Strongly compact algebras and composition operators

Joel H. Shapiro

Portland State University

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Lomonosov, 1980:

Strongly compact for an algebra of operators means

All bounded subsets strongly precompact.

Which algebras?

$\mathrm{alg}(T)$ = the *algebra generated* by operator T and the identity.

$\mathrm{com}(T)$ = the *commutant* of T.

Examples:

$\mathrm{alg}(I) = \mathbb{C}I$:	strongly compact.
$\mathrm{com}(I)$ = all operators:	not strongly compact.
$\mathscr{L}$(finite dim'l Hilbert space):	strongly compact.

Selected background

(1980) Lomonosov: Introduced "SC"
to study Invariant Subspace Problem.

(1990) Marsalli: Independently “discovered” SC.
Spectral suff. conditions for SC.
Characterized SC for self-adjoint algebras.

(2006) Lacruz, Lomonosov, Rodríguez-Piazza:
Examples, constructions, counterexamples,
normal operators, weighted shifts.

(2011) Fernández-Valles and Lacruz: Weighted shifts,
Cesàro operator, *composition operators.*

Composition Operators

φ holomorphic on $\mathbb{U}$, $\varphi(\mathbb{U}) \subset \mathbb{U}$

$$C_\varphi f = f \circ \varphi \qquad (f \in H(\mathbb{U}))$$

$$C_\varphi \colon H(\mathbb{U}) \to H(\mathbb{U}) \quad \text{linear transf.}$$

Littlewood Subord'n Thm. (1920's)

$$C_\varphi \colon H^2 \to H^2 \quad \text{(bounded) linear operator}$$

General Question

For which φ *is* $\mathrm{alg}(C_\varphi)$, $\mathrm{com}(C_\varphi)$ *strongly compact?*

For today: $\varphi \in \mathrm{LFT}(\mathbb{U})$

$\varphi \in \mathrm{LFT}(\mathbb{U})$ with a fixed point on $\partial\mathbb{U}$

Type of φ	Fixed pt. position	$\mathrm{alg}(C_\varphi)$	$\mathrm{com}(C_\varphi)$	Example: $\varphi(z) =$
PA	$\partial\mathbb{U}$ only	SC	not SC	$\frac{(1+i)z-1}{z+(-1+i)}$
PNA	$\partial\mathbb{U}$ only	SC	SC	$\frac{1}{2-z}$
HA	$\partial\mathbb{U}$ only	SC	not SC	$\frac{1+2z}{2+z}$
HNA	$\partial\mathbb{U}$ & $\mathbb{U}$	not SC	not SC	$\frac{z}{2-z}$
	$\partial\mathbb{U}$ & $\mathbb{U}_e$	SC	???	$\frac{1+z}{2}$

(Shaded results due to Lacruz and Fernández-Valles 2011)

Marsalli 1990: Sufficient conditions for SC

Equivalent Defn :

$\mathscr{A}_1 x$ is rel. compact in $H \quad \forall\, x \in H$ (or a dense subset of H).

Consequence—Useful Sufficient Condition:

$\exists$ family of finite dimensional $\mathscr{A}$-invariant subspaces that have dense linear span.

Corollaries:

Suff. for $\mathrm{alg}(T)$ SC:

$\exists$ *densely spanning family of eigenvectors.*

Suff. for $\mathrm{com}(T)$ SC:

$\exists$ *densely spanning family of finite dim'l eigenspaces.*

Application: φ elliptic

WLOG: $\varphi(z) = \omega z$, $|\omega| = 1$

- ω arbitrary

  Eigenvectors, Eigenvalues: z^n, ω^n $(n = 0, 1, 2, \ldots)$

  $\Rightarrow$ $\mathrm{alg}(C_\varphi)$ strongly compact.

- ω *not* a root of unity:

  All eigenvalues have multiplicity 1

  $\Rightarrow$ $\mathrm{com}(C_\varphi)$ strongly compact

- ω *is* a root of unity: $\omega^N = 1$

  $C_\varphi = I \oplus \omega I \oplus \cdots \oplus \omega^{N-1} I$

  & $\mathrm{com}(C_\varphi) \supset \mathscr{L}(H_0) \oplus \mathscr{L}(H_1) \oplus \cdots \oplus \mathscr{L}(H_{N-1})$

  $\Rightarrow$ $\mathrm{com}(C_\varphi)$ *not* strongly compact.

$\exists$ fixed point on $\partial\mathbb{U}$

Type of φ	Fixed pt. position	alg(C_φ)	com(C_φ)	Eigenvec, eigenval	Eigsp. dim'n	Rmks
PA	∂U only	SC	not SC	$e^{\lambda\frac{z+1}{z-1}}$, $e^{-\lambda a}$	∞	$\lambda \geq 0$ $\mathrm{Re}\, a = 0$
PNA	∂U only	SC	SC	$e^{\lambda\frac{z+1}{z-1}}$, $e^{-\lambda a}$	1	$\lambda \geq 0$ $\mathrm{Re}\, a > 0$
HA	∂U only	SC	not SC	$(\frac{1+z}{1-z})^\lambda$, a^λ	∞	$\lvert\mathrm{Re}\,\lambda\rvert < \frac{1}{2}$ $0 < a < 1$
HNA	∂U & U	not SC	not SC	*********	**	********
	∂U & U_e	SC	???	$(1-z)^\lambda$, $(\frac{1}{2})^\lambda$	∞	$\mathrm{Re}\,\lambda > -\frac{1}{2}$ $\varphi(z) = \frac{1+z}{2}$

Multiplier in the commutant

For $f \in H^\infty$: $\quad f \circ \varphi = f \implies C_\varphi M_f = M_f C_\varphi$

THEOREM. $\mathrm{alg}(M_f)$ strongly compact $\implies$

$\{\zeta \in \partial\mathbb{U} : |f(\zeta)| = \|f\|_\infty\}$ has measure zero.

Type of φ	Fixed pt. position	$\mathrm{alg}(C_\varphi)$	$\mathrm{com}(C_\varphi)$	Eigvec $f(z)$, Eigval	λ s.t. $C_\varphi f = f$	$\mathrm{alg}(M_f)$
PA	∂U only	SC	not SC	$e^{\lambda\frac{z+1}{z-1}}$, $e^{-\lambda a}$	$\lambda = \frac{2\pi}{\alpha}$ $a = i\alpha$ $\alpha > 0$	not SC
HA	∂U only	SC	not SC	$(\frac{1+z}{1-z})^\lambda$, a^λ	$\lambda = \frac{2\pi i}{\log a}$	not SC
$\frac{1+z}{2}$	∂U & U_e	SC	???	$(1-z)^\lambda$, $(\frac{1}{2})^\lambda$	$\lambda = \frac{2\pi i}{\log 2}$	*is* SC

Review

Type of φ	Fixed pt. position	$\mathrm{alg}(C_\varphi)$	$\mathrm{com}(C_\varphi)$	Example: $\varphi(z) =$
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PNA	$\partial\mathbb{U}$ only	SC	SC	$\dfrac{1}{2-z}$
HA	$\partial\mathbb{U}$ only	SC	not SC	$\dfrac{1+2z}{2+z}$
HNA	$\partial\mathbb{U}$ & $\mathbb{U}$	not SC	not SC	$\dfrac{z}{2-z}$
	$\partial\mathbb{U}$ & $\mathbb{U}_e$	SC	???	$\dfrac{1+z}{2}$

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HNA	∂U & U	not SC	not SC	**********	**	********

Suppose 0 and 1 fixed:

$$\therefore \quad \varphi(z) = \frac{s\,z}{1-(1-s)z} \qquad \left(= \tfrac{z}{2-z} \text{ if } s = \tfrac{1}{2}\right).$$

$$\therefore \quad C_\varphi = I_0 \oplus (C_\varphi|_{H_0^2}) \approx I_0 \oplus \underbrace{s\,C^*_{sz+(1-s)}}_{\text{Cowen's Adjoint Th.}}$$

THM. $\mathrm{alg}(C_\psi^*)$ SC $\iff$ ψ fixes a point of $\mathbb{U}$.

" $\therefore$ " $\mathrm{alg}(C_\varphi)$ not SC.

Some References

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