CYCLIC INNER FUNCTIONS IN BERGMAN SPACES*

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1. INTRODUCTION. Recently Boris Korenblum and James Roberts ([17], [18]) (independently) solved a problem in function theory that had been around for about fifteen years. These notes describe the problem, its solution, and some applications.

For $1 \le p < \infty$ and $\alpha > -1$, let A^p_α denote the space of functions f holomorphic in the open unit disc U for which

$$\|f\|^p_{p,\alpha} = \iint_U |f(z)|^p (1 - |z|)^\alpha \, dx\, dy < \infty .$$

The norm $\|\cdot\|_{p,\alpha}$ makes A^p_α into a Banach space called a (weighted) Bergman space. The forward shift S, defined on A^p_α by

$$(Sf)(z) = zf(z) \qquad (f \text{ in } A^p_\alpha, \quad z \text{ in } U)$$

is a bounded linear operator on A^p_α. A challenging unsolved problem is to describe the closed subspaces of A^p_α that are invariant under S. A special case of this problem—also unsolved—is to describe those $f \in A^p_\alpha$

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that are cyclic (for S, on A^p_α); that is, those f for which the subspace

$$\{pf : p \text{ a polynomial in } z\}$$

is dense. Note that for $f \in A^p_\alpha$, the closure of this subspace is the smallest closed subspace of A^p_α containing f and invariant under S.

If A^p_α is replaced by the Hardy space H^p, (in some sense the limiting case of A^p_α as $\alpha \to -1$) then the cyclic vectors are known: they are precisely the outer functions in H^p. This is part of Beurling's famous invariant subspace theorem for H^p. In particular this implies that if both f and $1/f$ are in H^p, then f is cyclic in H^p. It is not known if this last result is true when H^p is replaced by A^p_α: this gives an idea of how far we are from having a characterization of cyclic vectors in A^p_α (but see [1], and [14; pp. 111-115] for some partial results on this problem).

What Roberts and Korenblum have done is to complete a characterization, begun in 1967 by H. S. Shapiro, of the cyclic inner functions in A^p_α. To see what is involved, first observe that for each z in U the linear functional

$$f \to f(z) \qquad (f \text{ in } A^p_\alpha)$$

is continuous on A^p_α, so if f vanishes at some z_0 in U, then so does every member of the closure of $\{pf : p \text{ a polynomial}\}$. Thus if f is cyclic then it cannot vanish anywhere on U. In particular a cyclic inner function must be singular, that is it must have the form

(1) $$S\mu(z) = \exp\left\{\int_{\mathbb{T}} \frac{z+\omega}{z-\omega}\, d\mu(\omega)\right\}$$

where μ is a positive finite Borel measure on the unit circle $\mathbb{T}$ that is singular with respect to Lebesgue measure on $\mathbb{T}$.

So the problem is: characterize those positive singular measures μ on $\mathbb{T}$ for which $S\mu$ is cyclic in A^p_α. The solution, which occupies most of what follows is: $S\mu$ is cyclic if and only if μ places no mass on any Carleson set. Carleson sets are certain thin subsets of $\mathbb{T}$: they will be discussed shortly. The necessity of this Carleson set condition was proved by H. S. Shapiro in 1967 [12; Theorem 2]. His proof occupies section 4 (along with an introduction to Carleson sets). The sufficiency of this condition was proved in 1979 by Roberts. Korenblum recently informed me that he has actually known the result for several years: it follows from the work in his paper [7]. Here we present Roberts' proof, which seems more accessible (even though its starting point is the quantitative statement of the Corona Theorem!). The characterization of cyclic inner functions leads to an interesting factorization theorem, which is described in the last section.

Finally, it turns out that only certain elementary properties of Bergman spaces figure in the proofs, so the results actually hold in a more general setting. This is described in the next section.

2. <u>GENERAL SETTING FOR THE PROBLEM</u>. From now on B always denotes a complete metrizable topological vector space whose elements are functions holomorphic in U, and whose vector operations are the usual pointwise operations. We further assume that convergence in B implies uniform convergence on compact subsets of U, and that the metric of B is induced by a <u>quasinorm</u> $\|\cdot\|$. That is, $\|\cdot\|$ is a non-negative function on B with the following properties:

(Q1) $\|f\| = 0 \iff f \equiv 0$ on U

(Q2) There exists $C > 0$ such that

$$\|f+g\| \le C(\|f\| + \|g\|)$$

for all $f, g \in B$.

(Q3) $\|af\| = |a|\ \|f\|$, all $f \in B$ and $a \in \mathbb{C}$.

The Bergman space A^p_α in the quasinorm $\|\cdot\|_{p,\alpha}$ is an example of such a space B, even when $0 < p < 1$.

We will also assume that B obeys the following additional conditions:

(B1) $H^\infty \subset B$

(B2) The polynomials are dense in B

(B3) If $f \in B$ and $g \in H^\infty$, then $fg \in B$ and $\|fg\| \le \|f\|\ \|g\|_\infty$

(B4) If (g_n) is a uniformly bounded sequence in H^∞ which tends to zero pointwise on U, then $\|g_n f\| \to 0$ in B.

According to (B3) the shift operator S (= multiplication by z) defined in section 1 is a continuous linear operator on B; and it follows from (B2) and (B3) that if $f \in B$ then

$$[f] = \text{closure in } B \text{ of } \{pf : p \text{ a polynomial}\}$$

is the smallest S-invariant closed subspace of B containing f. We call $f \in B$ <u>cyclic</u> (for S, in B) if $[f] = B$.

We are going to prove the following results. From now on, μ is always a positive finite Borel measure on the unit circle that is singular with respect to Lebesgue measure, and S_μ is the singular inner function defined by formula (1) of section 1.

<u>THEOREM</u> N. (H. S. Shapiro [12, Theorem 2]). *Suppose in addition to* (B1) - (B4) *the space* B *satisfies*:

(B5) *There exists* $\alpha = \alpha(B) > 0$ *such that for each* $f \in B$ *with* $f(z) = \sum_0^\infty a_n z^n$,

$$|a_n| \le C\|f\| (n+1)^\alpha \qquad (n = 0, 1, 2, \cdots),$$

where $C = C(B)$ *is independent of* f *and* n.

If S_μ *is cyclic in* B, *then* $\mu(K) = 0$ *for every Carleson set* K.

<u>THEOREM</u> S. (Korenblum 1977, Roberts 1979). <u>Suppose in addition to</u> (B1) - (B4), <u>the space</u> B <u>satisfies</u>:

(B6) <u>There exists</u> $\beta = \beta(B) > 0$ <u>such that</u> $\|z^n\| \leq n^{-\beta}$ <u>for</u> $n = 2, 3, \cdots$.

<u>If</u> $\mu(K) = 0$ <u>for every Carleson set</u> K, <u>then</u> $S\mu$ <u>is cyclic</u>.

It is not difficult to check that the Bergman spaces A^p_α $(0 < p < \infty,\ \alpha > -1)$ all satisfy conditions (B1) - (B6).

3. PRELIMINARIES. Suppose the space B satisfies conditions (B1) - (B4). Here are some simple, but useful consequences.

PROPOSITION 1. Suppose $f \in B$

(a) If f is cyclic then f never vanishes on U.

(b) The following are equivalent:

(i) f is cyclic in B

(ii) $p_n f \to 1$ in B for some sequence (p_n) of polynomials

(iii) $g_n f \to 1$ in B for some sequence (g_n) in H^∞.

Proof. (a) We are also assuming that convergence in B implies uniform convergence on compact subsets of U, so if f vanishes at $z_0 \in U$, then so does every member of $[f]$.

(b) The implications (i) $\to$ (ii) $\to$ (iii) are obvious, and (iii) $\to$ (i) follows immediately from:

LEMMA 2. The polynomials are B-dense in H^∞.

Proof. For $f \in H^\infty$ there exist polynomials (p_n) with $\|p_n\|_\infty \le \|f\|_\infty$ and $p_n \to f$ pointwise on U. For example, take $p_n = n^{\text{th}}$ arithmetic mean of the partial sums of the Taylor series of f. By (B4), $p_n \to f$ in B.

PROPOSITION 3. Suppose q is an inner function.

(a) If $q_0 | q$ then $[q] \subset [q_0]$

(b) If q is singular and $\varepsilon > 0$, then: q is cyclic if and only if q^ε is cyclic.

Proof. (a) Write $q = q_0 q_1$ where q_1 is inner. Choose polynomials p_n such that $\|p_n\|_\infty \le 1$ and $p_n \to q_1$ pointwise on U (as in the proof of Lemma 2) so $p_n q_0 \to q$ in B by (B4), hence $q \in [q_0]$.

(b) We need only consider $0 < \varepsilon < 1$. If q is cyclic then so is q^ε, by part (a). Conversely, suppose q^ε is cyclic. Then $1 \in [q^\varepsilon]$, that is, 1 is a limit in B of polynomial multiples of q^ε. By (B3), q^ε is therefore a B-limit of polynomial multiples of $q^{2\varepsilon}$, hence $B = [q^\varepsilon] = [q^{2\varepsilon}]$. Continuing in this fashion we obtain $B = [q^{n\varepsilon}]$ for each positive integer n. When $n \ge 1/\varepsilon$ we have $[q^{n\varepsilon}] \subseteq [q]$ by part (a), hence q is cyclic.

PROPOSITION 4. ("Continuity of distance"). Suppose μ_n, μ are positive singular measures on $\mathbb{T}$, and $\mu_n \uparrow \mu$. Then for each $f \in B$:

$$\text{dist}_B(f, [S\mu_n]) \downarrow \text{dist}_B(f, [S\mu]) .$$

Remark. Here $\text{dist}_B(f, X) = \inf\{\|f-x\| : x \in S\}$ where $X \subset B$.

COROLLARY. If $\mu_n \uparrow \mu$ then $\bigcap_n [S\mu_n] = [S\mu]$.

Proof of Proposition. By Proposition 3 (a) we know $[S\mu_n] \supset [S\mu_{n+1}] \supset [S\mu]$ (since $S\mu_n | S\mu_{n+1} | S\mu$) so $d_n = \text{dist}_B(f, [S\mu_n])$ is certainly an increasing function of n, and $d_n \le d = \text{dist}(f, [S\mu])$ for each n. Fix $\varepsilon_n \downarrow 0$ and choose a polynomial g_n so that

$$\|f - g_n S\mu_n\| \le d_n + \varepsilon_n .$$

Write $\mu = \mu_n + \nu_n$, so ν_n is a positive singular measure on $\mathbb{T}$, and

$$\|S\nu_n f - g_n S_\mu\| = \|S\nu_n f - g_n S\mu_n S\nu_n\|$$

$$\leq \|f - g_n S\mu_n\| \qquad \text{(by (B3))}$$

$$\leq d_n + \varepsilon_n \ .$$

Therefore:

$$d \leq \|f - g_n S\mu\|$$

$$\leq \|f - S\nu_n f\| + \|S\nu_n f - g_n S\mu\|$$

$$\leq \|f - S\nu_n f\| + d_n + \varepsilon_n \ .$$

But $\nu_n \downarrow 0$, so $S\nu_n \to 1$ pointwise on $\mathbb{T}$, hence $\|f - S\nu_n f\| \to 0$ by (B4). Thus $d \leq \lim_n d_n$.

4. CARLESON SETS AND PROOF OF THEOREM N.

4.1. Boundary zeros of smooth analytic functions. In this section (and forever after), m denotes normalized Lebesgue measure on the unit circle $\mathbb{T}$, and A denotes the disc algebra: those continuous functions on the closed unit disc which are holomorphic on the open unit disc U. If $0 \not\equiv f \in A$, and K is the boundary zero set of f:

$$K = \{\omega \in \mathbb{T} : f(\omega) = 0\} ,$$

then K is a compact subset of $\mathbb{T}$. Moreover $m(K) = 0$, since the analyticity of f forces

$$\int_{\mathbb{T}} \log |f(\omega)| \, dm(\omega) > -\infty . \tag{1}$$

Conversely, it is a theorem of Fatou [5; page 80] that if K is a closed subset of $\mathbb{T}$ with $m(K) = 0$, then there exists $f \in A$ whose boundary zero set is K: in fact f can be taken to be an outer function.

However if we assume in addition that f satisfies a Lipschitz condition

$$|f(\omega_1) - f(\omega_2)| \leq M |\omega_1 - \omega_2|^{\alpha} \tag{2}$$

where M, α do not depend on $\omega_1, \omega_2 \in \mathbb{T}$; then the situation is different. To see what happens, suppose $0 \not\equiv f \in A$, and f satisfies (2). Let K be the boundary zero set of f, and for $\omega \in \mathbb{T}$ write

$$\rho_K(\omega) \overset{\text{def}}{=} \operatorname{dist}(\omega, K) = \inf\{|\omega - \omega'| : \omega' \in K\} . \tag{3}$$

Then given $\omega \in \mathbb{T}$, choose $\omega' \in K$ so that $\rho_K(\omega) = |\omega - \omega'|$. By the Lipschitz condition (2) :

$$\begin{aligned} |f(\omega)| &= |f(\omega) - f(\omega')| \\ &\leq M|\omega - \omega'| \\ &= M\rho_K(\omega) \quad . \end{aligned}$$

Substituting in (1) we obtain the following additional condition on K (beside $m(K) = 0$) :

$$\int_{\mathbb{T}} \log \rho_K \, dm > -\infty \quad . \tag{4}$$

This observation is due to Beurling [2] , and motivates :

<u>DEFINITION</u>. A compact subset K of $\mathbb{T}$ satisfying (4) is called a <u>Carleson set</u>.

So the boundary zero set of a nontrivial $f \in A$ with Lipschitz boundary function must be a Carleson set. Carleson's contribution is the following converse :

<u>CARLESON'S THEOREM</u> [3 ; Theorem 1] . Suppose $K \subset \mathbb{T}$ is a Carleson set and n is a positive integer. Then there exists an outer function $f \in A$ with n times differentiable boundary function such that $K = \{\omega \in \mathbb{T} : f(\omega) = 0\}$.

<u>Remark</u>. By replacing f by its $n+1^{\text{st}}$ power we can also insure that $K \subset \{\omega \in \mathbb{T} : f^{(j)}(\omega) = 0\}$ for $0 \leq j \leq n$. Here (and henceforth) $f^{(j)}$ denotes the j^{th} derivative of the boundary function.

Clearly every finite subset of $\mathbb{T}$ is a Carleson set. To get nontrivial examples, the following is useful (and easy).

PROPOSITION. *Suppose K is a compact subset of $\mathbb{T}$, and $\mathbb{T}\setminus K = \bigcup_n I_n$, the canonical decomposition of $\mathbb{T}\setminus K$ into an at most countable disjoint union of open arcs. Then K is a Carleson set $\Leftrightarrow$ $m(K) = 0$ and* $\sum_n m(I_n) \log \frac{1}{m(I_n)} < \infty$.

EXAMPLES. (a) The Cantor middle-thirds set is a Carleson set (use the Proposition).

(b) Every closed subset of a Carleson set is again one (use the definition).

(c) There exist countable closed subsets of $\mathbb{T}$ which are *not* Carleson sets. For example take $K = \{\omega_n\}_1^\infty \cup \{1\}$, where $\omega_n = e^{i\theta_n}$, $2\pi > \theta_n \downarrow 0$, and $\theta_{n+1} - \theta_n \sim \frac{1}{n(\log n)^2}$. Then use the Proposition.

The next section contains an elegant proof, due to Korenblum, of a "C^∞ version" of Carleson's Theorem. Right now, however, we want to show how Carleson's Theorem yields Shapiro's Theorem N .

Proof of Theorem N. Condition (B5) insures that if m is a positive integer $> \alpha+2$ (say) and $g \in C^{(m)}(\mathbb{T})$, so $g(\omega) = \sum_{-\infty}^{\infty} b_n \omega^n$ with $|b_n| = 0(|n|^{-m})$, then the equation

$$\lambda_g(f) = \sum_{n=0}^{\infty} a_n \overline{b_n} \qquad \left(f(z) = \sum_0^\infty a_n z^n, \quad f \in B\right)$$

defines a continuous linear functional on B .

Now suppose K is a Carleson set and μ is a positive singular measure on $\mathbb{T}$ with $\mu(K) > 0$. We will show that there is a nontrivial continuous linear functional λ_g as above which annihilates $[S\mu]$.

Suppose first of all that μ is actually <u>concentrated</u> on K. By Carleson's Theorem there exists an outer function $\varphi \in A$, not $\equiv 0$ on K for $0 \le j \le 2m$. Taylor's theorem implies that

$$\varphi^{(j)}(\omega) = 0(\rho_K(\omega)^{2m-j})$$

as $\omega \to K$, for $0 \le j \le 2m$, so it follows from Liebnitz' formula and elementary calculations that the function g defined by

$$g(\omega) = \begin{cases} \overline{\omega\varphi(\omega)}\, S\mu(\omega) & (\omega \in \mathbb{T}\setminus K) \\ 0 & (\omega \in K) \end{cases}$$

is m times differentiable on $\mathbb{T}$, and therefore defines a bounded linear functional λ_g on B. At least for $f \in H^\infty$ we can write

$$\lambda_g(f) = \int_{\mathbb{T}} f(\omega)\,\overline{g(\omega)}\,dm(\omega)$$

where $f(\omega)$ is the radial limit of f at ω. Thus for $n = 0, 1, 2, \cdots$; we have

$$\begin{aligned} \lambda_g(z^n S\mu) &= \int_{\mathbb{T}} \omega^n S\mu(\omega)\,\overline{g(\omega)}\,dm(\omega) \\ &= \int_{\mathbb{T}} \omega^{n+1}\varphi(\omega)\,dm(\omega) \\ &= \hat{\varphi}(-(n+1)) \quad \text{(Fourier coeff.)} \\ &= 0 \quad \text{since } \varphi \in A . \end{aligned}$$

Thus $\lambda_g \equiv 0$ on $[S\mu]$.

To see that λ_g is not trivial; suppose otherwise. Then for $n = 0, 1, 2, \cdots$;

$$0 = \lambda_g(z^n) = \int_{\mathbb{T}} \omega^n \overline{g(\omega)}\, dm(\omega)$$

$$= \int_{\mathbb{T}} \omega^{n+1} \varphi(\omega) \overline{S\mu(\omega)}\, dm(\omega) .$$

So if $h = \overline{S}\mu\, \varphi$ on $\mathbb{T}$, this last calculation shows that h is the boundary function of an H^∞ function—still called h. So $\varphi = S\mu\, h$ a.e. $[m]$ on $\mathbb{T}$, and since all functions involved are in H^∞, we have $\varphi \equiv S\mu h$ on U. But this contradicts the fact that φ is outer. Thus $\lambda_g \not\equiv 0$.

Finally, suppose that we only know $\mu(K) > 0$. Then $\mu = \nu + \sigma$ where ν is concentrated on K. So $S\mu = S_\nu S_\sigma$ (σ a positive singular measure) so by Proposition 3(a) of sec. 3, $[S\mu] \subset [S_\nu]$. By the last paragraph, $[S_\nu] \neq B$, and the proof is complete.

5. PROOF OF CARLESON'S THEOREM. In this section we prove a refinement of Carleson's theorem which has been obtained by a number of authors [8], [9], [10], [15], [16]. The proof below is due to Korenblum [6].

THEOREM. *If K is a Carleson set then there exists an outer function $f \in A$ with infinitely differentiable boundary function, such that K is the (boundary) zero set of f, and each derivative $f^{(j)}$ $(j = 1, 2, \cdots)$ also vanishes at each point of K.*

Proof. The idea is to construct a function H holomorphic on U, which has positive real part, and satisfies the following additional conditions:

(H1) H is holomorphic, with positive real part, at each point of $\mathbb{T}\backslash K$.

(H2) $\dfrac{\operatorname{Re}\{H(\omega)\}}{-\log \rho(\omega)} \to \infty$ as $\omega \to K$.

(H3) For each $n > 0$ there exists $k > 0$ such that $H^{(n)}(\omega) = 0(\rho(\omega)^{-k})$ as $\omega \to K$.

(H4) $H(z) = \int_{\mathbb{T}} \frac{\omega + z}{\omega - z} \{\operatorname{Re} H(\omega)\}\, dm(\omega)$

for each $z \in U$.

Once H is found, then $f = e^{-H}$ is the function we seek. For clearly f is bounded on U; holomorphic, bounded, and never zero on $\mathbb{T}\backslash K$ (by (H1)). From (H2) we have for each $n > 0$:

$$|f(\omega)| = 0(\rho(\omega)^n) \quad \text{as} \quad \omega \to K,$$

and this, along with (3.3) insures that

$$(5) \qquad |f^{(n)}(\omega)| = 0(\rho(\omega)^m) \qquad \text{as} \quad \omega \to K$$

for each $n, m \geq 0$. Thus the function φ defined on $\mathbb{T}$ by

$$\varphi(\omega) = \begin{cases} f(\omega) & \omega \in \mathbb{T}\setminus K \\ 0 & \omega \in K \end{cases}$$

is infinitely differentiable on $\mathbb{T}$, and coincides a.e. [m] with the radial limit function of f. Since f is bounded on U, it is the Poisson integral of φ, hence $f \in A$ and φ is its boundary function. Since (5) also implies that $\varphi^{(n)} \equiv 0$ on K, we see that f has the desired properties. f is outer by (H4).

In constructing H it is convenient to let $|I|$ denote the (non-normalized) arc length of I, and $\rho(\omega)$ the "arc length distance" from $\omega \in \mathbb{T}$ to K. Then if we write the canonical decomposition of $\mathbb{T}\setminus K$ into disjoint open intervals as $\mathbb{T}\setminus K = \bigcup_n I_n$, the hypotheses on K are:

$$(6) \qquad \sum |I_n| = 2\pi, \qquad \sum |I_n| \log \frac{1}{|I_n|} < \infty .$$

The first task is to decompose each I_n.

For n fixed, write I_n as a countable disjoint union of semi-open intervals $\{J_{n,k}\}_{k=1}^{\infty}$ such that

$$(7) \qquad |J_{n,k}| = \inf_{\omega \in J_{n,k}} \rho(\omega) = \frac{1}{2} \sup_{\omega \in J_{n,k}} \rho(\omega) .$$

This is easy: the point is that for each ω in $J_{n,k}$, $\rho(\omega)$ is comparable with $|J_{n,k}|$. We claim that:

(8) $$\sum_{n,k} |J_{n,k}| \log \frac{1}{|J_{n,k}|} < \infty \quad .$$

To see this, observe that for each fixed n :

$$\int_{I_n} \log \frac{1}{\rho(\omega)} \, dm(\omega) = \sum_k \int_{J_{n,k}} \log \left(\frac{1}{\rho(\omega)}\right) dm(\omega)$$

$$\geq \sum_k \int_{J_{n,k}} \log \left(\frac{1}{2|J_{n,k}|}\right) dm(\omega)$$

$$= 2\pi \sum_k |J_{n,k}| \log \left(\frac{1}{2|J_{n,k}|}\right) \quad .$$

So (8) follows upon summing both sides of this last inequality on n , and using the definition of Carleson set $\left(\int \log \frac{1}{\rho} \, dm < \infty\right)$.

Now we begin to construct H . Let $\alpha_{n,k} = \tau_{n,k}\, \omega_{n,k}$ where $\omega_{n,k}$ = midpoint of $J_{n,k}$ and $\tau_{n,k} = 1 + m(J_{n,k})$. Let

$$\varphi_{n,k}(z) = \frac{\alpha_{n,k} + z}{\alpha_{n,k} - z}$$

so $\varphi_{n,k}$ is holomorphic in the disc $|z| < \tau_{n,k}$, with positive real part. A routine calculation with Poisson kernels, using the definition of $\tau_{n,k}$, shows

(9) $$\operatorname{Re} \varphi_{n,k}(\omega) \geq \frac{M}{|J_{n,k}|} \; , \quad \text{all} \quad \omega \in J_{n,k}$$

where M is independent of n, k, ω .

At this point it is convenient to forget about the intervals (I_n) , and renumber $\{J_{n,k}\}$, $\{\varphi_{n,k}\}$, etc. as ordinary sequences: $\{J_n\}$, $\{\varphi_n\}$, $\cdots$. Then (8) becomes

$$\sum |J_n| \log \frac{1}{|J_n|} < \infty$$

so $\exists \; \lambda_n \uparrow \infty$ such that

$$\sum \lambda_n |J_n| \log \frac{1}{|J_n|} < \infty \quad .$$

Set

$$H(z) = \sum_n \{\lambda_n \mid J_n \mid \log \frac{1}{|J_n|}\} \varphi_n(z) \quad .$$

Then the series converges uniformly on compact subsets of $\mathbb{C}\setminus K$ to a holomorphic function H which satisfies condition (H1).

<u>H satisfies (H2)</u>. For if $\omega \in J_n$, then

$$\begin{aligned} \operatorname{Re} H(\omega) &\geq \lambda_n \mid J_n \mid \log \frac{1}{|J_n|} \operatorname{Re} \varphi_n(\omega) \\ &\geq M \lambda_n \log \frac{1}{|J_n|} \qquad \text{(by (9))} \\ &\geq C \lambda_n \log \frac{1}{\rho(\omega)} \qquad \text{(definition of } J_n) \quad . \end{aligned}$$

Since ω passes through infinitely many J_n on the way to any point of K, we have proved (H2).

<u>H satisfies (H3)</u>. Suppose $\omega \in \mathbb{T}\setminus K$. Write $\mu_k = \lambda_n |J_k| \log \frac{1}{|J_n|}$. We have for each $n = 1, 2, \cdots$

$$H^{(n)}(\omega) = \sum \lambda_k \mu_k \varphi_k^{(n)}(\omega) \quad .$$

But

$$|\varphi_k^{(n)}(\omega)| \le \frac{M_n}{|\tau_k - \omega|^{n+1}} \le \frac{M_n}{\rho(\omega)^{n+1}} ,$$

so

$$|H^{(n)}(\omega)| \le \frac{M_n}{\rho(\omega)^{n+1}} \sum_1^\infty \mu_k$$

which proves (H 3).

Finally, (H 4) follows from the fact that each φ_k has the desired representation, with $\{\text{Re } \varphi_k\}^\infty$, a bounded subset of $L^1(\mathbb{T})$.

This completes the construction of H, and the proof of the Theorem.

6. ROBERTS' PROOF OF THEOREM S.

6.1. Preliminary results. These are all well known.

6.1.1. Growth of Poisson integrals. Suppose μ is a positive, finite Borel measure on $\mathbb{T}$ with modulus of continuity ω_μ :

$$\omega_\mu(\delta) = \sup\{\mu(I) : m(I) < \delta, \quad I \text{ a sub-arc of } \mathbb{T}\},$$

where as usual m = normalized Lebesgue measure on $\mathbb{T}$. Let $P[\mu]$ = Poisson integral of μ.

LEMMA [11; Theorem 2 and following "Remark"]. For each $\frac{1}{4} \le r < 1$, $0 \le \theta < 2\pi$:

$$P[\mu]\,(re^{i\theta}) \le \frac{9\omega_\mu(1-r)}{1-r} .$$

Our interest in this Lemma is the following:

COROLLARY. Suppose μ as in the Lemma is also $\perp m$. Then for $\frac{1}{4} \le r < 1$ and $|z| = r$, we have

$$S\mu(z) \ge \exp\{-\frac{9\omega_\mu(1-r)}{1-r}\} .$$

In particular, if $\omega_\mu(\delta) \le C(\delta \log \frac{1}{\delta})$ for some $0 < \delta \le \frac{3}{4}$, then

$$|S\mu(z)| \ge (1 - |z|)^{9C} \qquad (|z| \le 1-\delta) .$$

Proof of Corollary. For $|z| = 1 - \delta$ the result follows immediately from the Lemma. But $S\mu$ never vanishes, so the result for $|z| < 1 - \delta$ follows upon applying the Maximum Principle to $\frac{1}{S\mu}$.

Proof of Lemma. Fix $\frac{1}{4} \le r < 1$ and choose a positive integer n so that $(n+1)^{-1} \le 1 - r < n^{-1}$. Let

$$I_k = \{e^{it} : \frac{2\pi k}{n+1} \le t < \frac{2\pi(k+1)}{n+1}\}$$

so $m(I_k) = (n+1)^{-1}$. It is enough to prove that

$$P[\mu](r) \le \frac{9\omega(1-r)}{1-r} .$$

Now

$$P[\mu](r) = \int_0^{2\pi} P_r(t)\, d\mu(t)$$

where

$$P_r(t) = \frac{1-r^2}{1 - 2r\cos t + r^2}$$

(the Poisson kernel). By standard calculations:

$$0 < P_r(t) \le \frac{2(1-r)}{(1-r)^2 + (\frac{t}{\pi})^2} \qquad (\tfrac{1}{4} \le r < 1)$$

so for e^{it} in I_k:

$$P_r(t) < \frac{2(1-r)}{(1-r)^2 + (\frac{k}{n+1})^2} < \frac{2}{1-r} \; \frac{1}{1 + \frac{k^2}{4}} ,$$

so

$$P[\mu](r) = \sum_{k=0}^{n} \int_{I_k} P_r(t)\, d\mu(t)$$

$$< \sum_{k=0}^{n} \frac{2}{1-r} \frac{\mu(I_k)}{1+\frac{k^2}{4}}$$

$$\leq \frac{2\,\omega_\mu(\frac{1}{(n+1)})}{1-r} \sum_{k=0}^{n} \frac{1}{1+\frac{k^2}{4}}$$

$$\leq \frac{2\,\omega_\mu(1-r)}{1-r} \sum_{k=0}^{n} \frac{1}{1+\frac{k^2}{4}}$$

But the sum on the right is $\leq$

$$1 + \int_0^\infty \frac{dx}{1+\frac{x^2}{4}} = 1 + \pi < 4\tfrac{1}{2}$$

and this completes the proof.

6.1.2. <u>THE CORONA THEOREM</u>. There exists $\kappa > 0$ such that: whenever $f_1, \cdots, f_n \in H^\infty$ with $\|f_i\|_\infty \leq 1$ for all i, and $|f_1| + \cdots + |f_n| \geq \delta$ on U, where $0 < \delta \leq \frac{1}{2}$, <u>then</u> there exists $g_1, \cdots, g_n$ in H^∞ with $\|g_i\|_\infty \leq \delta^{-\kappa}$ and $f_1 g_1 + \cdots + f_n g_n \equiv 1$ on U.

<u>Remark</u>. The original proof of the Corona Theorem, due to Carleson, is very complicated. However there is now a "simple" proof, due to Thomas H. Wolff. Although Wolff never published his proof, there have been a couple of expositions circulated around (see [13], for example), and a proof will be published in Paul Koosis' forthcoming book on H^p theory.

6.2. Distance estimates. For the rest of this section we assume—in addition to (B1) - (B4), that the space B satisfies condition (B6). μ (or μ_n) always denotes a positive, finite Borel measure on the unit circle, singular with respect to Lebesgue measure, and $S\mu$ is the singular inner function associated with μ by formula (1) of section 1.

Our goal is to show that $S\mu$ is cyclic whenever μ places no mass on any Carleson set. By Proposition 1 (b) of Section 3, $S\mu$ is cyclic iff $1 \in [S\mu]$. This suggests that we try to express the distance in B from 1 to $S\mu$ in terms of properties of μ.

Notation. For the rest of the paper:

$$C = \beta/27\kappa \quad \text{and} \quad N = \max(2, 4^{1/9C})$$

where β is the constant in (B6) and κ is the exponent in the Corona Theorem.

FIRST DISTANCE ESTIMATE. *Suppose* $n \geq N$ *is a fixed positive integer, and*

$$\omega_\mu\left(\frac{1}{n}\right) \leq C\,\frac{\log n}{n} \quad .$$

Then there exists $g \in H^\infty$ *with* $\|g\|_\infty \leq n^{\beta/3}$, *and*

$$\|1 - gS_\mu\|_B \leq n^{-2\beta/3} \quad .$$

Before giving the proof, let us note the following consequence, due to H. S. Shapiro (1967).

WEAK SUFFICIENT CONDITION FOR CYCLICITY ([12; Theorem 1]). If $\omega_\mu(\delta) = 0\,(\delta \log \frac{1}{\delta})$ as $\delta \to 0+$, then $S\mu$ is cyclic.

Since there exist singular measures with modulus of continuity $0(\delta \log \frac{1}{\delta})$, we see (for the first time) that at least some inner functions are cyclic in B.

Proof of First Distance Estimate. The idea is to obtain a lower bound for the quantity $|S\mu| + |z^n|$, and then apply the Corona Theorem (for two functions) with $f_1 = S\mu$, $f_2 = z^n$.

By Corollary 6.1. we have $|S\mu(z)| \geq n^{-9C}$ for $|z| \leq 1 - \frac{1}{n}$; and for $1 - \frac{1}{n} \leq |z| < 1$ we have, since $n \geq N$:

$$|z^n| \geq (1 - \tfrac{1}{n})^n \geq (1 - \tfrac{1}{2})^2 \geq n^{-9C} ,$$

hence

$$|S\mu(z)| + |z^n| \geq n^{-9C} \qquad (z \text{ in } U) .$$

We apply the Corona Theorem with $\delta = n^{-9C}$ (which is $\leq \frac{1}{4}$ because $n \geq N \geq 4^{1/9C}$), and obtain $g_1, g_2 \in H^\infty$ with $\|g_i\|_\infty \leq n^{9\kappa C}$ $(i = 1,2)$, and $g_1 S\mu + z^n g_2 \equiv 1$ on U. Thus:

$$\begin{aligned} \|1 - g_1 S\mu\|_B &= \|z^n g_2\|_B \\ &\leq \|z^n\|_B \, \|g_2\|_\infty && \text{(by (B3))} \\ &\leq n^{-\beta} n^{9\kappa C} && \text{(by (B6))} \\ &= n^{2\beta/3} && \text{(choice of } C) . \end{aligned}$$

The key to Roberts' proof of Theorem S is the following improvement of the First Distance Estimate. For notational convenience, whenever (n_i) is a finite or infinite sequence of positive integers, write

$$D[(n_i)] \overset{\text{def}}{=} \frac{1}{n_1^{2\beta/3}} + \sum_{i \geq 2} \left(\frac{n_1 \cdots n_{i-1}}{n_i^2}\right)^{\beta/3} .$$

<u>SECOND DISTANCE ESTIMATE</u>. <u>Suppose</u> μ <u>can be written as a finite or infinite sum</u>

$$\mu = \sum_i \mu_i$$

<u>where each</u> μ_i <u>satisfies the smoothness condition</u>

$$\omega_{\mu_i}\left(\frac{1}{n_i}\right) \leq C \, \frac{\log n_i}{n_i}$$

<u>and each</u> $n_i \geq N$. <u>Then</u>

$$\text{dist}\,(1, [S\mu]) \leq D[(n_i)] \quad .$$

<u>Proof</u>. (a) First suppose μ is decomposed into a finite sum: $\mu = \sum_{i=1}^{K} \mu_i$. We proceed by induction. Since the polynomials are B -dense in H^∞ (Lemma 2 of section 3), the case $K = 1$ follows immediately from the First Distance Estimate. Suppose the result is true for some $K \geq 1$. By the First Distance Estimate, there exists $g_1 \in H^\infty$ with $\|g_1\|_\infty \leq n^{\beta/3}$, such that

$$\|1 - g_1 S\mu_1\|_B \leq n_1^{-2\beta/3} \quad .$$

By the induction hypothesis there exists $g \in H^\infty$ such that

$$\| g S_{\mu_2 + \cdots + \mu_{K+1}} - 1 \|_B \le D[(n_2, n_3, \cdots, n_{K+1})] .$$

Thus:

$$\begin{aligned} \| g_1 g S_\mu - 1 \|_B &= \| (g_1 S_{\mu_1})(g S_{\mu_2 + \cdots + \mu_{K+1}} - 1) + (g_1 S_{\mu_1} - 1) \|_B \\ &\le \| g_1 S_{\mu_1} \|_\infty \| g S_{\mu_2 + \cdots + \mu_{K+1}} - 1 \|_B + \| g_1 S_{\mu_1} - 1 \|_B \\ &\le n_1^{\beta/3} D[(n_2, \cdots, n_{K+1})] + n_1^{-2\beta/3} \\ &= D[(n_1, n_2, \cdots, n_{K+1})] \end{aligned}$$

as desired.

(b) Suppose μ is decomposed into an infinite series $\mu = \sum \mu_i$. Let $\nu_k = \mu_1 + \cdots + \mu_k$, so $\nu_k \uparrow \mu$. Then by the "continuity of distance" lemma 6.1.2 we have:

$$\begin{aligned} \operatorname{dist}_B (1, [S\mu]) &= \lim_k \operatorname{dist}_B (1, [S\mu_k]) \\ &\le \lim_k D[(n_1, \cdots, n_k)] \\ &= D[(n_1, n_2, \cdots)] \end{aligned}$$

QED

6.3. Smoothly decomposable measures. We call μ "smoothly decomposable" if for each $\varepsilon > 0$ there exists a decomposition of μ into a finite or infinite sum $\mu = \sum \mu_i$, where each μ_i satisfies

(i) $\omega_{\mu_i}(\frac{1}{n_i}) \le C \frac{\log n_i}{n_i}$,

and

(ii) $D[(n_i)] < \varepsilon$, $n_i \geq N$ (all i) .

Here C and N are the constants of sec. 6.2. The second Distance estimate immediately yields:

STRONG SUFFICIENT CONDITION FOR CYCLICITY. If μ is smoothly decomposable, then $S\mu$ is cyclic.

In the next section we will show that any μ which is not smoothly decomposable places mass on some Carleson set. Thus: $S\mu$ not cyclic $\Longrightarrow \mu$ not smoothly decomp. $\Longrightarrow \mu(K) > 0$ for some Carleson K. This will complete the proof of Theorem S.

6.4. Completion of Proof of Theorem S. (a) Suppose μ is not smoothly decomposable. Then $\exists\, \varepsilon > 0$ such that whenever $\mu = \sum \mu_i$ is a decomposition of μ into a finite or infinite sum, and whenever (n_i) is a corresponding set of integers satisfying 6.3 (ii), then some μ_i fails to satisfy 6.3 (i).

(b) We must locate a Carleson set $K \subset \mathbb{T}$ with $\mu(K) > 0$. A simple calculation shows that if i_0 is a sufficiently large positive integer, then the sequence

$$n_i = 2^{2^{i_0+i}} \qquad (i = 1, 2, \cdots)$$

satisfies 6.3 (ii), with ε as in the last paragraph. (This is where we need $\underline{\underline{n_i^2}}$ in the denominator of the $i^{\underline{th}}$ term of the sum defining $D[(n_i)]$). Let $\mathcal{P}_i$ be a partition of $\mathbb{T}$ into n_i equal closed subarcs, done so that $\mathcal{P}_{i+1}$ refers $\mathcal{P}_i$ (possible since $n_i | n_{i+1}$).

(c) Separate the intervals of $\mathcal{P}_1$ into two classes: call $I \in \mathcal{P}_1$

$$\underline{\text{light}} \text{ if } \mu(I) \le \frac{C}{2} \frac{\log n_1}{n_1}$$

and

$$\underline{\text{heavy}} \text{ if } \mu(I) > \frac{C}{2} \frac{\log n_1}{n_1} .$$

For E a Borel subset of $I \in \mathcal{P}_1$, define

$$\mu_1(E) = \begin{cases} \mu(E) & \text{if } I \text{ is "light"} \\ \dfrac{\mu(E)}{\mu(I)} \dfrac{C}{2} \dfrac{\log n_1}{n_1} & \text{if } I \text{ is "heavy"} \end{cases}$$

Then μ_1 is a measure with these properties:

(i) $\mu - \mu_1$ is concentrated on the union of the heavy intervals in $\mathcal{P}_1$.

(ii) $\mu_1(I) = \frac{C}{2} \frac{\log n_1}{n_1}$ for each heavy $I \in \mathcal{P}_1$.

(iii) $\omega_{\mu_1}(\frac{1}{n_1}) \le C \frac{\log n_1}{n_1}$.

The last inequality holds because $\mu(I) \le \frac{C}{2} \frac{\log n_1}{n_1}$ for each $I \in \mathcal{P}_1$, and any arc J of length $\le \frac{1}{n_1}$ lies in the union of at most two adjacent arcs in $\mathcal{P}_1$. By (iii) we cannot have $\mu = \mu_1$: otherwise μ would satisfy 6.3 (i) with the "one-term decomposition" $\mu = \mu_1$, $(n_j) = n_1$. Thus

(iv) $\mu - \mu_1 \ne 0$.

We continue by induction. Having obtained $\mu_1, \cdots, \mu_{K-1}$; define μ_K

exactly as μ_1, but with $\mathcal{P}_K$ replacing $\mathcal{P}_1$, n_K replacing n_1, and $\mu - (\mu_1 + \cdots + \mu_{K-1})$ replacing μ. Let H_K denote the union of the heavy intervals of $\mathcal{P}_K$. Then just as above:

(i') $\quad \mu - (\mu_1 + \cdots + \mu_K)$ is concentrated on H_K,

(ii') $\quad \mu_K(I) = \frac{C}{2} \frac{\log n_K}{n_K} \quad \forall$ heavy $I \in \mathcal{P}_K$,

(iii') $\quad \omega_{\mu_K}(\frac{1}{n_K}) \le C \frac{\log n_K}{n_K}$,

(iv') $\quad \mu - (\mu_1 + \cdots + \mu_K) \neq 0$.

(d) Let $K = \bigcap_k H_k$. We will show that K is the set we are looking for. Since $H_1 \supset H_2 \supset \cdots$ (by (i')), and since H_k is closed for each k (being a finite union of closed intervals), it is clear that K is a closed, nonvoid subset of $\mathbb{T}$. Let $\nu = \sum \mu_k$. Then $\nu \neq 0$ and ν is concentrated on K, by (i'). So $\nu(K) > 0$.

<u>$\mu(K) > 0$</u>. To see this, observe that $\mu \neq \nu$, for otherwise we would have an infinite sum decomposition of μ satisfying 6.3 (i); and our hypothesis is that this cannot happen (note that the "continuity of distance" Proposition 4 of section 3 lurks behind this argument). Thus $\mu(K) \ge \nu(K) > 0$.

<u>$m(K) = 0$</u>. Rephrasing (ii') above: $\mu_k(I) = \frac{C}{2} m(I) \log n_k$ for each heavy $I \in \mathcal{P}_k$. Summing over those I's:

$$\|\mu_k\| \ge \mu_k(H_k) = \frac{C}{2} m(H_k) \log n_k \ .$$

Thus:

$$(*) \qquad \infty > \|\mu\| \geq \sum \|\mu_k\| \geq \frac{C}{2} \sum m(H_k) \log n_k \ .$$

In particular, $m(K) = \lim_k m(H_k) = 0$.

<u>K is a Carleson set</u>. Let L_k denote the union of the <u>interiors</u> of those light intervals of ρ_k which lie in H_{k-1}. Then the sets $\{L_k\}$ are pairwise disjoint, each is a finite disjoint union of open intervals, and $K' = \mathbb{T} \setminus \bigcup_k L_k$ is a compact subset of $\mathbb{T}$ which contains K. Now a point belongs to $K' \setminus K$ only if it is an endpoint of adjacent light intervals of some ρ_k, so $K' \setminus K$ is countable; hence by the last paragraph, $m(K') = 0$.

We claim that K' is a Carleson set, hence so is K (by "Example" (b), section 4.1). In view of the last paragraph, this amounts to showing that

$$\sum m(L_k) \log n_k < \infty \ .$$

Since $L_k \subset H_{k-1}$ (by the construction of μ_k) it is enough to show

$$\sum m(H_{k-1}) \log n_k < \infty \ .$$

But this is immediate from (*) above, and the fact that $\log n_k / \log n_{k-1} = 2$. The proof of Theorem S is now complete.

7. FACTORIZATION OF INNER FUNCTIONS. Throughout this section we assume that B satisfies all of the conditions (B1) - (B6). As usual, μ is a positive, finite Borel measure on $\mathbb{T}$, singular with respect to Lebesgue measure.

7.1. B-factorization. Call an inner function q B-inner if it divides every inner function in the invariant subspace [q]. For example, Blaschke products are B-inner, but inner functions with non-trivial cyclic factors are not. Our goal is to characterize B-inner functions, and to factor every inner function into B-inner and B-cyclic parts. The following result, due to Pat Ahern, is the first step.

THEOREM 1. *A singular inner function* S_μ *is* B*-inner if and only if* μ *is concentrated on a countable union of Carleson sets.*

The proof requires a preliminary result, also observed by Ahern.

PROPOSITION 2. Suppose μ is concentrated on a Carleson set and $f \in [S_\mu] \cap H^1$. Then S_μ divides f.

Proof. This is an adaptation of the proof of Theorem N. Suppose K is the Carleson set. Choose m and φ as in the proof of Theorem N (section 4). For $n > 0$ define g_n on π by

$$g_n(\omega) = \begin{cases} \overline{\omega^n \varphi(\omega)} \quad S_\mu(\omega) & (\omega \in \pi \setminus K) \\ 0 & (\omega \in K) . \end{cases}$$

As before, $g_n \in C^{(m)}(\mathbb{T})$ and the linear functional $\lambda_n = \lambda_{g_n}$ is continuous on B, non-trivial, and $\perp [S\mu]$. Now suppose $f \in H^1$ and $\lambda_n(f) = 0$ for all $n \geq 1$. Then

$$0 = \lambda_n(f) = \int_{\mathbb{T}} f\bar{g}\, dm$$

$$= \int_{\mathbb{T}} f(\omega)\, \varphi(\omega)\, \overline{S\mu(\omega)}\, \omega^n\, dm(\omega) \quad .$$

So $h(\omega) = f(\omega)\, \varphi(\omega)\, \overline{S\mu(\omega)}$ $(\omega \in \mathbb{T})$ coincides $[m]$ a.e. with the radial limit function of an H^1 function h, and $hS\mu = f\varphi$, both a.e. on $\mathbb{T}$ and identically on U. Because φ is outer, $S\mu \mid f$, as desired.

<u>Proof of Theorem 1</u>. Suppose μ is concentrated on a countable union of Carleson sets, say on $\bigcup_j K_j$. We may assume $K_1 \subset K_2 \subset \cdots$ because a finite union of Carleson sets is a Carleson set (perhaps the easiest way to see this is to think of Carleson sets as zero sets of smooth holomorphic functions). For fixed j define the singular measure μ_j by

$$\mu_j(E) = \mu(E \cap K_j)$$

for E a Borel subset of $\mathbb{T}$. Then $[S\mu_j] \supset [S\mu]$ (by Prop. 3 (a), section 3), and $\mu_j \uparrow \mu$.

Suppose q is an inner function in $[S\mu]$. Then for each j, $q \in [S\mu_j]$, so by Proposition 2, $q = q_j S\mu_j$ where q_j is inner. Since $\mu_j \uparrow \mu$, $S\mu_j \to S\mu$ uniformly on compact subsets of U. By normal families $q_j \to q_0$ uniformly on compact subsets of U, so q_0 is holomorphic on U and $q = q_0 S\mu$ (so necessarily q_0 is inner). Thus $S\mu$ is B-inner.

Conversely suppose μ is <u>not</u> concentrated on a countable union of Carleson sets. Let

$$\gamma = \sup \{\mu(K) : \ K \text{ a Carleson set}\}$$

and choose Carleson sets (K_j) so that $K_1 \subset K_2 \subset \cdots$ and $\mu(K_j) \uparrow \gamma$. Let $H = \bigcup_j K_j$ and set

$$\mu_0(E) = \mu(E \cap H)$$

for each Borel subset E of $\mathbb{T}$. Then μ_0 is concentrated on H, so $\mu_1 = \mu - \mu_0 \neq 0$ and clearly μ_1 gives no mass to any Carleson set (otherwise the definition of γ is violated). So $S\mu_1$ is cyclic by Theorem S, hence $P_n S\mu_1 \to 1$ in B for some sequence (P_n) of polynomials. By (B3), $P_n S\mu \to S\mu_0$, so $S\mu_0 \in [S\mu]$, but $S\mu \not| S\mu_0$. Thus $S\mu$ is not B-inner.

Remark. In the last paragraph of the above proof we showed that each singular measure μ can be written $\mu = \mu_0 + \mu_1$ where μ_0 is concentrated on a countable union of Carleson sets (so $S\mu_0$ is B-inner) and μ_1 gives no mass to any Carleson set (so $S\mu_1$ is B-cyclic). More generally we have:

THEOREM 2. (a) Every inner function q can be factored as $q = q_i q_c$ where q_i is B-inner and q_c is cyclic. The factorization is unique up to a multiplicative unimodular constant.

(b) More explicitly, $q_i = b S\mu_i$ where b is a Blaschke product or a constant and μ_i is concentrated on a countable union of Carleson sets; while $q_c = S\mu_c$ where μ_c gives no mass to any Carleson set.

Proof. Write $q = b S\mu$ where b is a Blaschke product or a constant, and $S\mu$ a singular inner function. We know that $\mu = \mu_i + \mu_c$ as in (b) above, so it is enough to show that $b S\mu_i$ is B-inner, and to check uniqueness.

$bS\mu_i$ is B-inner. Suppose $q \in [bS\mu_i]$. Then $q = bq_0$ where q_0 is inner, so we need only show $S\mu_i | q_0$. But $q \in [S\mu_i]$ (which contains $[bS\mu_i]$), and $S\mu_i$ is B-inner, so $S\mu_i | q$, by Proposition 2 of this section. Thus $S\mu_i | q_0$, as desired.

Uniqueness. Suppose $q = I_1 C_1 = I_2 C_2$ where q is inner, the I's are B-inner, and the C's are B-cyclic. Then $[q] = [I_1] = [I_2]$ so I_1 and I_2 divide each other (by the definition of "B-inner"). Thus I_1 is a constant (unimodular) multiple of I_2, hence the same is true of C_1 and C_2.

Remark. Part of statement (a) was obtained independently by Daniel Leucking without using the characterization of cyclic inner functions.

7.2. Weak factorization in H^p $(0 < p < 1)$. In 1969 Duren, Romberg, and Shields observed that when $0 < p < 1$ some inner functions q generate invariant subspaces qH^p which, although proper and closed, are weakly dense (that is, they are annihilated by no nontrivial continuous linear functional). Of course this phenomenon is associated with (and implies) the fact that when $0 < p < 1$ the space H^p is not locally convex.

Duren, Romberg, and Shields also observed the following: the weak closure $[qH^p]_w$ of qH^p is invariant under multiplication by z, and closed (since weakly closed) in H^p. By Beurling's Theorem, which holds even if $0 < p < 1$, we have $[qH^p]_w = q_w H^p$. So $q = q_w q_0$, where q_w is called the weak inner factor of q. This leads to the conjecture that q_0 is "weakly outer", i.e. $[q_0 H^p]_w = H^p$. Duren, Romberg, and Shields also asked if these notions of weak inner and weak outer depended on $p \in (0,1)$. The results of section 1 complete this picture.

THEOREM. (a) q is weakly inner iff it is B-inner.

(b) q is weakly outer iff it is B-cyclic.

(c) Each inner function q has (essentially) unique factorization $q = q_i q_0$ where q_i is weakly inner and q_0 is weakly outer.

Remark. In particular, this result shows that the concepts of weak inner and weak outer do not depend on $p \in (0,1)$.

Proof. According to [4; Theorem 7], the space $B = A^1_{\frac{1}{p}-2}$ contains H^p as a dense subspace and has the "same dual" in the sense that a linear functional on H^p is continuous if and only if it has a continuous linear extension to B. This immediately yields part (b) of the Theorem.

To prove (a), first suppose q is weakly inner. Then $H^p \cap [q]_B = qH^p$, so $q_1 \in [q]_B \Longrightarrow q_1 \in qH^p \Longrightarrow q | q_1$. Thus q is B-inner. Conversely, suppose q is B-inner. Then since $[qH^p]_w = q_w H^p$ we have $q_w \in [q]_B$, hence $q | q_w$. But $q_w | q$ trivially (since $q \in [qH^p]_w$), so q_w is a constant multiple of q, hence q is weak inner. This proves part (a).

Part (c) is an immediate consequence of the factorization Theorem of section 7.1.

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