CAUCHY TRANSFORMS AND BEURLING-CARLESON-HAYMAN THIN SETS

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1. INTRODUCTION

Let Δ denote the open unit disc of the complex plane, and T the unit circle. The *Cauchy transform* of a Schwartz distribution μ on T is the function C_μ holomorphic in Δ defined by

$$C_\mu(z) = \sum_{n=0}^{\infty} \hat{\mu}(n) z^n = \langle \mu, k_z \rangle \qquad (z \in \Delta),$$

where k_z is the Cauchy kernel $k_z(\tau) = (1 - \bar{\tau}z)^{-1}$, $(\tau \in T)$ and $\hat{\mu}$ is the Fourier transform of μ [9; Chapter I, sec. 7, Problem 5, pp. 43–44]. In this paper we characterize those closed subsets E of T for which C_μ has bounded characteristic in Δ whenever support $\mu \subset E$.

To get some feeling for this problem, observe that if E is a finite set and support $\mu \subset E$, then C_μ is a linear combination of derivatives of Cauchy kernels, hence simple estimates show that C_μ belongs to the Hardy space H^p for all sufficiently small p. In particular, C_μ has bounded characteristic in Δ. On the other hand, if $E = T$ then there are distributions which are "almost measures" for which C_μ is not of bounded characteristic [8; Chapter X, Prop. 2, p. 110].

Our characterization involves the decomposition of $T \setminus E$ into a countable disjoint union of open sub-arcs (I_n) of T. Letting ϵ_n denote the length of I_n, we can state the main result as follows:

THEOREM. *The following statements about E are equivalent:*

(i) *If μ is a distribution with support contained in E, then C_μ has bounded characteristic in Δ.*

(ii) *E has measure zero and* $\sum \epsilon_n \log \epsilon_n > -\infty$.

The sets E of measure zero for which $\sum \epsilon_n \log \epsilon_n > -\infty$ are usually called *Carleson sets,* and they have an interesting history. They were first introduced by A. Beurling [2], who observed that if a function f is continuous on the closed unit disc, holomorphic in the interior, and satisfies a Lipschitz condition on T, then

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$\{\tau \in T : f(\tau) = 0\}$ is a Carleson set. Later Carleson [3; Theorem 1] showed that every such set E is the boundary zero set of a function f_E continuous in the closed disc, holomorphic in the interior, and n times continuously differentiable on T; where n is any positive integer, but f_E depends on n. More recently Novinger [14] and Taylor and Williams [15] showed that f_E can be taken to be infinitely differentiable on T, Nelson [13] and Korenblum [11], gave shorter proofs of this; and Caveny and Novinger [4] obtained Beurling's result for functions with derivative in H^1.

Simultaneously with Carleson's work, Hayman [7] independently discovered Carleson sets as a means for rescuing the Block-Nevanlinna conjecture [5; Chapter 5, page 92]. He showed that if a function f of bounded characteristic extends over the complement of a Carleson set E to a domain D with certain geometric properties, and if for each $\alpha \in \mathbf{C}$ the equation $f(z) = \alpha$ does not have too many solutions in D away from ∂D, then each derivative of f has bounded characteristic. Recently P. Ahern [1] showed that it is necessary in Hayman's result that E be a Carleson set, thus improving an earlier result of P. B. Kennedy [10].

This paper continues in the direction of Hayman, Kennedy, and Ahern. It is not difficult to check that the Cauchy transform of a distribution μ on T extends holomorphically to $\hat{\mathbf{C}} \setminus$support μ ($\hat{\mathbf{C}}$ = extended complex plane), and no further. Since the derivatives of such Cauchy transforms are themselves Cauchy transforms, our first theorem states that if C_μ extends holomorphically over the complement of a Carleson subset of T, then C_μ and all its derivaties have bounded characteristic in Δ. This result, while similar to Hayman's, does not follow from his, since C_μ is not initially assumed to be of bounded characteristic.

In the other direction our result shows that if E is not a Carleson set, then there is a Cauchy transform C_μ which extends holomorphically to $\hat{\mathbf{C}} \setminus E$, but is not of bounded characteristic. In fact our C_μ will be the derivative of the Cauchy transform of a *measure* supported in E, thus providing yet another class of counterexamples to the Bloch-Nevanlinna conjecture. This part of our result is close in spirit to the work of Ahern [1], who constructs a Blaschke product in Δ which extends holomorphically across E, but has derivative not of bounded characteristic.

The paper is organized as follows: The main result is stated precisely in the next section, and Section 3 contains some preliminary work needed for its proof. In particular, elementary facts about Gaussian random series play a role and these are described in some detail. Finally the main theorem is proved in Section 4.

2. STATEMENT OF MAIN RESULT

From now on let m denote normalized Lebesgue measure on the unit circle T. If f is a complex valued function on the open unit disc Δ, and $0 \leq r < 1$, we define $f_r: T \to \mathbf{C}$ by $f_r(\tau) = f(r\tau)$, $(\tau \in T)$. We will be dealing with the following classical spaces of functions holomorphic in Δ. First there is the *Nevanlinna class* N consisting of functions f of *bounded characteristic,* that is:

$$\sup_{0\le r<1}\int_T \log^+|f_r|\,dm<\infty.$$

An important subclass of N is the *Smirnov Class* N^+, which is the collection of $f \in N$ for which

$$\lim_{r\to 1-}\int_T \log^+|f_r|\,dm=\int_T \log^+|f(\tau)|\,dm(\tau), \tag{2.1}$$

where $f(\tau)=\lim_{r\to 1-} f(r\tau)$, and the limit exists for $[m]$ a.e. $\tau \in T$ by Fatou's radial limit theorem [5; Theorem 2.2, page 17].

The collection of finite complex Borel measures on T will be denoted by $M(T)$, and the Schwartz distributions on T by $\mathscr{D}(T)$. All ideas and notations about these distributions will be as in [9; Chapter I, Sec. 7, pp. 43–44, Problem 5]. In particular:

(a) $\mathscr{D}(T)$ is the dual space of $C^\infty(T)$, with the pairing

$$\langle\mu,\varphi\rangle=\sum_{-\infty}^{\infty}\hat\mu(n)\hat\varphi(-n) \tag{2.2}$$

where $\mu \in \mathscr{D}(T)$ and $\varphi \in C^\infty(T)$. Note that in [9] Katznelson defines the pairing with $\hat\varphi(n)$ instead of $\varphi(-n)$; an inessential difference.

(b) The *support* of $\mu \in \mathscr{D}(T)$ is the complement of the largest open subset U of T for which $\langle\mu,\varphi\rangle = 0$ whenever $\varphi \in C^\infty(T)$ vanishes off U. If $\psi \in C^\infty(T)$ and $\psi \equiv 1$ on support μ, then $\psi\mu = \mu$.

(c) μ has *order n* if μ is a continuous linear functional on $C^{(n)}(T)$. The collection of distributions on T of order n is denoted by $\mathscr{D}^n(T)$ ($n = 0,1,2,\ldots$). In particular, $\mathscr{D}^0(T) = M(T)$, and the pairing (2.2) can be written $\langle\mu,\varphi\rangle = \int_T \varphi\,d\mu$ for $\varphi \in C^\infty(T)$ and $\mu \in \mathscr{D}^0(T)$. It is easy to see that $\mathscr{D}(T)=\bigcup_{n=0}^{\infty}\mathscr{D}^n(T)$.

(d) A simple calculation shows that if $\mu \in \mathscr{D}(T)$, then

$$izC'_\mu(z)=C_{\mu'}(z)\qquad (z\in\Delta) \tag{2.3}$$

where C'_μ is the derivative of C_μ with respect to z, and μ' is the (real variable) distributional derivative of μ, defined by the formula $\hat\mu'(n) = in\,\hat\mu(n)$ for all integers n. Note that support $\mu' \subset$ support μ, so (2.3) asserts that C'_μ is essentially the Cauchy transform of a distribution with support contained in that of μ.

(e) One final bit of notation is convenient. If E is a closed subset of T, and $X \subset \mathscr{D}(T)$, then $X_E = \{\mu \in X : \text{support } \mu \subset E\}$. In particular, $M_E(T)$ denotes all finite complex Borel measures on T with support in E.

We can now state the main result of this paper. In what follows, E is a closed subset of T with $T\setminus E = \cup I_n$ the canonical decomposition of the complement of E into disjoint open intervals.

THEOREM 1. *The following seven statements about E are equivalent:*

(i) $m(E) = 0$ *and* $\sum m(I_n)\log m(I_n) > -\infty$.

(ii) $C_\mu \in N^+$ *for every* $\mu \in \mathcal{D}_E(T)$.

(iii) $C_\mu \in N$ *for every* $\mu \in \mathcal{D}_E(T)$.

(iv) $C_\mu^{(n)} \in N^+$ *for every* $\mu \in M_E(T)$ $\quad (n = 0,1,2,\ldots)$.

(v) $C_\mu^{(n)} \in N$ *for every* $\mu \in M_E(T)$ $\quad (n = 0,1,2,\ldots)$.

(vi) $C'_\mu \in N^+$ *for every* $\mu \in M_E(T)$.

(vii) $C'_\mu \in N$ *for every* $\mu \in M_E(T)$.

We prove this result in Section 4, devoting the third section to the necessary preliminaries. The reader should note that in Theorem 1 the implications (ii) $\Rightarrow$ (iii), (iv) $\Rightarrow$ (v), (iv) $\Rightarrow$ (vi), (v) $\Rightarrow$ (vii), and (vi) $\Rightarrow$ (vii) are completely trivial, while (ii) $\Rightarrow$ (iv) and (iii) $\Rightarrow$ (v) follow immediately from successive applications of (2.3). So in Section 4 we will only have to prove (i) $\Rightarrow$ (ii) and (vii) $\Rightarrow$ (i). The former implication follows from a growth condition on C_μ, while the latter depends on elementary properties of Gaussian random series.

3. PRELIMINARY RESULTS

a. *Growth estimate for Cauchy transforms.* If E is a subset of T and $|z| \le 1$, write $\rho_E(z) = \operatorname{dist}(z,E) = \inf_{\tau\in E} |\tau - z|$. For $\eta > 0$, write

$$E_\eta \doteq \{\tau \in T: \rho_E(\tau) < \eta\} = \bigcup_{\tau\in E} I_\eta(\tau)$$

where $I_\eta(\tau)$ is the open interval of T centered at τ and having arc length 2η. For $\varphi \in C^\infty(T)$, let $\|\varphi\|_\infty = \max\{|\varphi(\tau)| : \tau \in T\}$. The following result is undoubtedly well known, but we have not been able to find a reference, so a proof is included.

LEMMA A. *Suppose E is a closed subset of T and* $\mu \in \mathcal{D}_E^n(T)$. *Then there is a constant* $A_\mu > 0$ *such that*

$$|C_\mu(z)| \le A_\mu \rho_E(z)^{-(n+1)}$$

for each z in Δ.

Proof. It is convenient to regard functions Φ in $C^\infty(T)$ as 2π-periodic functions φ on $\mathbf{R}$, by means of the formula $\Phi(e^{it}) = \varphi(t)$. Since $|\varphi'(t)| = |\Phi'(e^{it})|$, this identification will cause no difficulty in the estimates to follow.

Our hypothesis is that there is a positive constant B_μ such that

$$|\langle \mu,\varphi\rangle| \leq B_\mu \sup_{0\leq j\leq n} \|\varphi^{(j)}\|_\infty, \tag{3.1}$$

for each $\varphi \in C^\infty(T)$. Fix (forever) a nonnegative, even function $\varphi_1 \in C^\infty(T)$ with support $\varphi_1 \cap [-\pi,\pi] = [-1,1]$, and $\int \varphi_1\, dm = 1$. For $0 < \epsilon < 1$ define $\varphi_\epsilon \in C^\infty(T)$ by

$$\varphi_\epsilon(t) = \begin{cases} \epsilon^{-1}\varphi(t/\epsilon), & |t| \leq \epsilon \\ 0, & \epsilon < |t| \leq \pi. \end{cases}$$

Then $\int \varphi_\epsilon\, dm = 1$ for $0 < \epsilon \leq 1$.

Fix z in Δ, set $\epsilon = \rho_E(z)/6$, and define $\psi \in C^\infty(T)$ by

$$\psi(\theta) = \int_{E_{2\epsilon}} \varphi_\epsilon(\theta - t)\, dm(t)$$

Clearly $\psi \equiv 0$ off $E_{3\epsilon}$. Since $\psi \equiv 1$ on $I_\epsilon(z)$ for each $z \in E$, we also have $\psi \equiv 1$ on E_ϵ. Thus $\psi\mu = \mu$ since support $\mu \subset E$, hence:

$$C_\mu(z) = \langle \mu,k_z\rangle = \langle \psi\mu,k_z\rangle = \langle \mu,\psi k_z\rangle,$$

which, by (3.1) and Leibnitz' formula yields:

$$\begin{aligned} |C_\mu(z)| &\leq B_\mu \sup_{0\leq m\leq n} \|(\psi k_z)^{(m)}\|_\infty \\ &\leq B_\mu \sup_{0\leq m\leq n} \sum_{j=0}^{m} \binom{m}{j} \|\psi^{(j)} k_z^{(m-j)}\|_\infty. \end{aligned}$$

Since support $\psi \subset E_{3\epsilon}$, we have $\|\psi^{(j)} k_z^{(m-j)}\|_\infty \leq \|\psi^{(j)}\|_\infty \sup_{\tau\in E_{3\epsilon}} |k_z^{(m-j)}(\tau)|$. A straightforward calculation shows that for $\tau \in E_{3\epsilon}$:

$$|k_z^{(m-j)}(\tau)| \leq \rho_{E_{3\epsilon}}(z)^{-(m-j+1)} \leq (\rho_E(z)/2)^{-(m-j+1)},$$

by the definition of ϵ. Also:

$$\begin{aligned} |\psi^{(j)}(\theta)| &\leq \epsilon^{-j} \int_{E_{2\epsilon}} \epsilon^{-1} \left|\varphi_1^{(j)}\left(\frac{\theta - t}{\epsilon}\right)\right| dm(t) \\ &\leq \epsilon^{-j}\|\varphi_1^{(j)}\|_1 = (\rho_E(z)/6)^{-j}\|\varphi_1^{(j)}\|_1, \end{aligned}$$

where $\|\cdot\|_1$ denotes the norm in $L^1(T)$. From these estimates we obtain:

$$\|\psi^{(j)} k_z^{(m-j)}\|_\infty \leq 6^{m+1}\|\varphi_1^{(j)}\|_1\, \rho_E(z)^{-(m+1)},$$

hence:

$$|C_\mu(z)| \leq B_\mu \sup_{0\leq m\leq n} \sum_{j=0}^{m} \binom{m}{j} 6^{m+1} \|\varphi_1^{(j)}\|_1 \rho_E(z)^{-(m+1)}$$
$$\leq A_\mu \rho_E(z)^{-(n+1)},$$

which completes the proof, since φ_1 does not depend on z.

b. *Gaussian random vectors.* Suppose E is a complex vector space and $\mathscr{B}$ is a sigma-algebra of subsets of E. The pair $(E, \mathscr{B})$ is called a *measurable vector space* if the vector operations, addition: $E \times E \to E$, scalar multiplication: $\mathbf{C} \times E \to E$, are measurable when the product spaces have their natural product sigma-algebras. It is not difficult to see that if E is a separable, metrizable topological vector space, and $\mathscr{B}(E)$ denotes the collection of Borel sets of E, then $(E, \mathscr{B}(E))$ is a measurable vector space. However, if E is nonseparable, then it could happen that the product σ-algebra $\mathscr{B}(E) \otimes \mathscr{B}(E)$ is strictly smaller than $\mathscr{B}(E \times E)$, hence addition—which, by continuity, is measurable with respect to $\mathscr{B}(E \times E)$ —may not be measurable for $\mathscr{B}(E) \otimes \mathscr{B}(E)$.

Suppose $(\Omega, \mathscr{T}, P)$ is a probability space, and $(E, \mathscr{B})$ is a measurable vector space. An *E-valued random vector is* simply an $\mathscr{T} - \mathscr{B}$ measurable map $X: \Omega \to E$. Two such random vectors X and Y are said to be *similar* if

$$P\{X \in B\} = P\{Y \in B\}$$

for every $B \in \mathscr{B}$. An E-valued random vector is called *Gaussian* if, whenever Y and Z are independent and similar to X, then so are $(Y + Z)/\sqrt{2}$ and $(Y - Z)/\sqrt{2}$.

We will consider only measurable vector spaces $(E, \mathscr{B}(E))$ where E is a separable, metrizable topological vector space. The E-valued random vectors we deal with will all have the form

$$X = \sum \gamma_n u_n \tag{3.2}$$

where (u_n) is a sequence in E, and (γ_n) is a sequence of independent, normally distributed complex random variables with mean zero and variance one [8; Chapter XI, Sec. 3, page 118]; and the series (3.2) is assumed to converge with probability one. Henceforth we refer to (γ_n) as the *standard complex normal sequence,* following [8, Chapter XI, Sec. 3]. Since each γ_n is Gaussian in the sense of the last paragraph, it is not difficult to see that X, as given by (3.2), is an E-valued Gaussian random vector.

If $E = \mathbf{C}$ then a necessary and sufficient condition for the series (3.2) to converge almost surely is:

$$\sigma^2 = \sum |u_n|^2 < \infty, \tag{3.3}$$

in which case X has moments of all orders, and the distribution of X depends only on σ [8; Chapter XI, Sec. 3, page 118]. In particular, if $0 < p < \infty$ then there is a constant $C_p > 0$, not depending on X, such that

$$(3.4) \qquad (\mathscr{E}\{|X|^p\})^{1/p} = C_p\sigma = C_p(\mathscr{E}\{|X|^2\})^{1/2},$$

where $\mathscr{E}\{\cdot\} = \int (\cdot)\,dP$.

For general measurable spaces E the situation is not so simple, however E-valued Gaussian random vectors are still highly integrable. This is the content of the next result, essentially due to X. Fernique [6]. Let us call a function $\Lambda : E \to [0,\infty)$ *monotone* if $\Lambda(ax) \le \Lambda(x)$ whenever $x \in E$ and $a \in \mathbf{C}$ with $|a| \le 1$.

LEMMA B [6]. *Suppose $(E, \mathscr{B})$ is a measurable vector space and $\Lambda : E \to [0,\infty)$ is a measurable, monotone, subadditive function. If X is a Gaussian E-valued random vector, then there exist positive constants A and B such that for each $\lambda \ge 0$: $P\{\Lambda(X) > \lambda\} \le Ae^{-B\lambda}$. In particular, $\mathscr{E}\{\Lambda(X)\} < \infty$.*

Remark. In [6] Fernique assumes Λ is a *seminorm,* and gets a better exponential estimate on the distribution of $\Lambda(X)$. The proof given below is just a slight simplification of his: it is presented only for completeness.

Proof. Suppose Y and Z are independent and similar to X. Then for $t > s \ge 0$ we have, just as in [6]:

$$\begin{aligned} P\{\Lambda(X) \le s\} \cdot P\{\Lambda(X) > t\} &= P\{\Lambda(Y - Z)/\sqrt{2}) \le s \text{ and } \Lambda(Y + Z)/\sqrt{2}) > t\} \\ &\le P\{|\Lambda(Y/\sqrt{2}) - \Lambda(Z/\sqrt{2})| \le s \text{ and } \Lambda(Y/\sqrt{2}) + \Lambda(Z/\sqrt{2}) > t\} \\ &\le P\{t - s \le 2\Lambda(X/\sqrt{2})\}^2. \end{aligned}$$

From this inequality and the monotonicity of Λ we obtain:

$$(3.5) \qquad \frac{P\{\Lambda(X) > t\}}{P\{\Lambda(X) \le s\}} \le \left(\frac{P\{\Lambda(X) > (t-s)/2\}}{P\{\Lambda(X) \le s\}}\right)^2$$

Set $t_0 = s$ and $t_{n+1} = 2t_n + s$, so by induction, $t_n = (2^{n+1} - 1)s$. Let

$$x_n = P\{\Lambda(X) > t_n\}/P\{\Lambda(X) \le s\}$$

and choose s so that $P\{\Lambda(X) \le s\} > 1/2$. Then iteration of (3.5) yields:

$$P\{\Lambda(X) > (2^{n+1} - 1)s\} = x_n \le x_0^{2^n} = e^{-c \cdot 2^n} \qquad (n = 0, 1, 2, \dots),$$

where $c = -\log x_0 > 0$ by our choice of s. After a little routine calculation, this inequality yields the desired result.

c. *Gaussian Cauchy transforms.* Suppose (γ_n) is the standard complex normal sequence of section (b), and (m_n) is a sequence of positive numbers with $\sum m_n < \infty$. Since $\mathscr{E}\{|\gamma_n|^2\} = 1$ for each n, we have

$$\mathscr{E}\left\{\sum |\gamma_n| m_n\right\} = \sum \mathscr{E}\{|\gamma_n|\} m_n \leq \sum m_n < \infty,$$

which shows that $\sum \gamma_n m_n$ converges absolutely with probability one. Therefore if (ζ_n) is a sequence of points of T, and δ_{ζ_n} is the unit mass at ζ_n, then the expression

$$\mu = \sum \gamma_n m_n \delta_{\zeta_n} \tag{3.6}$$

almost surely defines a finite Borel measure on T. We will be concerned with the Cauchy transform

$$C_\mu(z) = \sum \frac{\gamma_n m_n}{1 - \bar{\zeta}_n z} \qquad (z \in \Delta), \tag{3.7}$$

of this random measure. We want to know when its derivative C'_μ is almost surely *not* in the Nevanlinna class N. Not surprisingly the answer depends on the quantity

$$\sigma^2(z) = \mathscr{E}\{|C'_\mu(z)|^2\} = \sum \frac{m_n^2}{|\zeta_n - z|^4} \qquad (|z| \leq 1). \tag{3.8}$$

LEMMA C. *If* $\int_T \log \sigma\, dm = \infty$, *then almost surely* $C'_\mu \notin N$.

Proof. Consider N in the topology $\mathscr{K}$ of uniform convergence on compact subsets of Δ – a separable metrizable topology on N – and let $\mathscr{B}$ denote the Borel sets for this topology. Let

$$\Lambda(f) = \sup_{0 \leq r < 1} \int_T \log(1 + |f_r|)\, dm$$

for $f \in N$. It is easy to check that Λ is a nonnegative, monotone, subadditive function on N. Moreover the sets $\{f \in N\colon \Lambda(f) \leq \epsilon\}$ $(\epsilon > 0)$ are easily seen to be $\mathscr{K}$-closed in N, hence Λ is $\mathscr{B}$-measurable.

Suppose $C'_\mu \in N$ almost surely. Since almost surely the series

$$C'_\mu(z) = \sum_n \frac{\gamma_n \bar{\zeta}_n m_n}{(1 - \bar{\zeta}_n z)^2}$$

converges uniformly on compact subsets of Δ, it follows from the discussion in section (b) that C'_μ is an N-valued Gaussian random vector; hence by Lemma B we have $\mathscr{E}\{\Lambda(C'_\mu)\} < \infty$. Write $f = C'_\mu$. Since $\log(1 + |f|)$ is subharmonic on Δ, the integral means $\int \log(1 + |f_r|)\, dm$ increase with r [5; Theorem 1.6, page 9], hence the monotone convergence theorem and Fubini's theorem yield:

$$\infty > \mathscr{E}\{\Lambda(f)\} = \mathscr{E}\left\{\lim_{r\to 1-}\int_T \log(1 + |f_r|)\,dm\right\}$$

$$= \lim_{r\to 1-}\int_T \mathscr{E}\{\log(1 + |f_r|)\}\,dm$$

$$\geq \lim_{r\to 1-}\int_T \mathscr{E}\{\log|f_r|\}\,dm.$$

Now suppose $\sigma(z)$ is given by (3.8). For $z \in \Delta$:

$$\mathscr{E}\{\log|f(z)|\} = \mathscr{E}\{\log|f(z)/\sigma(z)|\} + \log\sigma(z).$$

Since the random variable $f(z)/\sigma(z)$ has the same distribution as γ_n for each $z \in \Delta$, we have: $\mathscr{E}\{\log|f(z)|\} = \mathscr{E}\{|\gamma_n|\} + \log\sigma(z) = C + \log\sigma(z)$ where C is independent of n and z. Thus Fatou's lemma and the previous inequality yield

$$\infty > C + \lim_{r\to 1-}\int_T \log\sigma(r\tau)\,dm(\tau)$$

$$\geq C + \int_T \log\sigma(\tau)\,dm(\tau).$$

This shows that if $\int_T \log\sigma\,dm = \infty$, then $C'_\mu \notin N$ with positive probability; and it follows from the zero-one law [8; Chapter I, page 6] that $C'_\mu \notin N$ almost surely.

4. PROOF OF MAIN RESULT

In this section we prove Theorem 1. Recall from the discussion in Section 2 that only two implications need to be proved. Throughout this section E will denote a closed subset of T with $T\setminus E = \cup I_n$ the decomposition of the complement of E into at most countably many disjoint open intervals. As usual, m denotes normalized Lebesgue measure on T, and we write $\epsilon_n = m(I_n)$.

Proof of Theorem 1. (i) $\Rightarrow$ (ii). Suppose $m(E) = 0$ and $\sum \epsilon_n \log\epsilon_n > -\infty$. Then a straightforward calculation shows that

$$\int_T \log\rho_E(t)\,dt > -\infty. \tag{4.1}$$

We claim that for $0 \leq r < 1$ and $\tau \in T$:

$$\rho_E(r\tau) \geq \sqrt{r}\,\rho_E(\tau). \tag{4.2}$$

To see this, write $\tau = e^{i\theta}$ and choose $\tau_0 = e^{i\theta_0}$ in E so that $\rho_E(r\tau) = |\tau_0 - r\tau|$. Then

$$\begin{aligned}\rho_E(r\tau)^2 &= 1 - 2r\cos(\theta - \theta_0) + r^2 \\ &= (1-r)^2 + 4r\sin^2\left(\frac{\theta - \theta_0}{2}\right) \\ &\geq r\left[2\sin\left(\frac{\theta-\theta_0}{2}\right)\right]^2 \\ &= r|e^{i\theta} - e^{i\theta_0}|^2 \geq r\rho_E(\tau)^2,\end{aligned}$$

which proves (4.2).

Suppose $\mu \in \mathscr{D}^n_E(T)$. We want to show that $C_\mu \in N^+$. Assuming, as we may, that $A_\mu \geq 1$ in Lemma A, we have:

$$\begin{aligned}\log^+|C_\mu(r\tau)| &\leq \log^+[r^{-1/2}A_\mu \rho_E(\tau)^{-(n+1)}] \\ &= \log(r^{-1/2}A_\mu) - (n+1)\log\rho_E(\tau).\end{aligned} \tag{4.3}$$

Since $\log \rho_E \in L^1(T)$ by (4.1), the dominated convergence theorem yields (2.1) for $f = C_\mu$ hence $C_\mu \in N^+$ as desired.

(vii) $\Rightarrow$ (i). Suppose E does not satisfy condition (i). To complete the proof of Theorem 1 we must find $\mu \in M_E(T)$ so that $C'_\mu \notin N$. There are two cases to consider: (a) $m(E) > 0$, and (b) $m(E) = 0$ but $\sum \epsilon_n \log \epsilon_n = -\infty$. The first one is handled by the following proposition, which is nothing more than a slight modification of [8; Chapter X, Section 6, Proposition 2, page 110].

PROPOSITION 2. *Let E be a closed subset of T with $m(E) > 0$. Suppose (ζ_n) is a sequence of independent T-valued random variables, independent of the standard complex normal sequence (γ_n), with each ζ_n uniformly distributed in E. Suppose (m_n) is a sequence of positive numbers with $\sum m_n < \infty$, but $\sum m_n^{1/2} = \infty$. Then almost surely: $\mu = \sum \gamma_n m_n \delta_{\zeta_n} \in M_E(T)$, but $C'_\mu \notin N$.*

Proof. The hypothesis on the distribution of ζ_n is that

$$P\{\zeta_n \in B\} = m(B \cap E)/m(E)$$

for any Borel subset B of T. Since the sequences (ζ_n) and (γ_n) are independent of each other we can write the probability space $(\Omega, \mathscr{F}, P)$ as a product

$$(\Omega_\zeta \times \Omega_\gamma, \mathscr{F}_\zeta \otimes \mathscr{F}_\gamma, P_\zeta \otimes P_\gamma),$$

where (ζ_n) "lives" on Ω_ζ, and (γ_n) on Ω_γ. We also write $\mathscr{E}_\gamma\{\cdot\} = \int (\cdot)\, dP_\gamma$, and $\mathscr{E}_\zeta\{\cdot\} = \int (\cdot)\, dP_\zeta$.

For each $\omega' \in \Omega_\zeta$ we know that $\mu \in M_E(T)$ almost surely $[P_\gamma]$. It follows from Fubini's theorem that $\mu \in M_E(T)$ almost surely $[P]$, so it remains to show that $C'_\mu \notin N$ almost surely $[P]$.

For $\tau \in T$ and $\omega' \in \Omega_\zeta$, let $\sigma(\tau,\omega')$ be defined by (3.8):

$$\sigma^2(\tau) = \sigma^2(\tau,\omega') = \sum \frac{m_n^2}{|\zeta_n(\omega') - \tau|^4}.$$

Now suppose τ is a fixed *point of density* in E. We claim that $\sigma(\tau) = \infty$ almost surely $[P_\zeta]$. By the three series theorem [12; Chapter 2, Section 9, p. 34] it is enough to show that

$$\sum_n P_\zeta\left\{\frac{m_n^2}{|\zeta_n - \tau|^4} > 1\right\} = \infty, \tag{4.4}$$

and this follows from the calculation below:

$$\begin{aligned} P_\zeta\left\{\frac{m_n^2}{|\zeta_n - \tau|^4} > 1\right\} &= P_\zeta\{|\zeta_n - \tau| < \sqrt{m_n}\} \\ &= m\{I_{\sqrt{m_n}}(\tau) \cap E\}/m(E) \geq \sqrt{m_n}/2\pi \end{aligned}$$

where the last inequality holds for all n sufficiently large because τ is a point of density of E. Since $\sum \sqrt{m_n} = \infty$, we have (4.4). Thus for almost every $[m]$ point τ in E we have $\sigma(\tau) = \infty$ almost surely $[P_\zeta]$. By Fubini's theorem it follows that almost surely $[P_\zeta]$ we have: $\sigma(\tau) = \infty$ for $[m]$ a.e. τ in E, hence $\int_T \log \sigma(\tau)\, dm(\tau) = \infty$ so by Lemma C, $C'_\mu \notin N$ almost surely $[P_\gamma]$. Fubini's theorem again shows that almost surely $[P]$ we have $C'_\mu \notin N$, which completes the proof of Proposition 2.

To finish the proof of Theorem 1 we must show that whenever $m(E) = 0$ and $\sum \epsilon_n \log \epsilon_n = -\infty$, there exists $\mu \in M_E(T)$ with $C'_\mu \notin N$. This follows from the next result.

PROPOSITION 3. *Suppose* $m(E) = 0$ *and* $\sum \epsilon_n \log \epsilon_n = -\infty$. *Let* ζ_n *denote* *either endpoint of* I_n, *and set* $\mu = \sum \epsilon_n \gamma_n \delta_{\zeta_n}$. *Then almost surely* $\mu \in M_E(T)$ *but* $C'_\mu \notin N$.

Proof. Note that in this proof (ζ_n) is a *fixed* sequence of points in E, not a sequence of E-valued random variables. As before, $\mu \in M_E(T)$ almost surely, so by Lemma C, it is enough to show that $\int_T \log \sigma(\tau)\, dm(\tau) = \infty$ where σ is given by (3.8). The calculation in this case is similar to one in Ahern's paper [1]. For $\tau \in I_n$ we have

$$\log \sigma(\tau) \geq \log \left(\frac{\epsilon_n^2}{|\alpha_n - \tau|^4} \right)^{1/2} \leq -\log \epsilon_n,$$

hence

$$\begin{aligned} \int_T \log \sigma(\tau)\, dm(\tau) &= \sum_n \int_{I_n} \log \sigma\, dm \\ &\geq -\sum \epsilon_n \log \epsilon_n \\ &= \infty, \end{aligned}$$

which completes the proof of Proposition 3, and therefore of Theorem 1.

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