)\rvert^{-1}\}$

Dynamics of C_φ on H^2

Necessary for C_φ hypercyclic:

- φ has no fixed point in $\mathbb{U}$
- φ is 1–1 on $\mathbb{U}$

Theorem (w/ Paul Bourdon, 1990's):

For $\varphi \in \mathrm{LFT}(\mathbb{U})$ with no fixed point in $\mathbb{U}$:

C_φ hypercyclic unless φ parabolic, non-automorphic.

Examples:

- $\varphi(z) = \dfrac{2z-1}{2-z}$: C_φ is top. trans.
- $\varphi(z) = \dfrac{1}{2-z}$: C_φ not top. trans.

Linear Fractional Models

Boundary conformality at the DW pt.

$$\omega = \omega(\varphi) \;\in\; \partial\mathbb{U} \;\Longrightarrow\; \lim_{z\to\omega\angle} \frac{\varphi(z)-\omega}{z-\omega} \;:=\; \varphi'(\omega) \;\in (0,1]$$

The "LF-Model Theorem" (1880's—1980's):

For $\varphi : \mathbb{U} \to \mathbb{U}$ holomorphic, not elliptic:

- $\exists\,\psi \in \mathrm{LFT}(\mathbb{U})$ *&* $\sigma \in \mathrm{H}(\mathbb{U})$ *such that* $\sigma \circ \varphi = \psi \circ \sigma$
- φ *univalent* $\Longrightarrow$ σ *univalent.*

For $\omega \in \partial\mathbb{U}$:

- $\varphi'(\omega) < 1 \Longrightarrow \psi$ *hyperbolic*
- $\varphi'(\omega) = 1 \Longrightarrow \psi$ *parabolic*

LF Models & Hypercyclic C_φ

“Regular” maps

Assume φ univalent with $\omega = DW$ pt. on $\partial\mathbb{U}$, ... and

- φ *continuous on* $\overline{\mathbb{U}}$
- $\varphi(\overline{\mathbb{U}}) \subset \mathbb{U} \cup \{\omega\}$
- φ *is* $\in C^{(4)}$ *in nbd of* ω

Theorem (Bourdon & JHS, 1990's) For φ “regular”:

C_φ is hypercyclic unless φ of parabolic non-auto. type.

Selected References

1. J. H. Shapiro, *The essential norm of a composition operator,* Annals of Math. 125 (1987) 375-404.

2. P. Lefevre, D. Li, H. Queffelec, L. Rodriguez-Piazza, *Series of papers on compactness of composition operators*: 2004-12

3. Carol Kitai, *Invariant Closed Sets for Linear Operators,* Thesis, Univ. of Toronto 1982.

4. P. S. Bourdon and J. H. Shapiro, *Cyclic Phenomena for Comp. Ops.,* Memoirs AMS #596, Vol. 125, 1997, pp.1–105.

5. Frédéric Bayart and Étienne Materon, *Dynamics of Linear Ops.,* Cambridge Univ. Press 2009.

6. A. Peris and K-G. Grosse-Erdmann, *Linear Chaos,* Springer 2011.