

Eigenfunctions for hyperbolic composition operators—redux

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Abstract. The Invariant Subspace Problem (“ISP”) for Hilbert space operators is known to be equivalent to a question that, on its surface, seems surprisingly concrete: *For composition operators induced on the Hardy space H^2 by hyperbolic automorphisms of the unit disc, is every nontrivial minimal invariant subspace one dimensional (i.e., spanned by an eigenvector)?* In the hope of reviving interest in the contribution this remarkable result might offer to the studies of both composition operators and the ISP, I revisit some known results, weaken their hypotheses and simplify their proofs. *Sample results:* If φ is a hyperbolic disc automorphism with fixed points at α and β (both necessarily on the unit circle), and C_φ the composition operator it induces on H^2 , then for every $f \in \sqrt{(z - \alpha)(z - \beta)} H^2$, the doubly C_φ -cyclic subspace generated by f contains many independent eigenvectors; more precisely, the point spectrum of C_φ ’s restriction to that subspace intersects the unit circle in a set of positive measure. Moreover, this restriction of C_φ is hypercyclic (some forward orbit is dense). Under the stronger restriction $f \in \sqrt{(z - \alpha)(z - \beta)} H^p$ for some $p > 2$, the point spectrum of the restricted operator contains an open annulus centered at the origin.

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1. Introduction

More than twenty years ago Nordgren, Rosenthal, and Wintrobe [8] made a surprising connection between composition operators on the Hardy space H^2 and the Invariant Subspace Problem—henceforth, the “ISP”. The ISP asks if every operator on a separable Hilbert space has a nontrivial invariant subspace (following tradition: “operator” means “bounded linear operator,” “subspace” means “closed

linear manifold,” and for a subspace, “nontrivial” means “neither the whole space nor the zero-subspace”). Nordgren, Rosenthal, and Wintrobe proved the following [8, Corollary 6.3, page 343]:

Suppose φ is a hyperbolic automorphism of the open unit disc \mathbb{U} . Let C_φ denote the composition operator induced by φ on the Hardy space H^2 . Then the ISP has a positive solution if and only if every nontrivial minimal C_φ -invariant subspace of H^2 has dimension one.

It is easy to see that, for each nontrivial minimal invariant subspace V of a Hilbert space operator T , every non-zero vector $x \in V$ is cyclic, i.e., $\text{span}\{T^n x : n = 0, 1, 2, \dots\}$ is dense in V . If, in addition, T is invertible, then so is its restriction to V (otherwise the range of this restriction would be a nontrivial invariant subspace strictly contained in V , contradicting minimality). Thus for T invertible, V a nontrivial minimal invariant subspace of T , and $0 \neq x \in V$,

$$V = \overline{\text{span}}\{T^n x : n = 0, 1, 2, \dots\} = \overline{\text{span}}\{T^n x : n \in \mathbb{Z}\},$$

where now “ $\overline{\text{span}}$ ” means “closure of the linear span.”

The result of Nordgren, Rosenthal, and Wintrobe therefore suggests that for φ a hyperbolic disc automorphism we might profitably study how the properties of a function f in $H^2 \setminus \{0\}$ influence the operator-theoretic properties of $C_\varphi|_{D_f}$, the restriction of C_φ to the “doubly cyclic” subspace

$$D_f := \overline{\text{span}}\{C_\varphi^n f : n \in \mathbb{Z}\} = \overline{\text{span}}\{f \circ \varphi_n : n \in \mathbb{Z}\}, \quad (1.1)$$

with particular emphasis on the question of when the point spectrum of the restricted operator is nonempty. (Here, for n is a positive integer, φ_n denotes the n -th compositional iterate of φ , while φ_{-n} is the n -th iterate of φ^{-1} ; φ_0 is the identity map.)

Along these lines, Valentin Matache [4, 1993] obtained a number of interesting results on minimal invariant subspaces for hyperbolic-automorphically induced composition operators. He observed, for example, that if a minimal invariant subspace for such an operator were to have dimension larger than 1, then, at either of the fixed points of φ , none of the non-zero elements of that subspace could be both continuous and non-vanishing (since φ is a hyperbolic automorphism of the unit disc, its fixed points must necessarily lie on the unit circle; see §2.1 below). Matache also obtained interesting results on the possibility of minimality for invariant subspaces generated by inner functions.

Several years later Vitaly Chkliar [3, 1996] proved this result for hyperbolic-automorphic composition operators C_φ :

If $f \in H^2 \setminus \{0\}$ is bounded in a neighborhood of one fixed point of φ , and at the other fixed point vanishes to some order $\varepsilon > 0$, then the point spectrum of $C_\varphi|_{D_f}$ contains an open annulus centered at the origin.

Later Matache [5] obtained similar conclusions under less restrictive hypotheses.

In the work below, after providing some background (in §2), I revisit in §3 and §4 the work of Chkliar and Matache, providing simpler proofs of stronger

results. Here is a sample: for φ a hyperbolic automorphism of \mathbb{U} with fixed points α and β (necessarily on $\partial\mathbb{U}$):

- (a) If $f \in \sqrt{(z - \alpha)(z - \beta)} H^2 \setminus \{0\}$, then $\sigma_p(C_\varphi|_{D_f})$ intersects the unit circle in a set of positive measure.
- (b) If $f \in \sqrt{(z - \alpha)(z - \beta)} H^p \setminus \{0\}$ for some $p > 2$, then $\sigma_p(C_\varphi|_{D_f})$ contains an open annulus centered at the origin.

Note that the function $\sqrt{(z - \alpha)(z - \beta)}$ is an outer function, so the set of functions f being singled out in both parts (a) and (b) is dense in H^2 .

Finally, observe that, in the hypotheses of both (a) and (b), the exponent “ $\frac{1}{2}$ ” is best possible in the sense that for any smaller exponent the function $f \equiv 1$, for which D_f is the one dimensional subspace of constant functions, would satisfy both hypotheses. This comment applies throughout the sequel.

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2. Background material

2.1. Disc automorphisms

An *automorphism* of a domain in the complex plane is a univalent holomorphic mapping of that domain onto itself. Every automorphism of the open unit disc \mathbb{U} is a linear fractional map [9, Theorem 12.6, page 255].

Linear fractional maps can be regarded as homeomorphisms of the Riemann Sphere; as such, each one that is not the identity map has one or two fixed points. The maps with just one fixed point are the *parabolic* ones; each such map is conjugate, via an appropriate linear fractional map, to one that fixes only the point at infinity, i.e., to a translation. A linear fractional map that fixes two distinct points is conjugate, again via a linear fractional map, to one that fixes both the origin and the point at infinity, i.e., to a dilation $w \rightarrow \mu w$ of the complex plane, where $\mu \neq 1$ is a complex number called the *multiplier* of the original map (actually $1/\mu$ can just as well occur as the multiplier—depending on which fixed point of the original map is taken to infinity by the conjugating transformation). The original map is called *elliptic* if $|\mu| = 1$, *hyperbolic* if μ is positive, and *loxodromic* in all other cases (see, for example, [10, Chapter 0] for more details).

Suppose φ is a hyperbolic automorphism of \mathbb{U} . Then the same is true of its inverse. The fixed points of φ must necessarily lie on $\partial\mathbb{U}$, the unit circle. Indeed, if the attractive fixed point of φ lies outside the closed unit disc, then the compositional iterates of φ pull \mathbb{U} toward that fixed point, and hence outside of \mathbb{U} , which contradicts the fact that $\varphi(\mathbb{U}) = \mathbb{U}$. If, on the other hand, the attractive fixed point lies in \mathbb{U} , then its reflection in the unit circle is the repulsive fixed point, which is the attractive one for φ^{-1} . Thus φ^{-1} can't map \mathbb{U} into itself, another contradiction. Conclusion: both fixed points lie on $\partial\mathbb{U}$.

Let's call a hyperbolic automorphism φ of \mathbb{U} *canonical* if it fixes the points ± 1 , with $+1$ being the attractive fixed point. We'll find it convenient to move between the open unit disc \mathbb{U} and the open right half-plane Π^+ by means of the *Cayley transform* $\kappa : \Pi^+ \rightarrow \mathbb{U}$ and its inverse $\kappa^{-1} : \mathbb{U} \rightarrow \Pi^+$, where

$$\kappa(w) = \frac{w-1}{w+1} \quad \text{and} \quad \kappa^{-1}(z) = \frac{1+z}{1-z} \quad (z \in \mathbb{U}, w \in \Pi^+).$$

In particular, if φ is a canonical hyperbolic automorphism of \mathbb{U} , then $\Phi := \kappa^{-1} \circ \varphi \circ \kappa$ is an automorphism of Π^+ that fixes 0 and ∞ , with ∞ being the attractive fixed point. Thus $\Phi(w) = \mu w$ for some $\mu > 1$, and $\varphi = \kappa \circ \Phi \circ \kappa^{-1}$, which yields, after a little calculation,

$$\varphi(z) = \frac{r+z}{1+rz} \quad \text{where} \quad \varphi(0) = r = \frac{\mu-1}{\mu+1} \in (0, 1). \quad (2.1)$$

If φ is a hyperbolic automorphism of \mathbb{U} that is not canonical, then it can be conjugated, via an appropriate automorphism of \mathbb{U} , to one that is. This is perhaps best seen by transferring attention to the right half-plane Π^+ , and observing that if $\alpha < \beta$ are two real numbers, then the linear fractional map Ψ of Π^+ defined by

$$\Psi(w) = i \frac{w - i\beta}{w - i\alpha}$$

preserves the imaginary axis, and takes the point 1 into Π^+ . Thus it is an automorphism of Π^+ that takes the boundary points $i\beta$ to zero and $i\alpha$ to infinity. Consequently if Φ is any hyperbolic automorphism of Π^+ with fixed points $i\alpha$ (attractive) and $i\beta$ (repulsive), then $\Psi \circ \Phi \circ \Psi^{-1}$ is also hyperbolic automorphism with attractive fixed point ∞ and repulsive fixed point 0 . If, instead, $\alpha > \beta$ then $-\Psi$ does the job.

Since any hyperbolic automorphism φ of \mathbb{U} is conjugate, via an automorphism, to a canonical one, C_φ is similar, via the composition operator induced by the conjugating map, to a composition operator induced by a canonical hyperbolic automorphism. For this reason the work that follows will focus on the canonical case.

2.2. Spectra of hyperbolic-automorphic composition operators

Suppose φ is a hyperbolic automorphism of \mathbb{U} with multiplier $\mu > 1$. Then it is easy to find lots of eigenfunctions/eigenvalues for C_φ on H^2 . We may without loss of generality assume that φ is canonical, and then move, via the Cayley map, to the right half-plane where φ morphs into the dilation $\Phi(w) = \mu w$. Let's start by viewing the composition operator C_Φ as just a linear map on $\text{Hol}(\Pi^+)$, the space of all holomorphic functions on Π^+ . For any complex number a define $E_a(w) = w^a$, where $w^a = \exp(a \log w)$, and "log" denotes the principal branch of the logarithm. Then $E_a \in \text{Hol}(\Pi^+)$ and $C_\Phi(E_a) = \mu^a E_a$, i.e., E_a is an eigenvector of C_Φ (acting on $\text{Hol}(\Pi^+)$) and the corresponding eigenvalue is μ^a (again taking the principal value of the " a -th power"). Upon returning via the Cayley map to the unit disc, we see that, when viewed as a linear transformation of $\text{Hol}(\mathbb{U})$, the operator C_φ

has, for each $a \in \mathbb{C}$, the eigenvector/eigenvalue combination (f_a, μ^a) , where the function

$$f_a(z) = \left(\frac{1+z}{1-z} \right)^a \quad (z \in \mathbb{U}) \quad (2.2)$$

belongs to H^2 if and only if $|\operatorname{Re}(a)| < 1/2$. Thus the corresponding H^2 -eigenvalues μ^a cover the entire open annulus

$$A := \{ \lambda \in \mathbb{C} : \frac{1}{\sqrt{\mu}} < |\lambda| < \sqrt{\mu} \}. \quad (2.3)$$

In particular $\sigma(C_\varphi)$, the H^2 -spectrum of C_φ , contains this annulus, and since the map $a \rightarrow \mu^a$ takes the strip $|\operatorname{Re}(a)| < 1/2$ infinitely-to-one onto A , each point of A is an eigenvalue of C_φ having infinite multiplicity.

As for the rest of the spectrum, an elementary norm calculation shows that $\sigma(C_\varphi)$ is just the closure of A_μ . To see this, note first that the change-of-variable formula from calculus shows that for each $f \in H^2$ and each automorphism φ of \mathbb{U} (not necessarily hyperbolic):

$$\|C_\varphi f\|^2 = \int_{\partial\mathbb{U}} |f|^2 P_a \, dm \quad (2.4)$$

where m is normalized arc-length measure on the unit circle $\partial\mathbb{U}$, and P_a is the Poisson kernel for $a = \varphi(0)$; more generally, for any $a \in \mathbb{U}$:

$$P_a(\zeta) = \frac{1 - |a|^2}{|\zeta - a|^2} \quad (\zeta \in \partial\mathbb{U}) \quad (2.5)$$

(see also Nordgren's neat argument [7, Lemma 1, page 442], which shows via Fourier analysis that (2.4) holds for any inner function).

Now suppose φ is the canonical hyperbolic automorphism of \mathbb{U} with multiplier $\mu > 1$. Then φ is given by (2.1), so by (2.5)

$$P_r(\zeta) = \frac{1 - r^2}{|\zeta - r|^2} \leq \frac{1 + r}{1 - r} = \mu$$

which, along with (2.4) shows that

$$\|C_\varphi\| \leq \sqrt{\mu}. \quad (2.6)$$

Since also

$$P_r(\zeta) \geq \frac{1 - r}{1 + r} = \mu^{-1}$$

we have, for each $f \in H^2$

$$\|C_\varphi f\| \geq \frac{1}{\sqrt{\mu}} \|f\|,$$

which shows that (2.6) holds with C_φ replaced by C_φ^{-1} . Thus the spectra of both C_φ and its inverse lie in the closed disc of radius $\sqrt{\mu}$ centered at the origin, so by the spectral mapping theorem, $\sigma(C_\varphi)$ is contained in the closure of the annulus (2.3). Since we have already seen that this closed annulus contains the spectrum

of C_φ we've established the following result, first proved by Nordgren [7, Theorem 6, page 448] using precisely the argument given above:

Theorem 2.1. *If φ is a hyperbolic automorphism of \mathbb{U} with multiplier μ (> 1), then $\sigma(C_\varphi)$ is the closed annulus $\{\lambda \in \mathbb{C} : 1/\sqrt{\mu} \leq |\lambda| \leq \sqrt{\mu}\}$. The interior of this annulus consists entirely of eigenvalues of C_φ , each having infinite multiplicity.*

In fact the interior of $\sigma(C_\varphi)$ is precisely the point spectrum of C_φ ; see [6] for the details.

2.3. Poisson kernel estimates

Formula (2.5), giving the Poisson kernel for the point $a = \rho e^{i\theta_0} \in \mathbb{U}$, can be rewritten

$$P_a(e^{i\theta}) = \frac{1 - \rho^2}{1 - 2\rho \cos(\theta - \theta_0) + \rho^2} \quad (0 \leq \rho < 1, \theta \in \mathbb{R}).$$

We will need the following well-known estimate, which provides a convenient replacement (cf. for example [1, page 313]).

Lemma 2.2. *For $0 \leq \rho < 1$ and $|\theta| \leq \pi$:*

$$P_\rho(e^{i\theta}) \leq 4 \frac{(1 - \rho)}{(1 - \rho)^2 + (\theta/\pi)^2} \quad (2.7)$$

Proof.

$$P_\rho(e^{i\theta}) := \frac{1 - \rho^2}{1 - 2\rho \cos \theta + \rho^2} = \frac{1 - \rho^2}{(1 - \rho)^2 + \rho(2 \sin \frac{\theta}{2})^2} \leq \frac{2(1 - \rho)}{(1 - \rho)^2 + 4\rho(\theta/\pi)^2}$$

so, at least when $\rho \geq \frac{1}{4}$, inequality (2.7) holds with constant “2” in place of “4”. For the other values of ρ one can get inequality (2.7) by checking that, over the interval $[0, \pi]$, the minimum of the right-hand side exceeds the maximum of the left-hand side. \square

Remark 2.3. The only property of the constant “4” on the right-hand side of (2.7) that matters for our purposes is its independence of ρ and θ .

For the sequel (especially Theorem 3.5 below) we will require the following upper estimate of certain infinite sums of Poisson kernels.

Lemma 2.4. *For φ the canonical hyperbolic automorphism of \mathbb{U} with multiplier μ :*

$$\sum_{n=0}^{\infty} P_{\varphi_n(0)}(e^{i\theta}) \leq \frac{16\mu}{\mu - 1} \frac{\pi}{|\theta|} \quad (|\theta| \leq \pi). \quad (2.8)$$

In the spirit of Remark 2.3 above, the precise form of the positive constant that multiplies $\pi/|\theta|$ on the right-hand side of (2.8) is unimportant (as long as it does not depend on θ).

Proof. The automorphism φ is given by equations (2.1). For each integer $n \geq 0$ the n -th iterate φ_n of φ is just the canonical hyperbolic automorphism with multiplier μ^n , so upon substituting μ^n for μ in (2.1) we obtain

$$\varphi_n(z) = \frac{r_n + z}{1 + r_n z} \quad \text{where} \quad \varphi_n(0) = r_n = \frac{\mu^n - 1}{\mu^n + 1} \in (0, 1). \quad (2.9)$$

Thus $1 - r_n = 2/(\mu^n + 1)$, and so

$$\mu^{-n} < 1 - r_n < 2\mu^{-n} \quad (n = 0, 1, 2, \dots), \quad (2.10)$$

(in particular, r_n approaches the attractive fixed point $+1$ with exponential speed as $n \rightarrow \infty$; this is true of the φ -orbit of any point of the unit disc).

Fix $\theta \in [-\pi, \pi]$. We know from (2.7) and (2.10) that for each integer $n \geq 0$,

$$P_{r_n}(e^{i\theta}) \leq \frac{4(1 - r_n)}{(1 - r_n)^2 + (\theta/\pi)^2} \leq \frac{8\mu^{-n}}{\mu^{-2n} + (\theta/\pi)^2},$$

whereupon, for each non-negative integer N :

$$\begin{aligned} \frac{1}{8} \sum_{n=0}^{\infty} P_{r_n}(e^{i\theta}) &\leq \sum_{n=0}^{\infty} \frac{\mu^{-n}}{\mu^{-2n} + (\theta/\pi)^2} \\ &\leq \sum_{n=0}^{N-1} \frac{\mu^{-n}}{\mu^{-2n}} + \left(\frac{\pi}{\theta}\right)^2 \sum_{n=N}^{\infty} \mu^{-n} \\ &= \sum_{n=0}^{N-1} \mu^n + \left(\frac{\pi}{\theta}\right)^2 \sum_{n=N}^{\infty} \mu^{-n} \\ &= \frac{\mu^N - 1}{\mu - 1} + \left(\frac{\pi}{\theta}\right)^2 \mu^{-N} (1 - \mu^{-1})^{-1} \end{aligned}$$

where the geometric sum in the next-to-last line converges because $\mu > 1$.

We need a choice of N that gives a favorable value for the quantity in the last line of the display above. Let $\nu = \log_{\mu}(\pi/|\theta|)$, so that $\mu^{\nu} = \pi/|\theta|$. Since $|\theta| \leq \pi$ we are assured that $\nu \geq 0$. Let N be the least integer $\geq \nu$, i.e., the unique integer in the interval $[\nu, \nu + 1)$. The above estimate yields for any integer $N \geq 0$, upon setting $C := 8\mu/(\mu - 1)$ (which is > 0 since $\mu > 1$),

$$\sum_{n=0}^{\infty} P_{r_n}(e^{i\theta}) \leq C \left[\mu^{N-1} + \left(\frac{\pi}{\theta}\right)^2 \mu^{-N} \right] \leq C \frac{\pi}{|\theta|} \left[\frac{|\theta|}{\pi} \mu^{\nu} + \left(\frac{|\theta|}{\pi} \mu^{\nu}\right)^{-1} \right]. \quad (2.11)$$

By our choice of ν , both summands in the square-bracketed term at the end of (2.11) have the value 1 and this implies (2.8). \square

3. Main results

Here I extend work of Chkliar [3] and Matache [5] that provides, for a hyperbolic-automorphically induced composition operator C_{φ} , sufficient conditions on $f \in H^2$

for the doubly-cyclic subspace D_f , as defined by (1.1), to contain a rich supply of linearly independent eigenfunctions. I'll focus mostly on canonical hyperbolic automorphisms, leaving the general case for the next section. Thus, until further notice, φ will denote a canonical hyperbolic automorphism of \mathbb{U} with multiplier $\mu > 1$, attractive fixed point at $+1$ and repulsive one at -1 , i.e., φ will be given by equations (2.1).

Following both Chkliar and Matache, I will use an H^2 -valued Laurent series to produce the desired eigenvectors. The idea is this: for $f \in H^2$, and λ a non-zero complex number, if the series

$$\sum_{n \in \mathbb{Z}} \lambda^{-n} (f \circ \varphi_n) \quad (3.1)$$

converges strongly enough (for example, in H^2) then the sum F_λ , whenever it is not the zero-function, will be a λ -eigenfunction of C_φ that lies in D_f . Clearly the convergence of the series (3.1) will depend crucially on the behavior of φ at its fixed points, as the next result indicates. For convenience let's agree to denote by $A(R_1, R_2)$ the open annulus, centered at the origin, of inner radius R_1 and outer radius R_2 (where, of course, $0 < R_1 < R_2 < \infty$).

Theorem 3.1. (cf. [3]) *Suppose $0 < \varepsilon, \delta \leq 1/2$, and that*

$$f \in (z-1)^{\frac{1}{2}+\varepsilon} (z+1)^{\frac{1}{2}+\delta} H^2 \setminus \{0\}.$$

Then $\sigma_p(C_\varphi|_{D_f})$ contains, except possibly for a discrete subset, $A(\mu^{-\varepsilon}, \mu^\delta)$.

Proof. Our hypothesis on the behavior of f at the point $+1$ (the attractive fixed point of φ) is that $f = (z-1)^{\frac{1}{2}+\varepsilon} g$ for some $g \in H^2$, i.e., that

$$\infty > \int_{\partial\mathbb{U}} |g|^2 dm = \int_{\partial\mathbb{U}} \frac{|f(\zeta)|^2}{|\zeta-1|^{2\varepsilon+1}} dm(\zeta) \geq \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|f(e^{i\theta})|^2}{|\theta|^{2\varepsilon+1}} d\theta. \quad (3.2)$$

Upon setting $a = \varphi_n(0) := r_n$ in (2.4) we obtain

$$\|f \circ \varphi_n\|^2 = \int |f|^2 P_{r_n} dm, \quad (n \in \mathbb{Z}) \quad (3.3)$$

which combines with estimates (2.7) and (2.10) to show that if n is a non-negative integer (thus insuring that $r_n > 0$):

$$\begin{aligned} \|f \circ \varphi_n\|^2 &\leq 2\pi \int_{-\pi}^{\pi} |f(e^{i\theta})|^2 \frac{1-r_n}{(1-r_n)^2 + \theta^2} d\theta \\ &\leq 4\pi \int_{-\pi}^{\pi} |f(e^{i\theta})|^2 \frac{\mu^{-n}}{\mu^{-2n} + \theta^2} d\theta \\ &= 4\pi \mu^{-2n\varepsilon} \int_{-\pi}^{\pi} \frac{|f(e^{i\theta})|^2}{|\theta|^{1+2\varepsilon}} \left\{ \frac{(\mu^n |\theta|)^{1+2\varepsilon}}{1 + (\mu^n |\theta|)^2} \right\} d\theta \\ &\leq 4\pi \mu^{-2n\varepsilon} \sup_{x \in \mathbb{R}} \left\{ \frac{|x|^{1+2\varepsilon}}{1+x^2} \right\} \int_{-\pi}^{\pi} \frac{|f(e^{i\theta})|^2}{|\theta|^{1+2\varepsilon}} d\theta. \end{aligned}$$

By (3.2) the integral in the last line is finite, and since $0 < \varepsilon \leq 1/2$, the supremum in that line is also finite. Thus

$$\|f \circ \varphi_n\| = O(\mu^{-n\varepsilon}) \quad \text{as } n \rightarrow \infty,$$

which guarantees that the subseries of (3.1) with positively indexed terms converges in H^2 for all $\lambda \in \mathbb{C}$ with $|\lambda| > \mu^{-\varepsilon}$.

As for the negatively indexed subseries of (3.1), note from (2.1) that $\varphi^{-1}(z) = -\varphi(-z)$, so $\varphi_{-n}(z) = -\varphi_n(-z)$ for each integer n . Let $g(z) = f(-z)$, so our hypothesis on f implies that $g \in (z-1)^{\frac{1}{2}+\delta} H^2 \setminus \{0\}$. Let $\psi_n(z) = \varphi_n(-z)$ (the subscript on ψ does not now indicate iteration). Then for each positive integer n we have $\psi_n(0) = \varphi_n(0) = r_n$, hence:

$$\|f \circ \varphi_{-n}\|^2 = \|g \circ \psi_n\|^2 = \int_{\partial\mathbb{U}} |g|^2 P_{r_n} dm$$

so by the result just obtained, with g in place of f and ε replaced by δ ,

$$\|f \circ \varphi_{-n}\| = O(\mu^{-n\delta}) \quad \text{as } n \rightarrow \infty.$$

Thus the negatively indexed subseries of (3.1) converges in H^2 for all complex numbers λ with $|\lambda| < \mu^\delta$.

Conclusion: For each λ in the open annulus $A(\mu^{-\varepsilon}, \mu^\delta)$ the H^2 -valued Laurent series (3.1) converges in the norm topology of H^2 to a function $F_\lambda \in H^2$. Now F_λ , for such a λ , will be a C_φ -eigenfunction unless it is the zero-function, and—just as for scalar Laurent series—this inconvenience can occur for at most a discrete subset of points λ in the annulus of convergence (the relevant uniqueness theorem for H^2 -valued holomorphic functions follows easily from the scalar case upon applying bounded linear functionals). \square

Remark 3.2. Chkliar [3] has a similar result, where there are uniform conditions on the function f at the fixed points of φ (see also Remark 3.10 below); as he suggests, it would be of interest to know whether or not the “possible discrete subset” that clutters the conclusions of results like Theorem 3.1 can actually be nonempty.

Remark 3.3. The limiting case $\delta = 0$ of Theorem 3.1 still holds (see Theorem 3.6 below); it is a slight improvement on Chkliar’s result (see also the discussion following Theorem 3.6).

Remark 3.4. Note that the restriction $\varepsilon, \delta \leq 1/2$ in the hypothesis of Theorem 3.1 cannot be weakened since, as mentioned at the end of §2.2, the point spectrum of C_φ is the open annulus $A(\mu^{-\frac{1}{2}}, \mu^{\frac{1}{2}})$.

Here is a companion to Theorem 3.1, which shows that even in the limiting case $\delta = \varepsilon = 0$ (in some sense the “weakest” hypothesis on f) the operator $C_\varphi|_{D_f}$ still has a significant supply of eigenvalues.

Theorem 3.5. *If $f \in \sqrt{(z+1)(z-1)} H^2$ then $\sigma_p(C_\varphi|_{D_f})$ intersects $\partial\mathbb{U}$ in a set of positive measure.*

Proof. We will work in the Hilbert space $L^2(H^2, dm)$ consisting of H^2 -valued (m -equivalence classes of) measurable functions F on $\partial\mathbb{U}$ with

$$\|F\|^2 := \int_{\partial\mathbb{U}} \|F(\omega)\|^2 dm(\omega) < \infty.$$

I will show in a moment that the hypothesis on f implies

$$\sum_{n \in \mathbb{Z}} \|f \circ \varphi_n\|^2 < \infty. \quad (3.4)$$

Granting this, it is easy to check that the H^2 -valued Fourier series

$$\sum_{n \in \mathbb{Z}} (f \circ \varphi_n) \omega^{-n} \quad (\omega \in \partial\mathbb{U}) \quad (3.5)$$

converges unconditionally in $L^2(H^2, dm)$, so at least formally, we expect that for a.e. $\omega \in \partial\mathbb{U}$ we'll have $C_\varphi(F(\omega)) = \omega F(\omega)$. This is true, but a little care is needed to prove it. The ‘‘unconditional convergence’’ mentioned above means this: If, for each finite subset E of \mathbb{Z} ,

$$S_E(\omega) := \sum_{n \in E} (f \circ \varphi_n) \omega^{-n} \quad (\omega \in \partial\mathbb{U}),$$

then the net $(S_E : E \text{ a finite subset of } \mathbb{Z})$ converges in $L^2(H^2, dm)$ to F . In particular, if for each non-negative integer n we define $F_n = S_{[-n, n]}$, then $F_n \rightarrow F$ in $L^2(H^2, dm)$, hence some subsequence $(F_{n_k}(\omega))_{k=1}^\infty$ converges in H^2 to $F(\omega)$ for a.e. $\omega \in \partial\mathbb{U}$. Now for any n and any $\omega \in \partial\mathbb{U}$:

$$C_\varphi F_n(\omega) = \omega F_n(\omega) - \omega^{n+1} f \circ \varphi_{-n} + \omega^{-n} f \circ \varphi_{n+1}$$

which implies, since (3.4) guarantees that $\|f \circ \varphi_n\| \rightarrow 0$ as $n \rightarrow \infty$, that

$$C_\varphi F_n(\omega) - \omega F_n(\omega) \rightarrow 0 \text{ in } H^2 \quad (n \rightarrow \infty).$$

This, along with the a.e. convergence of the subsequence (F_{n_k}) to F , shows that $C_\varphi F(\omega) = \omega F(\omega)$ for a.e. $\omega \in \partial\mathbb{U}$. Now the H^2 -valued Fourier coefficients $f \circ \varphi_n$ are not all zero (in fact, none of them are zero) so at least for a subset of points $\omega \in \partial\mathbb{U}$ having positive measure we have $F(\omega) \neq 0$. The corresponding H^2 -functions $F(\omega)$ are therefore eigenfunctions of C_φ that belong to D_f , thus $\sigma_p(C_\varphi|_{D_f}) \cap \partial\mathbb{U}$ has positive measure.

It remains to prove (3.4). As usual, we treat the positively and negatively indexed terms separately. Since $f \in \sqrt{z-1} H^2$ we have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|f(e^{i\theta})|^2}{|\theta|} d\theta \leq \int_{\partial\mathbb{U}} \frac{|f(\zeta)|^2}{|\zeta-1|} dm(\zeta) < \infty$$

so successive application of (2.4) and (2.8) yields

$$\sum_{n=0}^{\infty} \|f \circ \varphi_n\|^2 = \int_{\partial\mathbb{U}} |f|^2 \left(\sum_{n=0}^{\infty} P_{r_n} \right) dm \leq \text{const.} \int_{-\pi}^{\pi} \frac{|f(e^{i\theta})|^2}{|\theta|} d\theta < \infty.$$

For the negatively indexed terms in (3.4), note that our hypothesis on f guarantees that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|f(e^{i(\theta-\pi)})|^2}{|\theta|} d\theta \leq \int_{\partial\mathbb{U}} \frac{|f(\zeta)|^2}{|\zeta+1|} dm(\zeta) < \infty. \quad (3.6)$$

Recall from the proof of Theorem 3.1 that $\varphi_{-n}(z) = -\varphi_n(-z)$ for $z \in \mathbb{U}$ and $n > 0$, and so

$$\|f \circ \varphi_{-n}\|^2 = \int |f|^2 P_{-r_n} dm = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{i\theta})|^2 P_{r_n}(\theta - \pi) d\theta.$$

Thus

$$\begin{aligned} \sum_{n=1}^{\infty} \|f \circ \varphi_{-n}\|^2 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{i\theta})|^2 \left(\sum_{n=1}^{\infty} P_{r_n}(\theta - \pi) \right) d\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{i(\theta-\pi)})|^2 \left(\sum_{n=1}^{\infty} P_{r_n}(\theta) \right) d\theta \\ &\leq \text{const.} \int_{-\pi}^{\pi} \frac{|f(e^{i(\theta-\pi)})|^2}{|\theta|} d\theta \\ &< \infty \end{aligned}$$

where the last two lines follow, respectively, from inequalities (2.8) and (3.6). This completes the proof of (3.4), and with it, the proof of the Theorem. \square

It would be of interest to know just how large a set $\sigma_p(C_\varphi|_{D_f})$ has to be in Theorem 3.5. Might it always be the whole unit circle? Might it be even larger? What I do know is that if the hypothesis of the Theorem is strengthened by replacing the hypothesis “ $f \in \sqrt{(z+1)(z-1)}H^2$ ” with the stronger “ $f \in \sqrt{(z+1)(z-1)}H^p$ for some $p > 2$ ”, then the conclusion improves dramatically, as shown below by the result below, whose proof reprises the latter part of the proof of Theorem 3.1.

Theorem 3.6. (cf. [5, Theorem 5.5]) *If $f \in \sqrt{(z+1)(z-1)}H^p \setminus \{0\}$ for some $p > 2$, then $\sigma_p(C_\varphi|_{D_f})$ contains, except possibly for a discrete subset, the open annulus $A(\mu^{-\varepsilon}, \mu^\varepsilon)$ where $\varepsilon = \frac{1}{2} - \frac{1}{p}$.*

Proof. I will show that the hypothesis implies that $f \in [(z-1)(z+1)]^{\frac{1}{2}+\delta}H^2$ for each positive $\delta < \varepsilon$. This will guarantee, by the proof of Theorem 3.1, that the series (3.1) converges in the open annulus $A(\mu^{-\delta}, \mu^\delta)$ for each such δ , and hence it converges in $A(\mu^{-\varepsilon}, \mu^\varepsilon)$, which will, just as in the proof of Theorem 3.1 finish the matter. The argument below, suggested by Paul Bourdon, greatly simplifies my original one. Our hypotheses of f imply that for some $g \in H^p$,

$$f = [(z-1)(z+1)]^{\frac{1}{2}+\delta} h \quad \text{where} \quad h = [(z-1)(z+1)]^{-(\frac{1}{2}+\delta)} g.$$

To show: $h \in H^2$. The hypothesis on δ can be rewritten: $2p\delta/(p-2) < 1$, so the function $[(z-1)(z+1)]^{-\delta}$ belongs to $H^{\frac{2p}{p-2}}$, hence an application of Hölder's inequality shows that h is in H^2 with norm bounded by the product of the H^p -norm of g and the $H^{\frac{2p}{p-2}}$ -norm of $[(z-1)(z+1)]^{-\delta}$. \square

In both [3] and [5, Theorem 5.3] there are results where the hypotheses on f involve uniform boundedness for f at one or both of the fixed points of φ . In [5, Theorem 5.4] Matache shows that these uniform conditions can be replaced by boundedness of a certain family of Poisson integrals, and from this he derives the following result.

[5, Theorem 5.5] *If $f \in (z-1)^{\frac{2}{p}} H^p$ for some $p > 2$, and f is bounded in a neighborhood of -1 , then $\sigma_p(C_\varphi|_{D_f})$ contains an open annulus centered at the origin.*

I'll close this section by presenting some results of this type, where uniform boundedness at one of the fixed points is replaced by boundedness of the Hardy-Littlewood maximal function. This is the function, defined for g non-negative and integrable on $\partial\mathbb{U}$, and $\zeta \in \partial\mathbb{U}$, by:

$$M[g](\zeta) := \sup \left\{ \frac{1}{m(I)} \int_I g \, dm : I \text{ an arc of } \partial\mathbb{U} \text{ centered at } \zeta \right\}.$$

The *radial maximal function* $R[g]$ of g at $\zeta \in \partial\mathbb{U}$ is the supremum of the values of the Poisson integral of g on the radius $[0, \zeta)$. It is easy to check that $M[g]$ is dominated pointwise on $\partial\mathbb{U}$ by a constant multiple of $R[g]$. What is perhaps surprising, but still elementary, is the fact that there is a similar inequality in the other direction: the radial maximal function of the non-negative integrable function g is dominated pointwise on $\partial\mathbb{U}$ by a constant multiple of its Hardy-Littlewood maximal function (see [9, Theorem 11.20, page 242]). This and (2.4) yield

Lemma 3.7. *For $f \in H^2$,*

$$M[|f|^2](-1) < \infty \implies \sup\{\|f \circ \varphi_n\| : n < 0\} < \infty.$$

To see that the hypotheses of Lemma 3.7 can be satisfied by functions in H^2 that are unbounded as $z \rightarrow -1$, one need only observe that

$$M[|f|^2](-1) \leq \text{const.} \int \frac{|f(\zeta)|^2}{|1+\zeta|} dm(\zeta),$$

hence, along with (2.4), the Lemma implies:

Corollary 3.8. *If $f \in \sqrt{z+1} H^2$ then $\sup\{\|f \circ \varphi_n\| : n < 0\} < \infty$.*

Thus if $f \in \sqrt{z+1} H^2$, or more generally if $M[|f|^2](-1) < \infty$, the negatively indexed subseries of (3.1) will converge in H^2 for all $\lambda \in \mathbb{U}$. We have seen in the proof of Theorem 3.1 that if $f \in (z-1)^{\frac{1}{2}+\varepsilon} H^2$ for some $\varepsilon \in (0, 1/2]$ then the positively indexed subseries of (3.1) converges for $|\lambda| > \mu^{-\varepsilon}$. Putting it all together we obtain the promised “ $\delta = 0$ ” case of Theorem 3.1:

Theorem 3.9. *Suppose $f \in (z + 1)^{\frac{1}{2}}(z - 1)^{\frac{1}{2} + \varepsilon} H^2 \setminus \{0\}$ for some $0 < \varepsilon < 1/2$. Then $\sigma_p(C_\varphi|_{D_f})$ contains, with the possible exception of a discrete subset, the open annulus $A(\mu^{-\varepsilon}, 1)$.*

Remark 3.10. By the discussion preceding this theorem, the hypothesis on f could be replaced by the weaker: “ $f \in (z - 1)^{\frac{1}{2} + \varepsilon} H^2 \setminus \{0\}$ and $M[|f|^2](-1) < \infty$,” (cf. [3]). If, in either version, the hypotheses on the attractive and repulsive fixed points are reversed, then the conclusion will assert that $\sigma_p(C_\varphi|_{D_f})$ contains, except for perhaps a discrete subset, the annulus $A(1, \mu^\varepsilon)$ (see §4.1, especially the discussion preceding Corollary 4.2).

Remark 3.11. Note how the previously mentioned Theorem 5.5 of [5] follows from the work above. Indeed, if $f \in (z - 1)^{2/p} H^p$ for some $p > 2$ then by Hölder’s inequality $f \in (z - 1)^{\frac{1}{2} + \varepsilon} H^2$, for each $\varepsilon < 1/p$. Thus, as in the proof of Theorem 3.1, the positively indexed subseries of (3.1) converges for $|\lambda| > \mu^{-1/p}$, and by Lemma 3.7 the boundedness of f in a neighborhood of -1 insures that the negatively indexed subseries of (3.1) converges in the open unit disc. Thus as in the proof of Theorem 3.1, $\sigma_p(C_\varphi|_{D_f})$ contains, with the possible exception of a discrete subset, the open annulus $A(\mu^{-1/p}, 1)$.

4. Complements and comments

In this section I collect some further results and say a few more words about the theorem of Nordgren, Rosenthal, and Wintrobe.

4.1. Non-canonical hyperbolic automorphisms

The results of §3, which refer only to canonical hyperbolic automorphisms φ , can be easily “denormalized”. Here is a sample:

Theorem 4.1. *Suppose φ is a hyperbolic automorphism of \mathbb{U} with attractive fixed point α , repulsive one β , and multiplier $\mu > 1$. Then*

- (a) (cf. Theorem 3.1) *Suppose, for $0 < \varepsilon, \delta < 1/2$ we have*

$$f \in (z - \alpha)^{\frac{1}{2} + \varepsilon} (z - \beta)^{\frac{1}{2} + \delta} H^2 \setminus \{0\}.$$

Then $\sigma_p(C_\varphi|_{D_f})$ contains, except possibly for a discrete subset, the open annulus $A(\mu^{-\varepsilon}, \mu^\delta)$.

- (b) (cf. Theorem 3.5) *If $f \in \sqrt{(z - \alpha)(z - \beta)} H^2$ then $\sigma_p(C_\varphi|_{D_f})$ intersects $\partial\mathbb{U}$ in a set of positive measure.*
- (c) (cf. Theorem 3.6) *If $f \in \sqrt{(z - \alpha)(z - \beta)} H^p \setminus \{0\}$ for some $p > 2$, then $\sigma_p(C_\varphi|_{D_f})$ contains, except possibly for a discrete subset, the open annulus $A(\mu^{-\varepsilon}, \mu^\varepsilon)$ where $\varepsilon = \frac{1}{2} - \frac{1}{p}$.*

Proof. I’ll just outline the idea, which contains no surprises. Suppose α and β (both on $\partial\mathbb{U}$) are the fixed points of φ , and—for the moment—that $\tilde{\alpha}$ and $\tilde{\beta}$ are any two distinct points of $\partial\mathbb{U}$. Then, as we noted toward the end of §2.1, there is

an automorphism ψ of \mathbb{U} that takes $\tilde{\alpha}$ to α and $\tilde{\beta}$ to β . Thus $\tilde{\varphi} := \psi^{-1} \circ \varphi \psi$ is a hyperbolic automorphism of \mathbb{U} that is easily seen to have attractive fixed point $\tilde{\alpha}$ and repulsive one $\tilde{\beta}$. Furthermore:

- $C_{\tilde{\varphi}} = C_{\psi} C_{\varphi} C_{\psi}^{-1}$, so $C_{\tilde{\varphi}}$ is similar to C_{φ} .
- For $f \in H^2$: $C_{\psi} D_f = D_{f \circ \psi}$.
- $F \in H^2$ is a λ -eigenvector for C_{φ} if and only if $C_{\psi} F = F \circ \psi$ is one for $C_{\tilde{\varphi}}$.
- For $f \in H^2$, $M[|f|^2](\beta) < \infty \iff M[|f \circ \psi|^2](\tilde{\beta}) < \infty$.
- For any $\gamma > 0$, $f \in (z - \alpha)^{\gamma} H^2 \iff C_{\psi} f \in (z - \tilde{\alpha})^{\gamma} H^2$.

Only the last of these needs any comment. If $f \in (z - \alpha)^{\gamma} H^2$ then

$$\begin{aligned} C_{\psi} f &\in (\psi(z) - \alpha)^{\gamma} C_{\psi}(H^2) \\ &= \left(\frac{\psi(z) - \psi(\tilde{\alpha})}{z - \tilde{\alpha}} \right)^{\gamma} (z - \tilde{\alpha})^{\gamma} H^2 \\ &= (z - \tilde{\alpha})^{\gamma} H^2 \end{aligned}$$

where the last line follows from the fact that the quotient in the previous one is, in a neighborhood of the closed unit disc, analytic and non-vanishing (because ψ is univalent there), hence both bounded and bounded away from zero on the closed unit disc. Thus $C_{\psi}((z - \alpha)^{\gamma} H^2) \subset (z - \tilde{\alpha})^{\gamma} H^2$, and the opposite inclusion follows from this by replacing ψ by ψ^{-1} and applying C_{ψ} to both sides of the result.

Theorem 4.1 now follows, upon setting $(\tilde{\alpha}, \tilde{\beta}) = (+1, -1)$, from Theorems 3.1, 3.5, and 3.6. \square

What happens if we interchange attractive and repulsive fixed points of φ in the hypotheses of Theorem 4.1(a)? Then the hypotheses apply to φ^{-1} , hence so does the conclusion. Since $C_{\varphi^{-1}} = C_{\varphi}^{-1}$, Theorem 4.1(a) and the spectral mapping theorem yield, for example, the following complement to Theorem 3.9:

Corollary 4.2. *If φ is a hyperbolic automorphism of \mathbb{U} with attractive fixed point α , repulsive one β , and multiplier $\mu > 1$. Suppose $f \in (z - \alpha)^{\frac{1}{2}}(z - \beta)^{\frac{1}{2} + \varepsilon}$ for some $\varepsilon \in (0, \frac{1}{2})$, then $\sigma_p(C_{\varphi}|_{D_f})$ contains, except possibly for a discrete subset, the open annulus $A(1, \mu^{\varepsilon})$*

The reader can easily supply similar “reversed” versions of the other results on the point spectrum of $C_{\varphi}|_{D_f}$.

4.2. The Nordgren-Rosenthal-Wintrobe Theorem

Recall that this result equates a positive solution to the Invariant Subspace Problem for Hilbert space with a positive answer to the question: “For φ a hyperbolic automorphism of \mathbb{U} , does every nontrivial minimal C_{φ} -invariant subspace of H^2 contain an eigenfunction?” The theorem comes about in this way: About forty years ago Caradus [2] proved the following elementary, but still remarkable, result:

If an operator T maps a separable, infinite dimensional Hilbert space onto itself and has infinite dimensional null space, then every operator on a separable Hilbert space is similar to a scalar multiple of the restriction of T to one of its invariant subspaces.

Consequently the invariant subspace lattice of T contains that of every operator on a separable Hilbert space.

Now all composition operators (except the ones induced by constant functions) are one-to-one, so none of these obeys the Caradus theorem's hypotheses. However Nordgren, Rosenthal, and Wintrobe were able to show that if φ is a hyperbolic automorphism, then for every eigenvalue λ of C_φ the operator $C_\varphi - \lambda I$, which has infinite dimensional kernel (recall Theorem 2.1), maps H^2 onto itself. Their restatement of the Invariant Subspace Problem follows from this via the Caradus theorem and the fact that C_φ and $C_\varphi - \lambda I$ have the same invariant subspaces.

4.3. Cyclicity

Minimal invariant subspaces for invertible operators are both cyclic and doubly invariant—this was the original motivation for studying the subspaces D_f . Thus it makes sense, for a given doubly invariant subspace, and especially for a doubly cyclic one D_f , to ask whether or not it is cyclic. Here is a result in that direction in which the cyclicity is the strongest possible: *hypercyclicity*—some orbit (with no help from the linear span) is dense. I state it for canonical hyperbolic automorphisms; the generalization to non-canonical ones follows from the discussion of §4.1 and the similarity invariance of the property of hypercyclicity.

Proposition 4.3. *Suppose φ is a canonical hyperbolic automorphism of \mathbb{U} and $f \in \sqrt{(z+1)(z-1)}H^2$. Then $C_\varphi|_{D_f}$ is hypercyclic.*

Proof. A sufficient condition for an invertible operator on a Banach space X to be hypercyclic is that for some dense subset of the space, the positive powers of both the operator and its inverse tend to zero pointwise in the norm of X (see [10, Chapter 7, page 109], for example; much weaker conditions suffice). In our case the dense subspace is just the linear span of $S := \{f \circ \varphi_n : n \in \mathbb{Z}\}$. As we saw in the proof of Theorem 3.5, our hypothesis on f insures that $\sum_{n \in \mathbb{Z}} \|f \circ \varphi_n\|^2 < \infty$ so both $(C_\varphi^n)_0^\infty$ and $(C_\varphi^{-n})_0^\infty$ converge pointwise to zero on S , and therefore pointwise on its linear span. \square

Remark 4.4. One can obtain the conclusion of Proposition 4.3 under different hypotheses. For example if f is continuous with value zero at both of the fixed points of φ , then the same is true of the restriction of $|f|^2$ to $\partial\mathbb{U}$. Thus the Poisson integral of $|f|^2$ has radial limit zero at each fixed point of φ (see [9, Theorem 11.3, page 244], for example), so by (3.3), just as in the proof of Proposition 4.3, $C_\varphi|_{D_f}$ satisfies the sufficient condition for hypercyclicity. In fact, all that is really needed for this argument is that the measure

$$E \rightarrow \int_E |f|^2 dm \quad (E \text{ measurable } \subset \partial\mathbb{U})$$

have symmetric derivative zero at both fixed points of φ (see the reference above to [9]).

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