A Formalism for Quantum Games and an Application

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Game Theory

Given

- a set of players {*1, 2, …, n*},
- a set S_i of *pure strategies* for each player
- a set Ω_i of possible *outcomes* or *payoffs* for each player
- a *game G* is a function

$$G : S_1 \times S_2 \times \cdots \times S_n \rightarrow \Omega_1 \times \Omega_2 \times \cdots \times \Omega_n$$

which assigns to each n-tuple of strategic choices an outcome to each player.

- Without loss of generality, the Ω_i's are all copies of **R**.

- A *strategy profile* is an n-tuple $(s_1, s_2, \ldots, s_n)$ where each $s_i \in S_i$ is a pure strategy chosen by player *i*.

When playing a game

Rational players seek to identify

- a strategy which guarantees them an outcome having maximal utility
- a *security strategy* : a strategy that guarantees a specific lower bound to the utility received
- for a fixed *n-1* tuple of opponents' strategies, a *best reply*: a choice $s_i \in S_i$ of strategy that delivers a utility at least as great of any other strategic choice.

In symbols, s_i is a best reply to the $n-1$ tuple

$$(s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$$

of opponents' strategies if and only if

$$G(s_1, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_n) \geq G(s_1, \ldots, s_{i-1}, s, s, \ldots, s_n)$$

for all $s \in S_i$.

Nash Equilibrium

A *Nash equilibrium* (also called a *solution* or just an *equilibrium*) is a strategy profile $(s_1, s_2, \ldots, s_n)$ such that each s_i is a best reply to the n-1 tuple of opponents' strategies.

The identification of Nash equilibria is a fundamental goal in the theory of games.

The Prisoner's Dilemma

The payoff matrix for this 2x2 game is

	t_1	t_2
s_1	$(3,3)$	$(0,5)$
s_2	$(5,0)$	$(1,1)$

$S_1 = \{s_1, s_2\}$; $S_2 = \{t_1, t_2\}$; $\Omega_1 = \Omega_2 = \mathbf{R}$

G: $\{s_1, s_2\} \times \{t_1, t_2\} \rightarrow \mathbf{R}^2$

For example: $G(s_1, t_2) = (0,5)$

	t_1	t_2
s_1	$(3,3)$	$(0,5)$
s_2	$(5,0)$	$(1,1)$

- Note that $G(s_2, t) \geq G(s_1, t)$ for all $t \in S_2$.
- Similarly, $G(s, t_2) \geq G(s, t_1)$ for all $s \in S_1$.
- Therefore, s_2 is a best reply to t_2 and t_2 is a best reply to s_2.
- Hence, (s_2, t_2) is the unique pure strategy Nash equilibrium of G.

A Caution

A game need not have a pure strategy equilibrium.
An example is the game known as Simplified Poker, whose payoffs are:

	t_1	t_2
s_1	$(5/4, -5/4)$	$(0, 0)$
s_2	$(0, 0)$	$(5/2, -5/2)$

The standard way to proceed is to extend G to a larger game G^{mix}.

Mixing Strategies

- Each player plays each of his or her pure strategies with a specific probability, determined by a probability distribution over his or her pure strategies, a so-called *mixed strategy*.
- Given a profile of such strategies we form the product distribution over the set of pure strategy profiles.
- Applying G , we obtain a probability distribution over ImG.
- Applying the expectation operator ε to this probability distribution, we then obtain the expected outcome for this strategic profile.
- Assigning the expected outcome to each mixed stategy profile gives G^{mix}.

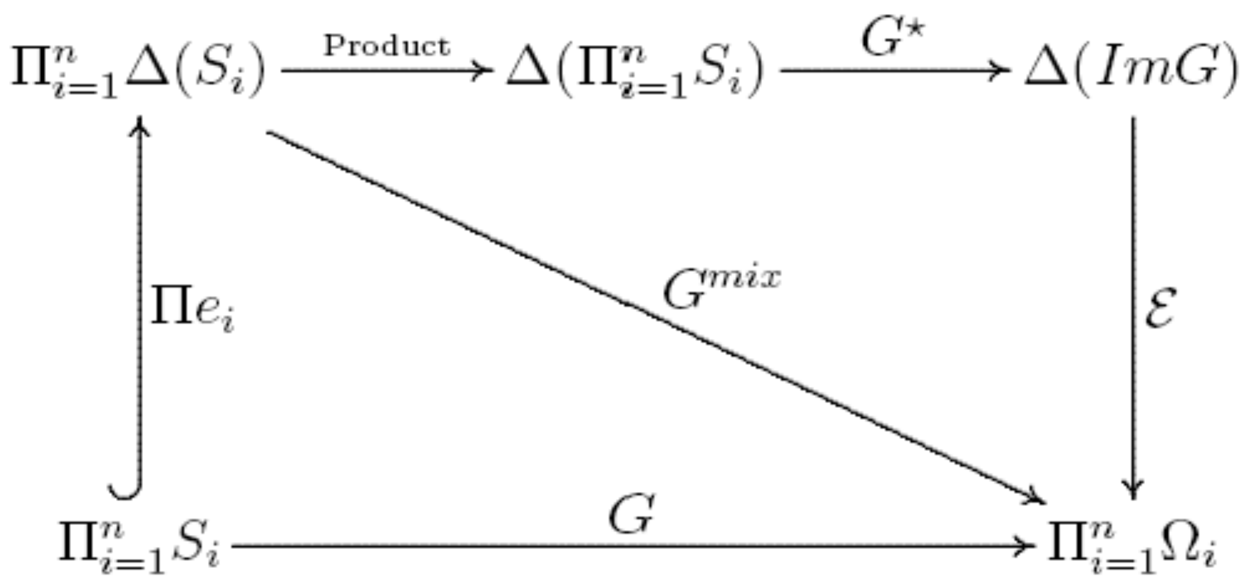
- Denote the set of probability distributions over S_i by $\Delta(S_i)$.

- The space S_i embeds in $\Delta(S_i)$ by considering each element of S_i as given by the probability distribution which assigns 1 to that element and 0 to the others.

- When S_i is finite, $\Delta(S_i)$ is just the set of real convex linear combinations of the elements of S_i.

Extending G to a new game G^{mix}

The extension of G by G^{mix} is indicated in the following commutative diagram



G^{mix} for Simplified Poker

- $\Delta(S_1) = \{ ps_1 + (1-p)s_2 \mid 0 \le p \le 1\} \equiv [0, 1]$.
- $\Delta(S_2) = \{ (1-q)t_1 + qt_2 \mid 0 \le q \le 1\} \equiv [0, 1]$
- $G^{mix}(p,q)=(5p/4 + 5q/2 – 15pq/4, -5p/4 - 5q/2 + 15pq/4)$
- $G^{mix}(2/3,1/3)=(5/6, -5/6)$
- A direct calculation shows that (2/3, 1/3) is the unique mixed strategy equilibrium of G^{mix}.

Nash's Theorem (1957)

- For a finite game *G*, there is always a Nash equilibrium in G^{mix}.

More extensions of G

- One begins by observing that the function

$$\prod_{i=1}^{n} \Delta(S_i) \rightarrow \Delta(\operatorname{Im} G)$$

is not onto. If player I plays his first strategy with probability *p*, and player II plays her second strategy with probability *q*, the resulting probability distribution over the Im*G* is given by

	t_1	t_2
s_1	$p(1-q)$	pq
s_2	$(1-p)(1-q)$	$(1-p)q$

- An easy exercise now shows that the element of $\Delta(\mathrm{Im}\, G)$ represented by

	t_1	t_2
s_1	1/2	0
s_2	0	1/2

is not realizable by any choice of *p* and *q.*

- Mediated classical communication addresses this issue.

Mediated Classical Communication

- Suppose a referee can be hired for a negligible cost.
- For a given *d* in $\Delta(\operatorname{Im} G)$ the referee enforces d by performing a random act with probability distribution *d*. This selects an outcome of *G*.
- Referee now communicates to the players their and only their strategic choice that produces this outcome.

- Note that the players are no longer playing the game *G* but in fact a larger game, described here for 2 x 2 games, but easily generalized to games with more players and strategies.

G_d^{com}

- Suppose the players' strategy spaces for *G* are represented by the set {A, B}.
- Strategy spaces for G_d^{com} are then represented by {A', B', C', D'}, where C' is always co-operate with referee, D' is always deviate, A' is play A always, and B' is play B always.

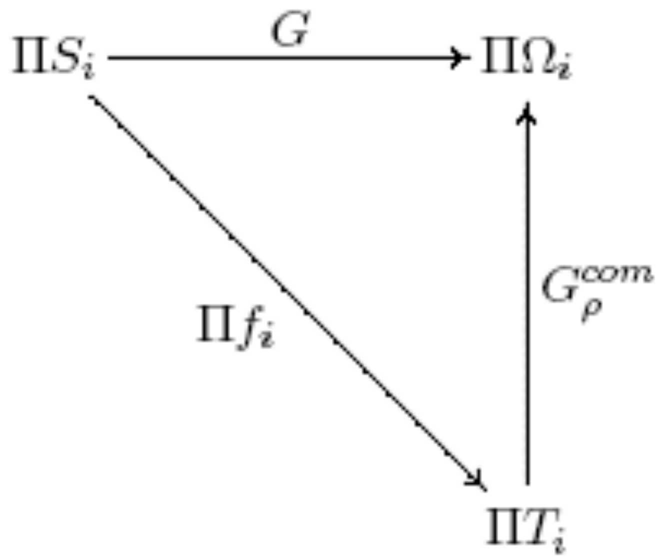
Note:

- If both players choose to play C' the outcome of G_d^{com} is exactly the expected outcome under *d*.

- The game G_d^{com} extends *G* as there is an embedding f_i for each player from {A, B} to T={A', B', C', D'} with f_i (A)=A' and f_i (B)= B' and such that

$$G = G_d^{com} \circ \prod_{i=1}^{2} f_i$$

Extending *G*



- Classical mediated communication thus gives a *family* of extensions of *G* indexed by $\Delta(\mathrm{Im}\, G)$.

Correlated Equilibrium

- Following Aumann, a *correlated equilibrium* for *G* occurs whenever (C′,C′) is a Nash equilibrium in G_d^{com} .
- This says the agreement to follow the referees recommendation is *self policing*, which means that no player has an incentive to deviate from the referees recommendation.
- The use of correlated equilibria may or may not improve the lot of the players.

Chicken

- A classic example of correlated equilibrium improving the players' lot is given by the following variant of Chicken:

	t_1	t_2
s_1	$(2, 2)$	$(0, 3)$
s_2	$(3, 0)$	$(-1, -1)$

	t_1	t_2
s_1	$(2, 2)$	$(0, 3)$
s_2	$(3, 0)$	$(-1, -1)$

- This game has two pure strategy equilibria and one mixed strategy equilibrium where each player plays their individual strategies with equal probabilities.
- The mixed strategy equilibrium pays out 1 to each player.
- Even without a referee any real convex linear combination of these three outcomes forms a self policing agreement between the players.

	t_1	t_2
s_1	$(2,2)$	$(0,3)$
s_2	$(3,0)$	$(-1,-1)$

- For example, players could jointly observe a fair coin and agree to play (s_1,t_2) if heads and (s_2,t_1) if tails. Outcome here is (1.5, 1.5) > (1,1).

- Even better and outside these linear combinations is the correlated equilibrium arising from $\frac{1}{3}(2,2)+\frac{1}{3}(0,3)+\frac{1}{3}(3,0)$ which has pay-out $\left(\frac{5}{3},\frac{5}{3}\right)$.

Prisoner's Dilemma and Simplified Poker

- In these games mediated communication does not improve the lot of the players.
- In Prisoner's Dilemma, because of the strong domination, players always have an incentive to deviate if *d* assigns a non-zero probability to any outcome other than (1,1).
- For Simplified Poker any deviation from the equilibrium strategies (2/3, 1/3) is fully exploitable by the other player and hence an incentive to deviate from any other potential correlated equilibrium strategy.

Quantization

- Of particular interest to us are extensions that use the generalized notion of probability distribution given by quantum superposition.
- For finite sets, a quantum superposition is given by a complex projective linear combination as opposed to a probability distribution which is given by a real convex linear combination.
- We will denote the set of quantum superpositions over a set *X* by *QS(X)*.

- For example, if *X* = {*x, y*} then a complex *projective* linear combination has form

$$\left\{ x\alpha + y\beta \mid (\alpha,\beta) \in \mathbb{C}^2 \setminus \{(0,0)\} \right\}$$

where *(xα + yβ) λ* ≡ *xα + yβ* for any nonzero complex number λ, called a *phase*.

- These become quantum superpositions once we identify *x* and *y* with an orthogonal basis of a quantum system.

Quantum measurement

- When the underlying complex vector space is a Hilbert space, we can assign a real length | | to each complex linear combination of basis vectors and assign to each component the ratio of the square of its coefficient to the square of the length of the combination.
- This allows us to obtain a real convex linear combination of our vectors via projection onto a basis. This is called *quantum measurement with respect to this basis* and denoted q_{meas} .
- For example, the real convex linear combination produced from $x\alpha + y\beta$ is

$$q_{meas}(x\alpha + y\beta) = \frac{|x|^2}{|x|^2+|y|^2}\alpha + \frac{\qquad}{\qquad}$$

Game Quantization

For a finite game G, we extend G to a new game G^Q as follows.

Given

- A collection of nonempty sets $Q_1, Q_2, \ldots, Q_n$
- A *protocol*, that is, a function $Q : \prod_{i=1}^{n} Q_i \rightarrow QS(\operatorname{Im} G)$
- Applying q_{meas} then gives a probability distribution over ImG.
- We can now form a new game G^Q by applying the expectation operator ε.

Proper and Complete Quantizations

- The Q_i's are the pure quantum strategy sets.

- If there exist embeddings

$$e_i' : S_i \to Q_i \quad with \quad G^Q \circ \Pi e_i' = G$$

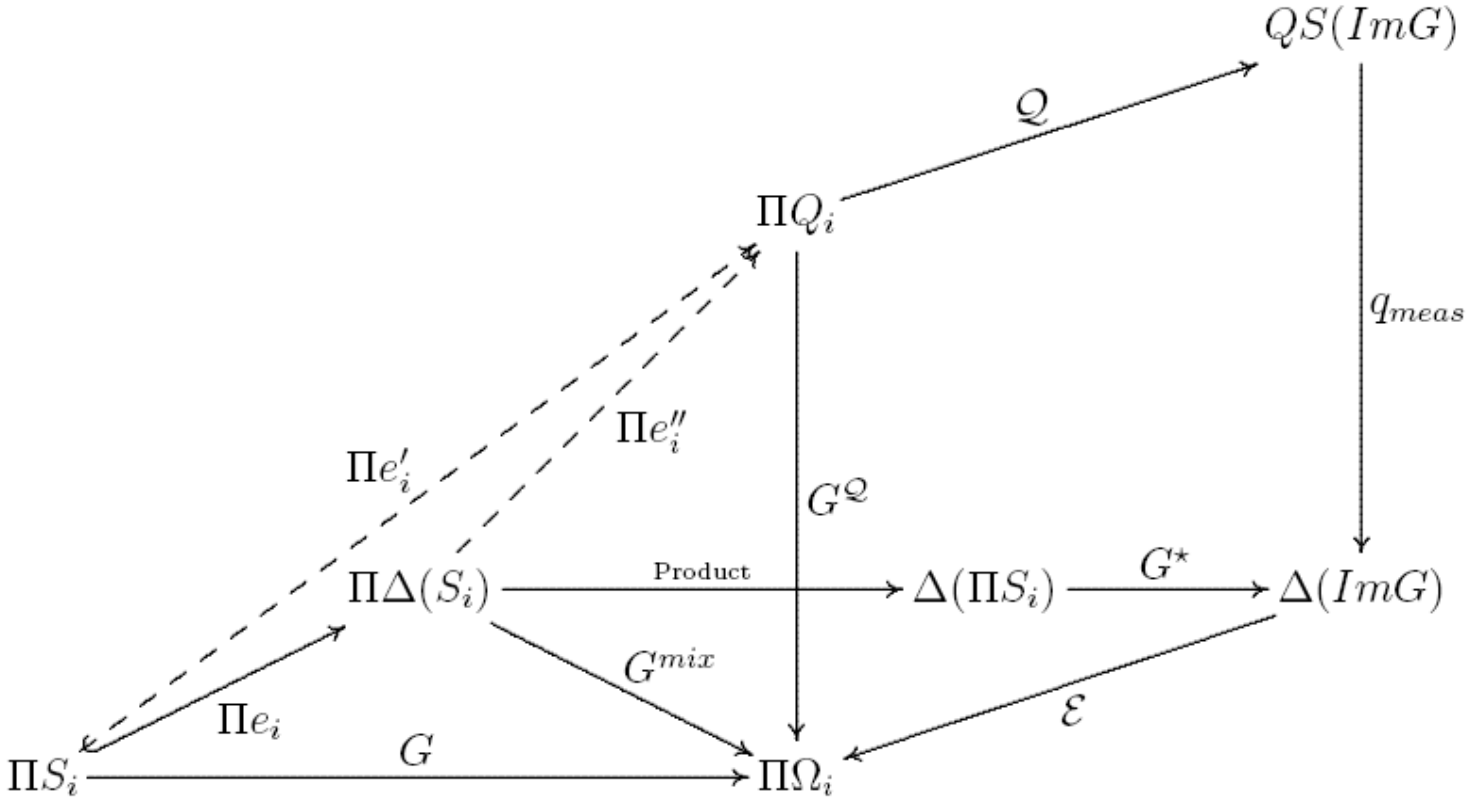
then G^Q is said to be a *proper* quantization of *G*.

- If there exist embeddings

$$e_i'' : \Delta(S_i) \to Q_i \quad with \quad G^Q \circ \Pi e_i'' \circ \Pi e_i = G$$

then G^Q is said to be a *complete* quantization of *G.*

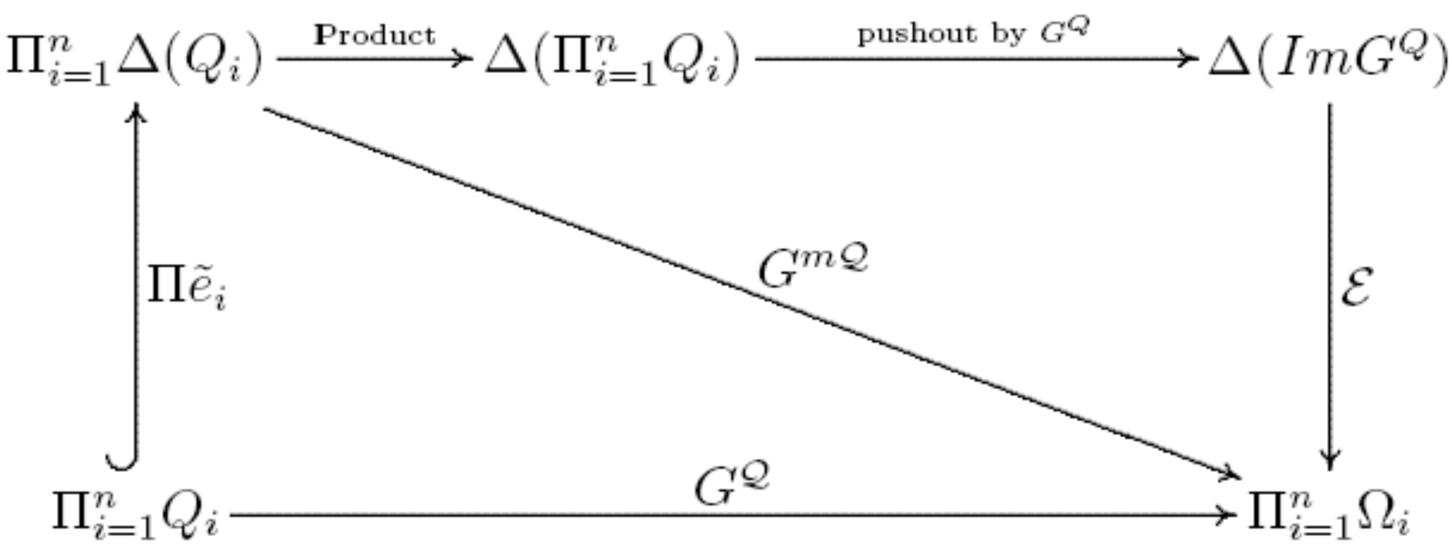
In pictures:



- A complete quantization is a proper quantization.
- Finding a proper quantization of a game *G* is just the usual mathematical problem of extending a function.

Mixed quantizations

Nothing prevents us from having G^Q play the role of G in the classical mixtures. This gives the *mixed quantization of G by the protocol Q*. Call this game G^{mQ}.



EWL protocols a.k.a Mediated Quantum Communication

- Eisert, Wilkens, and Lewenstein (EWL) put forward a specific family of protocols Q^{I} for the quantization of 2 player, 2 strategy games.
- The EWL protocols require the game *G* to be played with a referee who communicates with the players through a quantum channel, i.e., players and the referee can send superpositions of the two states |0> and |1> of a classical bit, a so-called *quantum bit* or *qubit* for short.

- Each player is sent a qubit (a two-state quantum system or a unit vector in a projective Hilbert space) by the referee. The qubits sent to the players are in a joint initial state, i.e., a quantum superposition of the joint states.
- For two players, this has form

$$\left|00\right\rangle \qquad \left|\ \right\rangle \qquad \left|\ \right\rangle \qquad \left|\ \right\rangle$$

- Each choice of the quadruple $(\alpha, \beta, \gamma, \delta)$ gives an initial state *I* and this initial state produces in turn a specific protocol Q^{I}.
- Players act on their qubit via elements of *SU(2)* and send back the final product. In particular, the two classical pure strategies available to the players are represented by *N* (*no flip*) and *F (flip)*.
- The referee expresses the final state in a specific basis of *QS(ImG)* and then assigns via q_{meas} the appropriate expected-payoffs to the players.

The Landsburg Classification

- S. Landsburg examines the quantum game arising from the EWL protocol applied to 2 player, 2 strategy games and corresponding to the initial state

$$I = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

- For this initial state the map

$$q_{\text{meas}} \circ Q^{I} : \prod \mathrm{Q}_i \to \Delta\left(\mathrm{Im}\, G\right)$$

  is onto.

- Landsburg determines the potential Nash equilibria in both the games G^{Q} and G^{mQ}. In particular, there never exists pure quantum equilibria; that is, there are no Nash equilibria in the game G^{Q}.

- However, there is always at least one Nash equilibria amongst the mixed quantum strategy profiles.

- In particular, when each player uses the uniform probability distribution over his or her pure quantum strategies, each player is guaranteed to receive at least the average of the four classical payouts.

A Fundamental Question

- Is the Nash equilibria in the quantized game truly new?
- Specifically, is the probability distribution that arises from the mixed quantum equilibrium different from that of a classical correlated equilibrium?

The Answer

- The mixed quantum strategy equilibrium requires the players to use the uniform probability distribution over his or her choice of pure quantum strategy thus producing the uniform distribution over the four classical payoffs.
- For the Prisoner's Dilemma this assigns a non-zero probability other than (1,1) and hence is <u>not</u> a classical correlated equilibrium.
- Note: the mixed quantum equilibrium <u>increases</u> the payoffs to both players.

- An even more amazing result holds for the *0-sum* game of Simplified Poker, where the uniformly mixed quantum equilibrium out performs the classical mixed strategy equilibrium payoffs for player I, yet is still a security strategy for player I against which player II has no recourse.

Conclusion

- Our discussion has shown that for Prisoner's Dilemma and Simplified Poker quantization indeed holds something new for Game Theory.
- Several other controversies in quantum games can be similarly resolved via the formalism presented here and new game theoretic interpretations of certain problems from both quantum computation and quantum logic synthesis can also be eliminated.
- Watch this space for details.

- The preprint for today's presentation is currently available at

  arXiv:0808.1389v1 [quant-ph]

Thanks, and have a great day !