FIXATED BY FIXED POINTS

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ABSTRACT. In these lectures I'll try to make the case that "$f(x) = x$" is the most important equation in all of mathematics.

Contents

1. Introduction

1.1. **What is a fixed point?** Suppose f is a map taking some set S into itself. To say a point $x \in S$ is a *fixed point* of f is just to say that $f(x) = x$.

A map f can have many fixed points (e.g. the identity map on any set) or no fixed points (e.g. the mapping of "translation-by-one," $x \to x + 1$, on the real line). On real intervals we have a nice way of picturing the fixed points of a function: they are the x-coordinates of the points where the graph of that function crosses the line $y = x$.

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1.2. **Example: Newton's Method.** Suppose for simplicity that f is a differentiable function $\mathbb{R} \to \mathbb{R}$, and that its derivative f' is continuous and never vanishes on $\mathbb{R}$. Consider the "Newton function" F of f:

$$F(x) := x - \frac{f(x)}{f'(x)} \qquad (x \in \mathbb{R}). \tag{1}$$

One can think of $F(x)$ as the horizontal coordinate of the point at which the line tangent to the graph of f at the point $(x, f(x))$ intersects the x-axis. Since f' doesn't vanish, F is a continuous mapping taking $\mathbb{R}$ into itself; note that the roots of f (those points $x \in \mathbb{R}$ such that $f(x) = 0$) are precisely the fixed points of F.

More to the point: Newton's method involves iterating the Newton function in the hope of generating approximations to the roots of f. One starts with an initial guess x_0, sets $x_1 = F(x_0)$, $x_2 = F(x_1)$, etc., and hopes that this sequence of "Newton iterates" (x_n) converges to a root. Geometrically it seems clear that if the Newton iterate sequence converges then it must converge to a root. This is true, and is in fact a special case of something quite general. Suppose, for example, that S is a metric space—at this point it's enough to think of S as subset of $\mathbb{R}^n$, or even $\mathbb{R}$—and that $F\colon S \to S$ is continuous. Suppose further that $x_0 \in S$ has the property that its *iterate sequence*

$$x_1 = F(x_0),\ x_2 = F(x_1),\ x_3 = F(x_2),\ \ldots$$

converges to a point $p \in S$. Then it is an easy exercise to show that F, thanks to its continuity, has p as a fixed point.

1.3. **Example: Initial value problems.** Suppose now that $f\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous. Then given a point $(x_0, y_0) \in \mathbb{R}^2$ we can consider the *initial value problem* (IVP)

$$y' = f(x, y), \qquad y(x_0) = y_0 \,. \tag{2}$$

Geometrically, the IVP (2) asks for a differentiable function y whose graph is a smooth curve in the plane that has the following properties:

- it passes through the point (x_0, y_0), and
- at each of its points (x, y) it has slope $f(x, y)$.

It's natural to attempt to solve the differential equation $y' = f(x, y)$ by integrating both sides with respect to x; with a little more care we can even build in the initial condition, arriving at an *integral equation*

$$y(x) = \int_{t=x_0}^{x} f(t, y(t))\, dt + y_0 \tag{3}$$

that's equivalent to the original IVP in the sense that a function y satisfies (3) for some interval of x's (containing x_0) iff it satisfies IVP for that same interval.

To make the connection with fixed points, let $C(\mathbb{R})$ denote the vector space of continuous, real valued functions on $\mathbb{R}$, and consider the *integral transform* $T\colon C(\mathbb{R}) \to C(\mathbb{R})$ defined by:

$$(Ty)(x) = \int_{t=x_0}^{x} f(t, y(t))\, dt + y_0 \qquad (x \in \mathbb{R}). \tag{4}$$

Then, a function $y \in C(\mathbb{R})$ satisfies our IVP (2) iff (as we've already noted) it satisfies integral equation (3), *iff it is a fixed point of the mapping T.*

We'll discuss the existence and uniqueness of such fixed points in §2.

1.4. **The Brouwer Fixed Point Theorem.** The most easily stated, and deepest of the fixed-point theorems we'll discuss, was proved in 1912 by the Dutch mathematician L. E. J. Brouwer [1]. Its initial setting is the closed unit ball B of Euclidean space $\mathbb{R}^n$.

The Brouwer Fixed Point Theorem. *Every continuous mapping of B into itself has a fixed point.*

It's easy to see that the result remains true if B is replaced by any of its homeomorphic image (i.e., any set $G = f(B)$ where f is continuous and one-to-one, and $f^{-1}\colon G \to B$ is also continuous).

For $n = 1$ the proof of Brouwer's theorem is easy. In this case we're talking about a continuous function f mapping $B = [-1, 1]$ into itself. We may suppose f doesn't fix either endpoint (otherwise we're done), so we must have $f(-1) > -1$ and $f(1) < 1$. In other words, the continuous function $g(x) := f(x) - x$ is positive at -1 and negative at 1. By the Intermediate Value Theorem, g must take the value zero at some point of $[-1, 1]$, and that point is a fixed point for f.

For $n > 1$ the proof is much more difficult, and there are many different versions. In §4 we'll give one that uses only methods of "advanced calculus."

1.5. **Application: Positive matrices.** Let's call a matrix A "positive" (written "$A > 0$") whenever all its entries are (strictly) positive.[1]

Theorem. *Every positive square matrix has a positive eigenvalue, to which corresponds a positive eigenvector.*

Proof. We'll view $\mathbb{R}^n$ as a space of column vectors, and to measure distances therein we'll use the "one-norm:"

$$\|x\|_1 := |\xi_1| + |\xi_2| + \ \dots \ + |\xi_n|$$

where ξ_j is the j-th coordinate of the vector x. Let K denote the set of non-negative vectors in the closed $\|\cdot\|_1$-unit sphere of $\mathbb{R}^n$, i.e, $K := \{x \in \mathbb{R}^n_+ : \|x\|_1 = 1\}$, where $\mathbb{R}^n_+$ denotes the set of vectors in $\mathbb{R}^n$, all of whose coordinates are non-negative.

Now suppose A is an $n \times n$ positive matrix. Then the map F defined by

$$F(x) \ := \ \frac{Ax}{\|Ax\|_1} \qquad (x \in K)$$

takes K continuously into itself (thanks to the positivity of A, which insures that on K the denominator in this definition is never zero). But K is homeomorphic to B^{n-1} (obvious for $n = 2$, intuitively clear for $n = 3$, and a nice little exercise otherwise), so by Brouwer's Theorem the mapping F has a fixed point $x_0 \in K$. Thus x_0 is a vector with non-negative coordinates, and $Ax_0 = \lambda x_0$, where $\lambda = \|Ax_0\|_1 > 0$.

We still need to check that x_0 is actually a *positive* vector. Suppose it isn't, i.e. that one of its coordinates—say the j-th one—is zero. Then that same coordinate of Ax_0 is zero. But the j-th coordinate of Ax_0 is the dot product of the j-th row of A with the (transpose of the) column vector x_0. Since the entries of A are all strictly positive, and the entries of x_0 are non-negative, and not all zero, this dot product can't be zero, so we have a contradiction. Conclusion: $x_0 > 0$. □

This result is part of a famous theorem of Perron (1907) which asserts that the eigenvalue λ we've produced above is, in fact, the maximum of the moduli of the *all* the (real or complex) eigenvalues of A. Perron further proved that this "Perron eigenvalue" has "geometric multiplicity one," in that the subspace of $\mathbb{R}^n$ of all vectors v with $Av = \lambda v$ has dimension one. In 1912 Frobenius extended

[1] *Warning:* This is not to be confused with the notion of "positive-definite", which is something completely different.

Perron's results to matrices with non-negative entries. The "Perron-Frobenius" theory is the subject of ongoing research, with an enormous literature across many scientific areas. For more on this see e.g. [10, Chapter 2], [23], or [24].

1.6. **The Banach Contraction Mapping Principle.** The theorem we're going to apply (in the next section) to both Newton's Method and the Initial Value Problem is perhaps the best known fixed-point theorem. It was proven in the 1920's by the Polish mathematician Stefan Banach [2] in his doctoral dissertation. Although its setting is more general than that of Brouwer's theorem, the theorem's proof is—perhaps paradoxically—a lot simpler.

Banach's theorem is set in a metric space (S, d) where S is a set and d is a *metric* on S i.e., a function $d\colon S \times S \to \mathbb{R}_+$ such that

- $d(x, y) = 0$ iff $x = y$,
- $d(x, y) = d(y, x) \quad \forall\, x, y \in S$
- $d(x, z) \leq d(x, y) + d(y, z) \quad \forall\, x, y, z \in S$.

The last property is called, for obvious reasons, "the triangle inequality." Think, for example, of S as R^n or any subset thereof, where $d(x, y)$ is the Euclidean distance between x and y.

The class of mappings addressed by Banach's Principle are called *strict contractions.*[2] To say $F\colon S \to S$ is one of these means that there is a positive "contraction constant" $c < 1$ for which

$$d(F(x), F(y)) \leq c d(x, y) \qquad \forall\, x, y \in S. \tag{5}$$

Clearly every strict contraction is continuous on S. With this in mind, here's the statement of Banach's Contraction Mapping Principle. It applies to *complete* metric spaces, i.e., those for which every Cauchy sequence converges.

The Banach Contraction Mapping Principle. *Suppose (S, d) is a complete metric space and $F\colon S \to S$ is a strict contraction. Then F has a* unique *fixed point, and every iterate sequence converges to this point.*

I'll prove this in the next section. For now, a few comments. Note first that the uniqueness is a simple consequence of the "strictness" of the contraction, for if $F(p) = p$ and $F(q) = q$ for some $p, q \in S$ then

$$d(p, q) = d(F(p), F(q)) \leq c d(p, q)$$

[2]These are often just called "contractions".

and since $0 < c < 1$ we must have $d(p,q) = 0$, whereupon $p = q$, as promised. However uniqueness can fail if in (5) we merely assume that the contraction constant c is 1. Example: the identity map on any metric space with more than one point. Similarly, if $c = 1$ then *existence* can fail, as shown by the example $F(x) = x + 1$ defined on the real line.

Finally, the *existence* part of the theorem can fail if the hypothesis of completeness is omitted. For example, if S is the semi-closed interval $[-1, 1)$ endowed with the usual metric of the real line, and $F \colon S \to S$ is defined by $F(x) = (1 + x)/2$, then F is a strict contraction on S (with $c = 1/2$), but has no fixed point (in S). This example also shows that "closed-ness" of the unit ball is essential in Brouwer's fixed-point theorem.

2. Banach's Contraction Mapping Principle

At the end of the last section I introduced the notion of a strict contraction on a metric space, stated the Banach Contraction Mapping Principle, discussed the uniqueness part of its statement, and showed how the result can fail if its hypotheses of completeness or "strictness" are omitted. In this section I'll complete the proof of the Contraction Mapping Principle, and give applications to both Newton's method and initial value problems.

2.1. **Proof of the Contraction Mapping Principle.**

First it's worth noting a crucial property of strict contraction mappings. In the last section we observed that if (S, d) is a metric space, $F \colon S \to S$ is continuous, and x_0 is a point of S for which the *iterate sequence* converges, then the limit of that sequence has to be a fixed point of F. If we make the stronger assumption that F be a *strict contraction,* then there results a very strong converse: *If p is a fixed point of F then* every *iterate sequence converges to p.*

To see why this is true, fix $x_0 \in S$ and define the iterate sequence (x_n) in the usual way: $x_1 = F(x_0), \ldots, x_n = F(x_{n-1}), \ldots$. Then, upon recalling that F satisfies (5) of the last section, with $c < 1$:

$$d(x_n, p) \;=\; d(F(x_{n-1}), F(p)) \;\le\; c d(x_{n-1}, p) \;\ldots\; \le c^n d(x_0, p),$$

so $d(x_n, p) \to 0$ as $n \to \infty$, i.e., (x_n) converges to p, as promised.

So to prove Banach's Principle there can be only one strategy: Fix a point $x_0 \in S$ and prove that its iterate sequence converges. Since our metric space is complete it's enough to show that (x_n) is a *Cauchy sequence,* i.e. that given

$\varepsilon > 0$ there is a positive integer N such that whenever the indices m and n are $\geq N$ we have $d(x_n, x_m) < \varepsilon$.

To this end, consider a pair of indices $m < n$ and observe, from the triangle inequality that

$$d(x_n, x_m) \;\leq\; \sum_{j=m}^{n-1} d(x_{j+1}, x_j)$$

while from the definition of "iterate sequence" and the strict contractiveness of the mapping F:

$$d(x_{j+1}, x_j) \;=\; d(F(x_j), F(x_{j-1}) \;\leq\; c d(x_j, x_{j-1}) \;\ldots\; \leq\; c^j d(x_1, x_0)$$

whereupon (since $c < 1$)

$$d(x_m, x_n) \;\leq\; \sum_{j=m}^{n-1} c^j d(x_1, x_0) \;\leq\; d(x_1, x_0) \sum_{j=m}^{\infty} c^j \;=\; \frac{d(x_1, x_0)}{1-c}\, c^m$$

Now given $\varepsilon > 0$, we may choose N so that $\frac{d(x_1,x_0)}{1-c}\, c^N < \varepsilon$, which insures, by the above chain of inequalities, that $N \leq m < n \Rightarrow d(x_m, x_n) < \varepsilon$, hence our iterate sequence (x_n) is Cauchy. □

2.2. **Application to Newton's Method.** Suppose f is a real-valued function defined on a finite, closed interval $[a, b]$ of the real line, and that we know f has a root somewhere in the open interval (a, b). We're going to use the Contraction Mapping Principle to show that, under suitable hypotheses on f, Newton's method, for any starting point, converges to this root.

More precisely, suppose $f \in C^2(I)$, with f' never zero on I, and that f has different signs at the endpoints of I, say (without loss of generality) $f(a) < 0$ and $f(b) > 0$. Then f has a unique root x^* in the interior (a, b) of I. Under these hypotheses we have:

Theorem. *There exists $\delta > 0$ such that for any x_0 in the interval $[x^* - \delta, x^* + \delta]$, Newton's method with starting point x_0 converges to x^*.*

In other words, under reasonable hypotheses on f, if you start close enough to a root of f then the iteration sequence for the Newton function (1) will converge to that root.

Proof. Let M denote the maximum of $|f(x)''|$ as x ranges through I, and let m denote the corresponding minimum of $|f'(x)|$. By continuity, and the hypothesis that f' never vanishes on I we know that M is finite and $m > 0$.

According to (1) the Newton function for f is:

$$F(x) \;=\; x \;-\; \frac{f(x)}{f'(x)} \qquad (x \in I)$$

whereupon the quotient rule for differentiation gives

$$F'(x) \;=\; \frac{f(x)\,f''(x)}{f'(x)^2} \qquad (x \in I).$$

This formula, along with our bounds on f' and f'', yields the estimate

$$|F'(x)| \;\leq\; \frac{M}{m^2}|f(x)| \qquad (x \in I_\delta)$$

so upon shrinking δ enough to insure that

$$|f(x)| \;\leq\; \frac{m^2}{2M} \qquad \text{for} \qquad x \in I_\delta := [x^* - \delta, x^* + \delta]$$

(possible because f is continuous at x^* and takes the value zero there) we see that $|F'(x)| \leq 1/2$ for each $x \in I_\delta$.

This estimate on F' does the trick! For starters, if $x, y \in I_\delta$ then, along with the mean value theorem of differential calculus, it shows that

$$|F(x) - F(y)| \;\leq\; \frac{1}{2}\,|x - y| \qquad \forall x, y \in I_\delta\ .$$

Thus F is a strict contraction on I_δ—once we know F maps that interval into itself. But it does, since the same inequality shows that for each $x \in I_\delta$ (upon noting that the root x^* of f is a fixed point of F):

$$|F(x) - x^*| \;=\; |F(x) - F(x^*)| \;\leq\; \frac{1}{2}\,|x - x^*| \;\leq\; \frac{1}{2}\,\delta \;<\; \delta$$

so $F(x) \in I_\delta$, as desired.

Thus Banach's Contraction Mapping Principle applies to the strict contraction F acting on the complete metric space $I_\delta = [x^* - \delta, x^* + \delta]$, and guarantees that for any starting point in I_δ the corresponding F-iteration sequence converges to the fixed point of F, which must necessarily be the unique root of f in I. □

2.3. **Application to initial value problems.** Consider once again the initial value problem (IVP) given by equations (2) of Section 1.3. Recall that a function y satisfies this IVP if and only if it satisfies the equivalent integral equation (3), and that this means y is a fixed point of the integral transformation T, as defined on some appropriate interval by (4). We'll show that, under appropriate hypotheses, this integral transform is a contraction mapping on a suitable complete metric space of functions, and this will guarantee, by Banach's Contraction Mapping Principle, a unique solution to our IVP within that metric space.

The "suitable space" in question is considerably more complicated than the real intervals on which we analyzed Newton's Method. Instead of the real line, our initial setting will be $C(I)$, the space of real-valued functions continuous on a finite, closed, real interval I. On $C(I)$ we define the "max-norm"

$$\|f\| := \max_{x\in I}|f(x)| \qquad (f \in C(I))$$

and use this to define a metric d by:

$$d(f,g) = \|f-g\| \qquad (f,g \in C(I)).$$

In this metric a sequence converges (resp. is Cauchy) if and only if it converges (resp. is Cauchy) uniformly on I. A fundamental property of uniform convergence is that every sequence in $C(I)$ that is uniformly Cauchy on I converges to a function in $C(I)$ [30, Thms. 7.14 & 7.15, pp. 150–151], i.e. the metric space $(C(I), d)$ is complete. Our task will be to find an appropriate interval I and closed subset $\mathcal{S}$ of $C(I)$ on which the Contraction Principle applies. This will take a little work.

Here's another difference between what we're about to do and what we did in applying Banach's Principle to Newton's Method. We knew in advance that the Newton function had a unique fixed point in some initial interval (namely: the root of the original function f, which we decreed to be at the center of that interval), and we used Banach's principle to show that, upon shrinking this interval appropriately, we could guarantee that the Newton iteration sequence (i.e. "Newton's Method") converged to the fixed point. For initial value problems, however, we don't know in advance that solutions exist, so we'll be relying on Banach's Principle to produce both a unique solution in an appropriate metric space of functions, and provide an algorithm for approximating this solution.

In what follows we'll consider the initial value problem (2) in the following setting. The domain of the real-valued function f, instead being the whole plane as it was in §1.3, will be defined only a compact rectangle R in $\mathbb{R}^2$ centered at (x_0, y_0), with sides parallel to the coordinate axes. More precisely, $R = [x_0-\rho, x_0+\rho]\times[y_0-h, y_0+h]$ for some positive numbers h and ρ. We'll assume f is continuous on R and that $\frac{\partial f}{\partial y}$ exists at every point of R and is continuous there.[3]

The Picard-Lindelöf Theorem. *For f and R as described above, there exists a positive number $r \leq \rho$ such that on the interval $[x_0 - r, x_0 + r]$ the initial value problem*

$$y' = f(x, y), \qquad y(x_0) = y_0 \tag{6}$$

has a solution and on that interval this solution is unique.

Strategy of proof. We're going to use the fact, discussed in §1.3 that a function defined on an interval I centered at x_0, and of length $\leq 2\rho$, is a solution of our initial value problem if and only if it is a fixed point of the integral mapping

$$Tu(x) = \int_{t=x_0}^{x} f(t, u(t))\, dt + y_0 \qquad (x \in I).$$

where u runs through some appropriate class of functions continuous on I. At the very least all the points $(t, u(t))$, as t runs through I, must lie in R, which demands that

$$|u(t) - y_0| \leq h \text{ for every } t \in I. \tag{7}$$

Since I is compact, the absolute value of each $u \in C(I)$ has a maximum, which we'll denote by $\|u\|$. Then there's a natural metric on $C(I)$ defined by

$$d(u, v) = \|u - v\| \qquad (u, v \in C(I)),$$

and since convergence (resp. Cauchy-ness) in the metric d is the same as uniform convergence (resp Cauchy-ness), the metric space $(C(I), d)$ is complete (see [30, Theorem 7.15, page 151] for example). In this setting inequality (7) above can be described succinctly as:

> *u belongs to the closed ball $B(y_0, h)$ in $C(I)$ of radius h, centered at the constant function y_0.*

Thus, in order to use the Banach Contraction Mapping Principle in finding a fixed point for the mapping T, we will have to show that I can be chosen, first so that T maps the ball $B(y_0, h)$ into itself, and next—by appropriately shrinking I if necessary—that T is a strict contraction on that ball.

Proof of Theorem. Let's temporarily fix a positive number $r \leq \rho$, to be determined shortly; and work for the moment on the interval $I := [x_0 - r, y_0 + r]$.

[3] To say a derivative exists on a closed set means here that the function being differentiated is actually defined and differentiable on some open superset.

The integral operator T is defined on $B := B(y_0, h)$, the closed ball in $C(I)$ having center at the constant function y_0 and radius h, since for functions in that ball the graph over I is contained in R. Thus for every function u in B and every $x \in I$:

$$|Tu(x) - y_0| = \left| \int_{t=x_0}^{x} f(t, u(t))\, dt \right| \leq \int_{t=x_0}^{x} |f(t, u(t))|\, dt \leq M|x - x_0| \leq Mr$$

where M denotes the maximum of the continuous function $|f|$ over the compact rectangle R. Thus we can insure that T takes B into itself by requiring $r \leq h/M$.

The next task is to show that we can further restrict r so as to guarantee that T is a strict contraction on B (of course B is changing with r). For this, observe that—upon writing M' for the maximum of $|\frac{\partial f}{\partial y}|$ over R—we obtain

$$\begin{aligned}
|Tu(x) - Tv(x)| &= \left| \int_{t=x_0}^{x} [f(t, u(t)) - f(t, v(t))]\, dt \right| \\
&= \left| \int_{t=x_0}^{x} \left[\int_{s=v(t)}^{u(t)} \frac{\partial f}{\partial y}(t, s)\, ds \right] dt \right| \\
&\leq \left| \int_{t=x_0}^{x} M' \, |u(t) - v(t)|\, dt \right| \\
&\leq M' \, \|u - v\| \, |x - x_0| \\
&\leq M' r \|u - v\| \, .
\end{aligned}$$

Thus upon choosing r to be the minimum of h/M and $1/(2M')$ we guarantee that if $I = [x_0 - r, x_0 + r]$ then the integral operator T takes B strictly contractively into itself. Since this ball is a closed subset of the complete metric space $C(I)$, Banach's Contraction Mapping Principle guarantees the T has a fixed point therein, and this fixed point is a solution to our initial value problem, in fact the only one in B. We've already noted that *any* solution of the initial value problem on the interval I has to belong to B, so the fixed point provided by Banach's Principle is in fact the *only* solution to our initial value problem. □

2.4. **Remarks.** (a) The result seems to originate in Lindelöf's 1894 paper [21], in which he generalizes earlier work of Picard. The iteration associated with Banach's principle is often called in this in this special case "Picard Iteration." Note that the proof given above will still work if the differentiability of f in the second variable is replaced by to a "Lipschitz condition"

$$|f(x, y_1) - f(x, y_2)| \leq M'|y_2 - y_1| \qquad ((x, y_1), (x, y_2) \in R).$$

(b) It's instructive to illustrate the "local" nature of the result above by considering the simple initial-value problem $y' = a(1+y^2), y(0) = 0$, where $a > 0$. The unique solution is easily seen to be: $y(x) = \tan(ax)$, for which the maximal interval of existence is $(-\frac{\pi}{2a}, \frac{\pi}{2a})$. Thus, even though the right-hand side $f(x,y) = 1 + y^2$ of our differential equation is infinitely differentiable (even real-analytic) *on the entire plane,* the solution exists only on a finite interval which, for large a, is very small. Thus in nonlinear situations we must be aware that singularities can arise "unexpectedly."

(c) The interval of existence/uniqueness promised us by Banach's Principle could be very small. There is, however, always a *maximal* such interval, and this interval has the property that *the solution's graph continues out to the boundary of the region on which the function f is defined and continuously differentiable.* More precisely: Suppose f is continuously differentiable on an open set G that contains the point (x_0, y_0), and let I now denote the above-mentioned maximal interval of existence/uniqueness for the IVP (6). Then the graph over I of this solution leaves every compact subset of G. For details see, e.g. [28, §2.4].

(d) Our restriction to first order differential equations may seem severe, but in fact it's not. For example, the second order problem on the real interval I:

$$y'' \;=\; f(x,y,y'), \qquad y(x_0) \;=\; y_0,\; y'(x_0) \;=\; y_1 \qquad (x \in I)$$

can be rewritten as:

$$Y' \;=\; F(x,Y), \qquad Y(x_0) = Y_0 \qquad (x \in I) \tag{8}$$

where $Y = (y, y')$ is a function taking I into $\mathbb{R}^2$, $Y_0 = (y_0, y_1)$ is a vector in $\mathbb{R}^2$, and $F(x,Y) = (y', f(x,y,y'))$ maps the original domain of f (a subset of $\mathbb{R}^3$) into $\mathbb{R}^2$.

It's not difficult to check that the proof given above for our original "scalar-valued" IVP works almost *verbatim* (with the Euclidean norm replacing the absolute value) for the vector valued one, and produces a unique solution for the second order IVP. Of course the idea generalizes readily to initial value problems of order > 2.

(e) In a similar vein our analysis of Newton's Method can be generalized to higher dimensions. Suppose the function f maps some open set G of $\mathbb{R}^n$ into itself, and that $f(p) = 0$ for some point $p \in G$. If we assume that all first and second order partial derivatives of the components of f are continuous, and that the derivative f', which is now a linear transformation on $\mathbb{R}^n$, is nonsingular at every point of G, then, just as in the single-variable case, we can form the "Newton function"

$$F(x) = x - f'(x)^{-1} f(x)$$

where on the right-hand side we see the inverse of the linear transformation $f'(x)$ acting on the vector $f(x)$. A bit more work than before shows that, when restricted to a suitable closed rectangle centered at p, the function F is a strict contraction, so for any point in that rectangle the Newton iteration converges to p.

3. Toward a proof of Brouwer's Theorem: Retractions

Let's say that a metric space (S, d) has the "fixed point property" if every continuous mapping of the space into itself has a fixed point. With this terminology the Brouwer Fixed Point Theorem discussed in §1.4 can be restated:

The Brouwer Fixed Point Theorem. *For every positive integer n, the closed unit ball of $\mathbb{R}^n$ has the fixed point property.*

We'll prove this theorem in the next section. The proof will involve reducing the theorem to an equivalent result about an important class of mappings called *retractions.* This reduction, and these mappings, form the subject matter of this section.

3.1. **Retractions.** Suppose S is a metric space and A is a subset of S. To say that a continuous mapping $P\colon S \to A$ is a *retraction* of S onto A means that $P(S) = A$ and the restriction of P to A is the identity map on A. An equivalent way to restate "$P|_A =$ identity on A" is: "$P = P \circ P$."

Perhaps the most familiar example of a retraction is a projection of $\mathbb{R}^n$ onto a subspace. Here are two such examples, where $S = \mathbb{R}^2$ and A is the horizontal axis. Write $x \in \mathbb{R}^2$ in real coordinates $x = (x_1, x_2)$.

(a) Let $P(x) = (x_1, 0)$. Here P is the *orthogonal* projection of $\mathbb{R}^2$ onto the horizontal axis.
(b) Let $P(x) = (x_1 + x_2, 0)$. Now P projects x at a 45° angle onto the horizontal axis.

Here's another example that's more immediately relevant to our program. Let S be the closed annulus in $\mathbb{R}^2$ centered at the origin, with inner radius 1/2, and outer radius 1. Let A be the outer boundary of S, i.e., the unit circle.

For $x \in S$ let $P(x) = x/\|x\|$. Then P takes S onto A, and its restriction to A is the identity map on that set. It's easy to check (draw a picture) that P is continuous, hence it's a retraction of the annulus S onto its outer boundary, the unit circle. The reason this is of interest to us is that no such mapping exists for the unit disc

The unit circle not *a retract of the closed unit disc.*

This is a special case of the following result, the proof of which will occupy §4.

The "No Retraction" Theorem". *For each positive integer n, there is no retraction taking the closed unit ball of $\mathbb{R}^n$ onto its boundary.*

3.2. **No Retraction Theorem ⇒ Brouwer Fixed Point Theorem.** Here we'll show that the No Retraction Theorem implies the Brouwer Fixed Point Theorem. Since we've already proved, in §1.4, the Brouwer Theorem for $n = 1$, we'll work in $\mathbb{R}^n$ with $n > 1$.

Suppose, for the sake of contradiction, that the closed unit ball B of $\mathbb{R}^n$ does *not* have the fixed point property, i.e., that there is a continuous map $f : B \to B$ that has no fixed point. We'll use f to construct a retraction of B onto its boundary. The retraction is easily visualized: for x in B we know that $f(x) \neq x$, so we can draw the half-line L that starts at $f(x)$ and passes through x. Let $P(x)$ be the point at which L intersects ∂B. Clearly P is a map taking B onto its boundary, and fixing every point of that boundary. It's easy to convince yourself—e.g. by drawing a picture for the case $n = 2$—that P is continuous (proof in a moment), and so is a retraction taking B onto its boundary, thus contradicting the No Retraction Theorem.

So to complete the proof of Brouwer's theorem—assuming the No Retraction Theorem—we need only verify, independent of pictures, the continuity of the map P, whose definition we can write, for $x \in B$, as:

$$P(x) \; := \; x + \lambda(x)u(x) \tag{9}$$

where $u(x)$ is the unit vector in the direction from $f(x)$ to x:

$$u(x) \; := \; \frac{x - f(x)}{\|x - f(x)\|} \tag{10}$$

and $\lambda(x)$ is a non-negative scalar chosen to make $\|P(x)\| = 1$.

Since $x - f(x)$ is continuous on B, and never zero there (because f has no fixed point), the denominator on the right-hand side of (10) is bounded away from zero (compactness of B), hence u inherits the continuity of f. As for $\lambda = \lambda(x)$, it is the non-negative solution to the equation

$$0 = \|P(x)\|^2 - 1 \;=\; \|x + \lambda u(x)\|^2 - 1 \;=\; \lambda^2 + 2b\lambda - c$$

where $b = \langle x, u(x)\rangle$ (the dot product of the vectors x and $u(x)$) and $c = 1 - \|x\|^2$, which is ≥ 0. From the quadratic equation there's only one such solution:

$$\lambda \;=\; -b + \sqrt{b^2 + c} \;=\; -\langle x, u(x)\rangle \;+\; \sqrt{\langle x, u(x)\rangle^2 + (1 - \|x\|^2)} \tag{11}$$

Note that the condition "$\lambda = 0$ when $\|x\| = 1$" is satisfied because in this case $\langle x, u(x)\rangle > 0$, reflecting the fact that—as a simple picture shows—in this case the line segment from $f(x)$ to x makes an acute angle with the one from the origin to x. However, hewing to our vow to prove the continuity of P independently of pictures, the following calculation is perhaps preferable:

$$\begin{aligned} 0 \leq \langle x, u(x)\rangle \;=\; \frac{\|x\|^2 - \langle x, f(x)\rangle}{\|x - f(x)\|} \quad &\Longleftrightarrow \quad 0 \;\leq\; \|x\|^2 - \langle x, f(x)\rangle \\ &\Longleftrightarrow \quad \langle x, f(x)\rangle \leq \|x\|^2 = 1 \end{aligned}$$

where the last inequality holds thanks to the Cauchy-Schwartz inequality.[4]

Thus $\lambda(x)$ depends continuously on $x \in B$, hence so does $P(x)$. □

3.3. **More on retractions.** The argument above, reducing the Brouwer Fixed Point Theorem to the No-Retraction Theorem, might seem at first to be just a clever trick. In fact:

The Brouwer Theorem is equivalent *to the No-Retraction Theorem.*

Proof. Suppose the No-Retraction Theorem fails, i.e.that there exists a retraction P of B onto its boundary. Then $Q := -P$ is a continuous map taking B into itself which is easily seen to have no fixed point. So the Brouwer theorem fails. Thus "Brouwer" implies "No Retraction." □

Retractions and fixed points are intimately bound up in other ways, as shown by the following result.

Theorem 3.1. *Every retract of a space with the fixed point property has the fixed point property.*

[4] I thank Paul Bourdon for this calculation.

Proof. Suppose S is a metric space with the fixed point property, A is a subset of S, and $P\colon S \to A$ is a retraction of S onto A. Let $f : A \to A$ be a continuous map. We need to show that f has a fixed point. Since $g := f \circ P$ maps S into itself it has a fixed point. Since g maps S into A, this fixed point, call it a, belongs to A. But the restriction of P to A is the identity map, so

$$a = g(a) = f(P(a)) = f(a)$$

so a is a fixed point of f. □

Which spaces have the fixed point property? Every one-point space has it (trivially), and for each positive integer n the closed unit ball of $\mathbb{R}^n$ has it (nontrivially—this is Brouwer's Theorem, whose proof we haven't yet completed). Thanks to Theorem 3.1, we can use the Brouwer Theorem to exhibit more examples. The one below is not homeomorphic to any closed ball in any Euclidean space (exercise).

Example 3.2. In $\mathbb{R}^2$ let S be the union of the closed intervals of length two along the x and y axes respectively, centered at the origin. Then S has the fixed point property.

Proof. Let B denote the closed unit ball of $\mathbb{R}^2$ (a.k.a "the closed unit disc"). Then $S \subset B$, so by Brouwer's theorem and Theorem 3.1 above, we need only show that S is a retract of B. We'll accomplish this by modifying the "non-orthogonal" projections introduced above in §3.1. The set S divides B into four quadrants. Bisect each of these quadrants by 45° lines, and project each point in B onto S by moving it parallel to the closest of these quadrant bisectors. Thus, each point of one of the bisectors goes to the origin, each point of the region $y > |x|$ to that part of S above the y-axes, ..., and, by this recipe, the points of S get left alone. The result is a map P that takes B onto S, and whose restriction to S is the identity. I leave it to you to convince yourself that P is continuous. □

Theorem. *Every compact convex subset of $\mathbb{R}^n$ has the fixed point property.*

Proof. Let C be a compact convex subset of R^n. In case $n = 1$ our set is just a closed interval, which we already know has the fixed point property, so let's assume (although we're not really going to need to) that $n > 1$.

Claim. C is a retract of $\mathbb{R}^n$.

Even though $\mathbb{R}^n$ does not have the fixed point property, this will prove our result. Indeed, since C is compact it is contained in a closed ball B, which by Brouwer's theorem, has the fixed point property. The CLAIM gives us a retraction P of $\mathbb{R}^n$ onto C, and the restriction of P to B is a retraction of B onto C. The result will now follow—once we've proved our CLAIM—from Theorem 3.1.

Proof of the CLAIM. The retraction we're about to produce is of crucial importance in its own right—it is the *Closest Point Map.* Suppose $x \in \mathbb{R}^n$. Since C is compact there is at least one point $\kappa \in C$ with $\|x-\kappa\| = \inf\{\|x-c\|\colon c \in C\}$ (proof: There is a a sequence (c_j) of points in C for which $\|x - c_j\|$ converges to the infimum in question. By the compactness of C, this sequence has a convergent subsequence, whose limit is a point closest to x. Now replace the whole minimizing sequence by this convergent one).

CLAIM: This "closest point" κ is unique.

Indeed, suppose $k \in C$ is another point "closest to x." For convenience let

$$d = \inf\{\|x - c\|\colon c \in C\} = \|x - \kappa\| = \|x - k\|$$

Let $v = x - \kappa$ and $w = x - k$. By the Parallelogram Law:

$$\|v + w\|^2 \;+\; \|v - w\|^2 \;=\; 2\|v\|^2 \;+\; 2\|w\|^2 \;=\; 4d^2\,.$$

On the other hand, $\|(v + w)/2\| = \|(\kappa + k)/2 - x\| \geq d$, since, by convexity, $(\kappa + k)/2 \in C$. This, along with the last display, yields

$$4d^2 \;+\; \|v - w\|^2 \;\leq\; 4d^2$$

so $\|v - w\| = 0$, i.e. $v = w$, i.e. $\kappa = k$. This proves the CLAIM.

Now that we know there's a unique closest point in C to x, let's give it a name: $P(x)$. Thus P maps $\mathbb{R}^n$ onto C, and fixes each point of C. To show that P retracts $\mathbb{R}^n$ onto C we need only verify its continuity. This follows quickly from the "closest-point uniqueness" from which the mapping P owes its definition.

Proposition. *Suppose (X, d) is a metric space with a compact subset A for which every $x \in X$ has a unique closest point $P(x)$ in A. Then P is a retraction of X onto A.*

Proof of Proposition.[5] Define the function "distance to A" by:

$$d_A(x) := \inf\{a \in A\colon d(x, a)\} \qquad (x \in X).$$

[5] I thank Steve Silverman and Mau Nam Nguyen for showing me this argument.

Note first that $d_A\colon X \to [0,\infty)$ is continuous; in fact, it is "non-expansive" in the sense that

$$|d_A(x) - d_A(y)| \le d(x,y) \qquad (x,y \in X). \tag{12}$$

To see why this is true, fix x and y in X; suppose (without loss of generality) that $d_A(x) \ge d_A(y)$. Then for every $a \in A$ $d_A(x) \le d(x,a) \le d(x,y) + d(y,a)$, from which follows (thanks to the fact that a was an arbitrary element of A) that $d_A(x) \le d(x,y) + d_A(y)$.

Note: this continuity of "distance-to-A" is quite general; we have used neither the compactness of A, nor the uniqueness, nor even the existence, of the "closest point in A to x."

Now let's return to our compact subset A that *does* have the "unique closest point" property, and the map $P(x) =$ "closest point in A to x." We're trying to show that P is continuous, so fix $x_0 \in X$ and suppose (x_n) is a sequence in X that converges to x_0. Our goal is to show that $P(x_n) \to P(x_0)$. Since A is compact, the sequence $(P(x_n))$ of closest points has a subsequence convergent to a point—call it y_0—of A. To keep notation under control, let's replace (temporarily) the whole sequence by this subsequence, so that $P(x_n) \to y_0$. Then:

$$\begin{aligned} d_A(x_0) &= \lim_n d_A(x_n) && \text{(continuity of } d_A) \\ &= \lim_n d(x_n, P(x_n)) && \text{(definition of } P) \\ &= d(x_0, y_0) && \text{(definition of } y_0) \end{aligned}$$

so y_0 is a closest point in A to x_0, hence by uniqueness, $y_0 = P(x_0)$. This argument actually proves that if x_0 is any point of x and (x_n) is any sequence that converges to x_0, then every subsequence of (x_n) has a further subsequence whose image under P converges to $P(x_0)$. Thus $P(x_n) \to P(x_0)$, as desired. This completes the proof of the Proposition, and with it the proof that the closest-point mapping of $\mathbb{R}^n$ onto the compact convex subset C is continuous, and is therefore a retraction. □

Remarks. (a) With a little more care we can weaken the compactness hypothesis on the convex set C to just "closed-ness." The idea is that an application, similar to the one above, of the Parallelogram Law shows that the "minimizing sequence" (c_j) discussed above is actually a Cauchy sequence, and therefore converges, its limit being the unique closest point in C to x. In case C is a linear subspace of $\mathbb{R}^n$, this closest point turns out to be the orthogonal projection of x onto C. These arguments generalize, with no essential changes, to the setting of infinite dimensional Hilbert space (see [35, §3.2, page 26 ff.], for example).

(b) For a closed convex subset of $\mathbb{R}^n$ (or more generally of any Hilbert space) the "closest point map" P is more than just continuous: it is "non-expansive" in the sense that

$$\|P(x) - P(y)\| \leq \|x - y\|$$

for all $x, y \in \mathbb{R}^n$; see for example, [15, Theorem 3.13, page 118] for the details.[6]

4. Proof of the Brouwer Fixed-Point Theorem

In 3.2 we showed that the *Brouwer Fixed Point Theorem* (henceforth "BFPT") is equivalent to the *No-Retraction Theorem* (henceforth "NRT") in the sense that each implies the other. In this section we'll complete the proof of the BFPT by proving a "C^1-version" of the NRT, and showing that this suffices to establish the BFPT. The argument is a modification, due to to C. A. Rogers [29], of an idea originally due to John Milnor [25]. Here is the outline. First we'll show that:

$$(*) \qquad C^1\text{-NRT} \implies C^1\text{-BFPT} \implies \text{BFPT},$$

where, the prefix "C^1-" means that the result is only being claimed for maps whose (real-valued) coordinate functions have continuous first order partial derivatives on some open set that contains B. Then we'll get down to business and give the Milnor-Rogers proof of the C^1-NRT, i.e. we'll prove that there is no C^1-retraction taking the closed unit ball B of $\mathbb{R}^n$ onto its boundary.

4.1. **Proof of (*).** (a) C^1-BFPT $\implies$ BFPT: This is a straightforward argument that is based on a well-known approximation theorem. We are assuming that every C^1-mapping of B into itself has a fixed point, and desire to prove that the same is true of any continuous map. The key is the following well-known approximation theorem:

> *Given $f\colon B \to \mathbb{R}^n$ continuous and $\varepsilon > 0$ there exists a C^1-map $g : B \to \mathbb{R}^n$ with $|f(x) - g(x)| \leq \varepsilon$ for every $x \in B$.*

[6] I thank Mau Nam Nguyen for this reference.

For example the Stone-Weierstrass Theorem (see e.g. [30, Theorem 7.6, page 159]) guarantees that the coordinate functions of g can even be chosen to be polynomials (in n variables).

Suppose $f\colon B \to B$ is a continuous map. We desire to show that f has a fixed point. To this end let (ε_k) be any sequence of positive numbers convergent to zero, and for each natural number k choose—by the above-mentioned approximation result—a C^1-map $f_k : B \to \mathbb{R}^n$ with

$$|f_k(x) - (1-\varepsilon_k)f(x)| \le \varepsilon_k \qquad (x \in B). \tag{13}$$

By the "reverse triangle inequality" we have $|f_k(x)| - (1-\varepsilon_k)|f(x)| \le \varepsilon_k$ for every $x \in B$ and each index, i.e.

$$|f_k(x)| \le \varepsilon_k + (1-\varepsilon_k)|f(x)| \le \varepsilon_k + (1-\varepsilon_k) = 1.$$

Thus f_k maps B into itself, so by our assumption that the C^1-BFPT holds, f_k has a fixed point $p_k \in B$. By the (ordinary) triangle inequality, for every $x \in B$:

$$\begin{aligned} |f_k(x) - f(x)| &= |f_k(x) - (1-\varepsilon_k)f(x) - \varepsilon_k f(x)| \\ &\le |f_k(x) - (1-\varepsilon_k)f(x)| + \varepsilon_k|f(x)| \\ &\le \varepsilon_k + \varepsilon_k = 2\varepsilon_k \end{aligned}$$

so in particular

$$|f(p_k) - p_k| = |f(p_k) - f_k(p_k)| \le 2\varepsilon_k \qquad (k \in \mathbb{N}), \tag{14}$$

i.e., p_k is a "$2\varepsilon_k$-approximate fixed point of f." The compactness of B insures that the sequence (p_k) has a subsequence convergent to a point $p \in B$. After appropriately renaming the subscripted f's, p's, and ε's, we may assume that $p_k \to p$, from which (14) and the continuity of f imply that $f(p) = p$. Thus f has a fixed point, as we wished to prove.

This last part of the argument will be needed later on, so it seems worthwhile to isolate what it actually proves.

The Approximate Fixed Point Lemma. *Suppose f is a continuous map of a compact metric space into itself, and that for each $\varepsilon > 0$ there is a point x_ε in the space such that the distance from $f(x_\varepsilon)$ to x_ε is $< \varepsilon$. Then f has a fixed point.*

(b) C^1-NRT $\implies$ C^1-BFPT: Suppose C^1-BFPT fails, so there exists a C^1-map $f\colon B \to B$ with no fixed point. We'll show that the retraction P given by equations (9)–(11) in the proof that NRT $\implies$ BFPT is, in this case, a

C^1-map. The only issue here is the C^1-nature of the parameter $\lambda(x)$ on the right-hand side of equation (9), but this follows immediately from equation (11) and the fact that, on the right-hand side of that equation the quantity under the radical sign is C^1 and—as was noted at the end of §3.2—strictly positive for each $x \in B$. □

4.2. **Proof of C^1-NRT.** This is the heart of our proof of the BFPT. Suppose C^1-NRT is false, i.e. suppose there exists a C^1-retraction P taking B onto its boundary. We will show that this leads to a contradiction. The argument comprises three steps.

Step I: *A bridge from the identity map to P.*

For $0 \leq t \leq 1$ define the map P_t on B by

$$P_t(x) = (1-t)x + tP(x) \qquad (x \in B). \tag{15}$$

Directly from this definition it follows that:

* P_0 is the identity map on B, while $P_1 = P$.
* Each P_t is a C^1-map that—since each of its values is a convex combination of two elements of B—takes B into itself.
* Each map P_t fixes every point of ∂B.

For the next step let B° denote the open unit ball of $\mathbb{R}^n$, i.e. the interior of B.

Step II: *For all sufficiently small t, P_t is a homeomorphism of B° onto itself.* In short, for t sufficiently close to zero P_t behaves a lot like the identity map P_0. Let's defer the proof of this statement until we've seen how it leads to the desired contradiction.

Step III: *Deriving the contradiction.*

Define $h : [0,1] \to \mathbb{R}$ by the multiple Riemann integral:

$$h(t) := \int_{B^\circ} \det {P_t}'(x)\, dx \qquad (0 \leq t \leq 1), \tag{16}$$

where $P_t'(x)$ is the derivative of P_t evaluated at $x \in B^\circ$. Here we view $P_t'(x)$ as an $n \times n$ matrix whose entries are continuous real-valued functions on B°.

By Step II we know (or will know once we prove it) that there exists $t_0 \in (0,1)$ such that P_t maps B° homeomorphically onto itself for all $t \in [0, t_0]$. Thus by (16) and the change-of-variable formula for multiple integrals,

$$h(t) = \int_{P_t(B^\circ)} dx = \text{Volume of } B^\circ \qquad (0 \leq t \leq t_0).$$

On the other hand, $\det P_t{}'$ is a polynomial in t with continuous real-valued coefficients, so h is a polynomial in t with value equal to the volume of B° on the interval $[0, t_0]$, and therefore has that constant value for *all* values of t. In particular, $h(1) > 0$. But $P_1 = P$ maps B° into the unit sphere ∂B, a subset of $\mathbb{R}^n$ that has no interior, so by the inverse function theorem (see [30, Thm. 9.24, page 221], for example) its derivative matrix $P_1(x)'$ is singular for every $x \in B^\circ$. Thus for $t = 1$ the integrand on the right-hand side of (16) is identically zero, hence $h(1) = 0$, which is the desired contradiction.

PROOF OF STEP II. This takes place in several stages, each of which expresses the fact that as we restrict t to increasingly smaller values, P_t inherits successively more properties of P_0, the identity map.

STEP IIA: *For all t sufficiently small, P_t is one-to-one on B.*

Proof. Because P is a C^1-map on B, for any pair x, y of points in B the Lipschitz estimate

$$\|P(x) - P(y)\| \le L\|x - y\|$$

where L is a constant independent of x and y. Thus, for $x, y \in B$ and $0 \le t \le 1$:

$$\begin{aligned} \|P_t(x) - P_t(y)\| &= \|(1-t)(x-y) + t[P(x) - P(y)]\| \\ &\ge (1-t)\|x-y\| - t\|P(x) - P(y)\| \\ &\ge (1-t)\|x-y\| - tL\|x-y\| \\ &= [1 - t(1+L)]\,\|x-y\| \ . \end{aligned}$$

Conclusion: $0 \le t < 1/(1+L) \implies P_t$ *is one-to-one.*[7] □

STEP IIB: *For all t sufficiently small, P_t is a homeomorphism of B° onto $P_t(B^\circ)$.*

Proof. From the definition (15) of P_t we see that for each $t \in [0,1]$:

$$P_t{}'(x) = (1-t)\,I + t\,P'(x) \qquad (x \in B)$$

where I denotes the $n \times n$ identity matrix. Thus the "C^1-ness" of the retraction P translates into continuity for the map $(t, x) \to P_t{}'(x)$ as it takes the compact product space $[0,1] \times B$ into the space of $n \times n$ real matrices (endowed with the topology of $\mathbb{R}^{n^2}$). Since continuous functions on compact metric spaces are uniformly continuous, the function $(t, x) \to \det P_t{}'(x)$ is a uniformly continuous real-valued function on $[0,1] \times B$. Since $P_0{}'(x)$ is the $n \times n$ identity matrix for each $x \in B$, it follows from this uniform continuity that there exists $0 < t_0 <$

[7] I thank Paul Bourdon for this improvement of my original argument.

$1/C$ (where C is the constant of Step IIa) such that $\det P_t'(x) \geq 1/2$ for each $(t, x) \in [0, t_0] \times B$, hence $P_t'(x)$ is invertible for all those pairs (t, x).

Now fix $t \in [0, t_0]$. By the inverse function theorem, for every $x \in B^\circ$ there is an open ball centered at x and contained in B° such that P_t is a homeomorphism of this ball onto its P_t-image—this image being contained in B° thanks to the injectivity of P_t. Thus $P_t(B^\circ)$ is an open subset of B°. In summary: the map P_t is both injective and a local homeomorphism on B°, so it is a homeomorphism onto $P_t(B^\circ)$. □

Step IIc: *For all t sufficiently small, $P_t(B^\circ) = B^\circ$.*

Proof. Suppose $P_t(B^\circ) \neq B^\circ$. Then there is a point $y_0 \in B^\circ$ that belongs to the boundary of $P_t(B^\circ)$. One can therefore choose a sequence (y_k) of points in $P_t(B^\circ)$ with $y_k \to y_0$. Thus there exists a sequence (x_k) in B° with $P_t(x_k) = y_k$ for each index k. By passing to an appropriate subsequence we may assume, thanks to the compactness of B, that (x_k) converges to some point $x_0 \in B$. By the continuity of P_t we have $y_0 = P_t(x_0)$, so $x_0 \in B^\circ$ (else $x_0 \in \partial B$ and we'd have $y_0 = x_0$—since P_t fixes each point of ∂B—contradicting the assumption that $y_0 \in B^\circ$). Thus $y_0 = P_t(x_0)$ lies in $P_t(B^\circ)$, contradicting our assumption that y_0 lies on the boundary of that open set. □

This completes the proof of Step II, and with it, that of the Brouwer Fixed Point Theorem.

5. The Schauder Fixed-Point Theorem

The material of this section is based on Steve Silverman's lecture of November 30, 2012.

5.1. **An infinite dimensional Brouwer Theorem?** Does the Brouwer Fixed Point Theorem generalize to infinite dimensional situations? The answer is "yes and no." In the previous section we proved two versions of Brouwer's Theorem:

> Version 1. *The closed unit ball of $\mathbb{R}^n$ has the fixed point property,*

and more generally (but, in fact, equivalently):

VERSION 2. *Every compact convex subset of $\mathbb{R}^n$ has the fixed point property.*

If $\mathbb{R}^n$ is replaced by an infinite dimensional normed linear space, then Version 1 fails spectacularly, as was shown in 1951 by Dugundji [12, Theorem 6.3, page 362] (see also [20] for further generalizations).

For any infinite dimensional normed linear space, the closed unit ball does not have the fixed point property.

In contrast, Schauder [33, 1930] showed that Version 2 of Brouwer's theorem *survives* in infinite dimensional normed spaces. This is the "Schauder Fixed Point Theorem;" its proof and its application to initial value problems will be occupy the rest of this section—after we engage in a little more stage-setting.

The proof of the Brouwer theorem depended implicitly on compactness of the closed unit ball of $\mathbb{R}^n$. No infinite dimensional normed space has this property (cf. the result of Dugundji mentioned above); see, e.g. [31, Theorem 1.22, page 17] for an an even more general result. To observe this non-compactness in action in infinite dimensional Hilbert space, note that for any orthonormal sequence the distance between two distinct elements is $\sqrt{2}$, hence such a sequence, which belongs to the closed unit ball, has no convergent subsequence, thus rendering that closed ball noncompact.

For a concrete example of the infinite dimensional failure of Brouwer Version 1, let's work in the particular Hilbert space ℓ^2 consisting of square-summable real sequences. Let $(e_n)_1^\infty$ be the standard orthonormal basis in that space (e_n is the sequence with 1 in the n-th position and zero elsewhere), and—following Kakutani [17, 1943]—define the map T on ℓ^2 by

$$Tf := (1 - \|f\|)e_1 + \sum_{n=1}^{\infty} f(n)e_{n+1} = (1 - \|f\|, f(1), f(2), \ldots) \qquad (f \in \ell^2)$$

where we regard an element of ℓ^2 both as a real-valued function on the set of natural numbers (the first equation), and as a list of real numbers (the second one). Then for $\|f\| \leq 1$:

$$\|Tf\| \leq \|(1 - \|f\|)e_1\| + \left\| \sum_{n=1}^{\infty} f(n)e_{n+1} \right\| = (1 - \|f\|) + \|f\| = 1,$$

so T takes the closed unit ball B of ℓ^2 into itself, and—it's easy to see—does so continuously. But:

T has no fixed point in B.

Indeed, if $f \in B$ then upon equating components in the equation $Tf = f$ we see that $f(n) = 1 - \|f\|$ for each $n \in \mathbb{N}$. But $f \in \ell^2$, so $f(n) \to 0$. Since f is a constant function, this forces $f(n)$ to be 0 for all n. But then $\|f\| = 1 - f(1) = 1$, a contradiction.

We'll commence below the proof of the Schauder Fixed Point Theorem. The major step will involve showing each compact subset of a normed linear space can be "almost" embedded in the (necessarily compact) convex hull of a finite subset of its points. Application of the Brouwer theorem to these finite dimensional convex hulls will then produce, for the map in question, approximate fixed points which—thanks to compactness—can be chosen to converge to an actual fixed point.

5.2. **Preliminaries.** The proof of the Schauder theorem requires some standard preparatory results. For the rest of this section we'll let X and Y denote a (real) normed (linear) spaces, and will denote the norms in these spaces by $\|\cdot\|$, letting the context determine the space in which the norm is operating.

Proposition. *A linear map between two normed linear spaces is continuous if and only if it is bounded on some ball centered at of the origin.*

Proof. Let $T\colon X \to Y$ be a linear transformation. Suppose T is bounded on some ball B of X. Thanks to the homogeneity of the mapping T we can assume that B is the open unit ball of X. Let $M = \sup\{\|Tx\|\colon x \in B\}$. Then for any two points x_1 and x_2 in X we have $x_1 - x_2 \in \rho B$, where $\rho = \|x_1 - x_2\|$, whereupon, thanks again to homogeneity,

$$\|Tx_1 - Tx_2\| = \|T(x_1 - x_2)\| \leq M\|x_2 - x_1\|$$

thus establishing the continuity of T.

Conversely, if T is continuous then the inverse image of the open unit ball in Y is an open subset of X that contains the origin, and so contains, for some $\varepsilon > 0$, the open ball of radius ε in X centered at the origin. Thus the values of T on this ball are bounded in norm by 1. □

Proposition. *If* $\dim(X) = n < \infty$ *then X is linearly homeomorphic to $\mathbb{R}^n$.*

Proof. Let $(e_j : j = 1, 2, \ldots n)$ be the standard unit vector basis in $\mathbb{R}^n$ (so e_j is the vector with 1 in the j-th coordinate and zeros elsewhere), and let

$(x_j : j = 1, 2, \ldots n)$ be any basis for X. Define the map $T\colon \mathbb{R}^n \to X$ by $Tv = \sum_j \lambda_j(v) x_j$ where $\lambda_j(v)$ is the j-th coordinate of $v \in \mathbb{R}^n$ (i.e. T is the linear map that takes e_j to x_j for $1 \le j \le n$). Thus T is a linear isomorphism of $\mathbb{R}^n$ onto X. Since each of the linear functionals λ_j is continuous on $\mathbb{R}^n$ it follows that T is continuous. So left to prove is the continuity of T^{-1}.

To establish this, recall the notation from the previous sections: B denotes the closed unit ball of R^n and ∂B is its boundary, the unit sphere. Now ∂B is a compact subset of $\mathbb{R}^n$ that does not contain the origin. Thus $T(\partial B)$, thanks to the continuity and injectivity of T, is a compact subset of X that does not contain the origin, hence $T(\partial B)$ is disjoint from some open ball W in X that is centered at the origin.

I claim that $\Omega := T^{-1}(W)$ is contained in B°, the open unit ball of $\mathbb{R}^n$. This will establish that T^{-1} is bounded on W, thus—by the previous Proposition—insuring its continuity. To see that $\Omega \subset B^\circ$, note that Ω:

- is convex, thanks to the linearity of T^{-1}, hence arcwise connected,
- contains the origin, and,
- does not intersect ∂B, thanks to the injectivity of T^{-1}.

That does it! If Ω were not contained entirely in B° it would have to pass through ∂B, which it does not. □

To see this same argument at work in far greater generality, see [31, Theorem 1.21, pp. 16-17].

Corollary. *In any normed linear space the convex hull of a finite set of points is compact.*

Proof. Each such convex hull is contained in a finite dimensional subspace of our normed space. This finite dimensional subspace is, by the Proposition above, linearly homeomorphic to a finite dimensional Euclidean space. This linear homeomorphism takes the convex hull in question to the convex hull of a finite set in Euclidean space, which is clearly bounded, easily seen to be closed, and therefore compact. Thus the original convex hull is compact. □

Here is the approximation result we'll need. Recall that to say a subset S of a metric space is *totally bounded* means that for every $\varepsilon > 0$ there is a finite subset $F_\varepsilon \subset S$ such that every point of S lies within ε of some element of F_ε, i.e., S is contained in the union of the ε-balls with centers in F_ε. The set F_ε

is called an *ε-net* for S. With this terminology it's easy to see that: *Every compact subset of a metric space is totally bounded.*

5.3. **The Schauder Projection.** We can now formulate the key step in the proof of the Schauder Fixed Point Theorem. We employ the notations: conv(A) for the convex hull of a subset A of a normed space, and for any metric space we denote by $B(x, r)$ the open ball of radius r centered at the point x.

Proposition. *Suppose K is a compact convex subset of a normed linear space X. Then given $\varepsilon > 0$ and an ε-net F_ε contained in K, there exists a continuous map $P_\varepsilon \colon K \to \mathrm{conv}(F_\varepsilon)$ such that $\|P_\varepsilon(x) - x\| < \varepsilon$ for every $x \in K$.*

Proof. We are assuming that $F_\varepsilon = \{x_1, x_2, \ldots x_N\}$, where $K \subset \cap_{j=1}^{N} B(x_j, \varepsilon)$. For each index j define $\tau_j \colon K \to [0, \infty)$ by

$$\tau_j(x) := \begin{cases} \varepsilon - \|x - x_j\| & \text{if} \quad x \in K \cap B(x_j, \varepsilon) \\ \\ 0 & \text{if} \quad x \in K \backslash B(x_j, \varepsilon) \end{cases}$$

(The graph of τ_j is a sort of "tent" over the part of K that lies within ε of x_j; this is most easily visualized when K is a subset of the real line). Since each τ_j is continuous on K, and strictly positive on $K \cap B(x_j, \varepsilon)$, it follows that the function τ defined by

$$\tau(x) := \sum_{j=1}^{N} \tau_j(x) \qquad (x \in K)$$

is strictly positive and continuous. Thus we can define, for each index j, the continuous function $\lambda_j : K \to [0, 1]$, by

$$\lambda_j(x) := \frac{\tau_j(x)}{\tau(x)} \qquad (x \in K).$$

The function λ_j is strictly positive on $B(x_j, \varepsilon) \cap K$, and identically zero off that set, and the sum of the λ_j's is identically 1 on K. Thus the map P_ε defined on K by

$$P_\varepsilon(x) := \sum_{j=1}^{N} \lambda_j(x) x_j \qquad (x \in K)$$

maps K continuously into $\mathrm{conv}(F_\varepsilon)$, with

$$\|P_\varepsilon(x) - x\| = \|\sum_{j=1}^{N} \lambda_j(x)(x_j - x)\| \leq \sum_{j=1}^{N} \lambda_j(x)\|x_j - x\| \qquad (x \in K).$$

In the last sum on the right, $\lambda_j(x)$ is zero whenever $\|x - x_j\|$ is $\geq \varepsilon$, hence for every $x \in K$:

$$\|P_\varepsilon(x) - x\| < \varepsilon \sum_{j=1}^{N} \lambda_j(x) = \varepsilon$$

as we wished to show. □

The final estimate above is perhaps better viewed like this: The coefficient $\lambda_j(x)$ vanishes for those vectors x_j that lie outside the ball $B(x, \varepsilon)$, so $P_\varepsilon(x) - x$ is a "subconvex" combination of points in that ball, and so also lies in that ball.

The map P_ε constructed above of often called the "Schauder Projection" of K onto $\mathrm{conv}(\mathsf{F}_\varepsilon)$.

5.4. **Proof of the Schauder Fixed Point Theorem.** We're given a compact, convex subset C of a normed space X, and a continuous map $f\colon C \to C$. We wish to show that f has a fixed point. By the "Approximate Fixed Point Lemma" of §4.1 it's enough to show that given $\varepsilon > 0$ there exists $x_\varepsilon \in C$ such that $\|f(x_\varepsilon) - x_\varepsilon\| < \varepsilon$.

So let $\varepsilon > 0$ be given, choose an ε-net $F_\varepsilon \subset C$, and let P_ε be the Schauder projection of C onto $\mathrm{conv}(\mathsf{F}_\varepsilon)$ that's promised by the last Proposition. Then $g_\varepsilon := P_\varepsilon \circ f$ maps C continuously into $\mathrm{conv}(\mathsf{F}_\varepsilon)$, and so maps $\mathrm{conv}(\mathsf{F}_\varepsilon)$ continuously into itself. Since $\mathrm{conv}(\mathsf{F}_\varepsilon)$ is a compact, convex subset of a finite dimensional subspace of X, it is homeomorphic (even linearly) to a compact, convex subset of a finite dimensional Euclidean space, so by "Version 2" of the Brouwer Fixed Point Theorem, g_ε has a fixed point x_ε that lies in $\mathrm{conv}(\mathsf{F}_\varepsilon)$, and hence in C. Thus:

$$\|f(x_\varepsilon) - x_\varepsilon\| = \|f(x_\varepsilon) - g_\varepsilon(x_\varepsilon)\| = \|f(x_\varepsilon) - P_\varepsilon(f(x_\varepsilon)\| < \varepsilon$$

where the final estimate follows from the Proposition of the last section. This completes the proof of the Schauder Fixed Point Theorem. □

5.5. **Return to initial value problems.** In §2.3 we used the Banach Contraction Mapping Principle to prove the Picard-Lindelöf theorem on solutions of first order initial value problems. This theorem required an extra smoothness condition on the right-hand side of the differential equation in (2), in return for which the Banach Principle guaranteed uniqueness, as well as existence. Thanks to the Schauder Fixed Point Theorem we'll be able to prove existence of solutions without any extra smoothness, but at the cost of giving up uniqueness. The setting will be the same as in §2.3: f will be a real-valued function that is continuous on a compact rectangle $R = [x_0 - r_0, x_0 + r_0] \times [y_0 - h, y_0 + h]$ and M will denote the maximum of the absolute value of f over R. Under just these hypotheses the Schauder Fixed Point Theorem will provide:

Peano's Theorem. *Let $r = \min\{h/M, r_0\}$. Then the initial value problem*

$$y' = f(x, y), \qquad y(x_0) = y_0$$

has a solution on the interval $[x_0 - r, x_0 + r]$.

Proof. Let I denote the interval $[x_0 - r, x_0 + r]$. From our proof of the Picard-Lindelöf theorem we know that, thanks to our choice of the parameter r, the integral operator T defined by:

$$Tu(x) = y_0 + \int_{t=x_0}^{x} f(t, u(t))\, dt \qquad u \in C(I)$$

maps the closed ball $B := B(y_0, h)$—those functions in $C(I)$ with distance at most h from the constant function y_0—into itself. We also know that in order to show that the initial value problem

$$y' = f(x, y) \qquad y(x_0) = y_0$$

has a solution on I it's sufficient (and necessary) to show that T has a fixed point.

Now B, although closed in $C(I)$, is not compact (exercise), so we can't apply the Schauder theorem directly. However, we'll be able to show that $T(B)$ is *relatively compact,* and this will turn out to be enough. To prove this relative compactness it's enough to show, by the Arzela-Ascoli Theorem (see e.g. [30, Theorem 7.25, page 158]), that $T(B)$ is *bounded* in $C(I)$, and *equicontinuous* on the interval I, i.e. for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$x, y \in I \ \&\ |x - y| < \delta \implies |Tu(x) - Tu(y)| < \varepsilon \qquad \underline{\text{for}}\ \underline{\text{all}}\ u \in B.$$

The boundedness of $T(B)$ has already been established, since we've chosen the half-length r of I so that $T(B) \subset B$. To prove equicontinuity, fix $u \in B$

and observe that for each pair of points $x, y \in I$ with $|x - y| < \delta$,

$$
\begin{aligned}
|Tu(x) - Tu(y)| &= \left| \int_{t=x_0}^{x} f(t, u(t))\, dt - \int_{t=x_0}^{y} f(t, u(t))\, dt \right| \\
&= \left| \int_{t=x}^{y} f(t, u(t))\, dt \right| \\
&\leq \left| \int_{t=x}^{y} |f(t, u(t))|\, dt \right| \\
&\leq M|x - y| \; .
\end{aligned}
$$

Thus if $\varepsilon > 0$ is given and $|x - y| < \varepsilon/M$, then $|Tu(x) - Tu(y)| < \varepsilon$ for every $u \in B$, thus establishing the equicontinuity of $T(B)$.

Now the equicontinuity of $T(B)$ carries immediately over to conv(T(B)), which, being contained in B, is also bounded. Thus conv(T(B)) is relatively compact, and so its closure, K, in $C(I)$ is compact. Since $T(B) \subset K$, the Schauder theorem applies to the restriction of T to K, and furnishes the desired fixed point. □

5.6. **Non-Uniqueness in Peano's Theorem.** In contrast to the existence and uniqueness provided under the additional smoothness required by the Banach Contraction Mapping Principle, the Schauder theorem, which requires only a continuity hypothesis, yields the existence of solutions without any guarantee of uniqueness. The example below shows that non-uniqueness can naturally occur. Consider, for example, the initial value problem

$$y' = -2y^{1/2}, \qquad y(1) = 0.$$

$y(t) = (1 - t)^2$ is a solution for $t \geq 0$, but so is $y \equiv 0$, and more realistically, so is

$$y(t) = \begin{cases} (1 - t)^2 & (0 \leq t \leq 1) \\ 0 & (t \geq 1). \end{cases}$$

This "realistic" solution expresses a physical phenomenon discovered by Evangalista Toricelli (1608–1647). *Toricelli's Law* states that water issues from a small hole in the bottom of a container at a rate that is proportional to the square root of the water's depth (see e.g. [11] for more details). This leads to an IVP like the one above for which $y(t)$ is the depth of the water in the container at time $t \geq 0$. In our "realistic" solution the water starts out at height 1 when $t = 0$, the container runs dry at $t = 1$, and thereafter stays dry.

5.7. **"Generalization" of the Schauder Fixed Point Theorem.** The last part of our proof of Peano's theorem provides an extension of The Schauder Fixed Point Theorem to maps defined on closed convex sets that are not necessarily compact.

Corollary. *If C is a closed convex subset of a normed linear space, and $f: C \to C$ is a continuous map with $f(C)$ relatively compact, then f has a fixed point.*

Here "relatively compact" means that the closure is compact, i.e., that the original set is totally bounded. For the proof of this corollary we need a useful result which generalizes our previously noted fact that, in a normed space, the convex hull of a finite set of points is compact.

Lemma. *The convex hull of a relatively compact subset of a normed linear space is relatively compact.*

Proof of Lemma. In our normed space let B_r denote the open ball of radius r centered at the origin. Given subsets A and B of that space, we'll denote by $A + B$ the collection of sums $a + b$ where a ranges through A and b through B.

Suppose A is a totally bounded subset of our normed space. We wish to show that conv(A) is totally bounded. To this end let $\varepsilon > 0$. Then A has an $\varepsilon/2$-net F, i.e. $A \subset F + B_{\varepsilon/2}$. Then conv(F), being the convex hull of a finite set, is compact, hence totally bounded, and therefore possesses an $\varepsilon/2$-net G, i.e. $\mathrm{conv(F)} \subset \mathrm{G} + \mathrm{B}_{\varepsilon/2}$. Thus

$$A \ \subset\ F + B_{\varepsilon/2} \ \subset\ \mathrm{conv(F)} + \mathrm{B}_{\varepsilon/2}$$

and since the latter set, being the sum of two convex sets, is convex, we have $\mathrm{conv(A)} \subset \mathrm{conv(F)} + \mathrm{B}_{\varepsilon/2}$. Putting it all together:

$$\mathrm{conv(A)} \ \subset\ \mathrm{conv(F)} + \mathrm{B}_{\varepsilon/2} \ \subset\ \mathrm{G} + \mathrm{B}_{\varepsilon/2} + \mathrm{B}_{\varepsilon/2} \ =\ \mathrm{G} + \mathrm{B}_{\varepsilon}\,.$$

Thus G is an ε-net for conv(A), so conv(A) is totally bounded. □

Proof of Corollary. Since $f(C)$ is a relatively compact subset of the convex set C, by the last Lemma its convex hull is relatively compact. Thus the closure, K, of conv(f(C)) is compact in our normed space, and since C is closed, $K \subset C$. Now $f(K) \subset f(C) \subset K$, so Schauder's Theorem applies to the restriction of f to K, yielding the desired fixed point. □

6. Fixed-point theorems for commuting maps

The Schauder Fixed Point Theorem assures us that every continuous map taking a compact convex subset of a normed space into itself has a fixed point. Suppose, more generally, that we have a *family* of such maps, each of which commutes under composition with all the others. Might the Schauder theorem be extended to provide us with a *common* fixed point for all these maps? Encouragement for such a possibility comes from the fact that for any commuting family of self-maps of a set, the (possibly empty) fixed point set of each map in the family is taken into itself by all the other maps. In particular, if one of the maps has a *unique* fixed point (e.g. if the underlying set is a complete metric space and the map in question is a strict contraction), then that point is also fixed by every map in the family.

However, the problem of extending the Schauder theorem, even to a pair of commuting continuous maps on the closed unit interval, apparently first floated In the 1950's by several mathematicians independently,[8] defied solution until 1969 when Boyce [5] and Hunecke [16] independently produced (highly non-trivial) counterexamples.

On the other hand, in the 1930's Markov [22] used an extension due, to Tychonov, of the Schauder theorem to show that common fixed points do exist in very general settings for commutative families of *affine* mappings on compact convex sets. A couple of years later Kakutani [18] gave a direct proof that works in even more generality; this result, and some of its many applications, forms the content of the present section.[9]

6.1. **Commuting families of affine maps.** The setting for this section is a general (real) Hausdorff *topological vector space,* i.e. a vector space V over the real field on which there is a topology that "respects" the vector operations, in the sense that addition, when viewed as a map from the product space $V \times V$ into V, is continuous, as is scalar multiplication, viewed as a map $\mathbb{R} \times V \to V$.

If C is a convex subset of V, a map $f : C \to V$ is called *affine* if, whenever x and y belong to C and $0 \leq t \leq 1$, then $f(tx + (1-t)y) = tf(x) + (1-t)f(y)$. Clearly such a map preserves convexity, and, by a straightforward induction, respects more general "convex sums." Affine maps arise most commonly as the

[8] See the first paragraph of [5].

[9] The material of this section and the next is based on Steve Silverman's lecture of January 11, 2013.

restrictions of linear maps to convex sets, and these are the only ones that will concern us here. Finally, let us say that a family of self-maps of some set is *commuting* if any pair of maps in the family commutes under composition.

With this terminology in place, here is the main result of this section.

The Markov-Kakutani Fixed Point Theorem. *Suppose V is a Hausdorff topological vector space inside of which K is a compact, convex subset. Suppose $\mathscr{A}$ is a commuting family of continuous affine maps taking K into itself. Then there exists a point $p \in K$ such that $Ap = p$ for every $A \in \mathscr{A}$.*

Proof. This is Kakutani's proof from [18]. We may, without loss of generality, assume that the identity map I on K belongs to $\mathscr{A}$. For $f \in \mathscr{A}$ and $n \in \mathbb{N}$ let f^n denote the composition of f with itself n times, and set $f^0 = I$. Then each f^n is again an affine, continuous mapping of K into itself, as is

$$f_N \ := \ \frac{1}{N+1}\sum_{n=0}^{N} f^n \qquad (N = 0, 1, 2, \cdots).$$

Let

$$\mathscr{S} := \{f_N(K) : N = 0, 1, 2, \cdots, \text{and } f \in \mathscr{A}\}$$

Clearly every common fixed point of $\mathscr{A}$ belongs to $\bigcap \mathscr{S}$. Conversely, if $p \in \bigcap \mathscr{S}$ then for each $N = 0, 1, 2, \cdots$ there exists $q_N \in K$ with $p = f_N(q_N)$, after which a simple calculation based on the fact that f respects convex sums shows that for each non-negative integer N:

$$f(p) - p \ = \ \frac{1}{N+1}\left[q_N - f^{N+1}(q_N)\right] \ \in \ \frac{1}{N+1}(K - K). \tag{17}$$

Claim: $f(p) = p$.

To see why this is true, first note that $K - K$ (the set of algebraic differences of all pairs of elements of K), being the image of the compact set K under the map $(x, y) \to x + (1)y$ that takes $V \times V$ into V, is compact in V.

Next, fix an open neighborhood U of the origin in V, and observe that, thanks to the continuity of scalar multiplication, for each $x \in V$ there is a positive scalar ε such that $\varepsilon x \in U$. Or, turning things around, $V = \bigcup_{\varepsilon>0}(1/\varepsilon)U$. In particular, the compact set $K - K$ is covered by the collection of open sets $\{(1/\varepsilon)U : \varepsilon > 0\}$, so thanks to the previously noted compactness a finite subcollection of these open sets covers $K - K$ as well. In other words, for all sufficiently small $\varepsilon > 0$ we have $\varepsilon(K-K) \subset U$. In particular, from equation (17) we see that $f(p) - p \in U$ for all sufficiently large N. Since U is an arbitrary open neighborhood of the origin, and since the topology of our ambient vector space V is Hausdorff, we see that $f(p) = p$, as desired.

Thus $\bigcap \mathscr{S}$ is precisely the set of common fixed points for $\mathscr{A}$.

It remains to show that $\bigcap \mathscr{S}$ is non-empty. Since each of the sets in the family $\mathscr{S}$ is a closed subset of the compact set K, it's enough to show that each finite subfamily of $\mathscr{S}$ has non-empty intersection. To this end, fix such a finite subfamily $\mathscr{S}_0 := \{f_m(K), g_n(K), \cdots, h_k(K)\}$ and note that

$$K_0 := (f_m \circ g_n \cdots \circ h_k)(K)$$

is a nonempty subset of $f_m(K)$. By commutativity, K_0 is equally well a subset of $g_m(K), \cdots, h_k(K)$, and so belongs to the intersection of all the sets in $\mathscr{S}_0$, hence as desired, this intersection is nonempty. □

6.2. **Application: Haar measure for compact abelian groups.** By analogy with the definition of topological vector space, a *topological group* is a group G endowed with a topology that renders both its group operation $G \times G \to G$ and its operation of inversion $G \to G$ continuous. We'll assume here that all topological groups are Hausdorff. Recall that a positive measure of total mass one on a sigma algebra of subsets of some set is called a *probability measure*.

Theorem 6.1. *Every compact abelian topological group has a unique probability measure μ on its Borel sets that is G-*invariant *in the sense that $\mu(gE) = \mu(E)$ for every Borel subset E of G.*

Here the notation gE denotes the collection of all elements ge of G, where e runs through G. The measure μ promised by this theorem is called the *Haar measure* of G. For a familiar example, take G to be the unit circle, with complex multiplication as its group operation, whereupon Haar measure is just normalized Lebesgue arc-length measure.

Proof. UNIQUENESS: If μ and ν are Borel probability measures on G invariant in the sense described above, then for each $f \in C(G)$ we have, by the change of variable formula of measure theory [14, Theorem C, page 163]:

$$\int f(x)\,d\mu(x) = \int f(gx)\,d\mu(x)$$

for every $g \in G$ (where "$\int$" denotes "$\int_G$"). Upon integrating both sides of the above equation with respect to ν, and applying Fubini's theorem, we obtain

$$\begin{aligned}\int f(x)\,d\mu(x) &= \int\int f(gx)\,d\mu(x)\,d\nu(g) = \int\int f(gx)\,d\nu(g)\,d\mu(x) \\ &= \int\int f(g)\,d\nu(g)\,d\mu(x) = \int f(g)\,d\nu(g)\end{aligned}$$

with the first and last equalities follow from the G-invariance of μ and the fact that both μ and ν have total mass one, the second one from Fubini's theorem, and the third from the commutativity of G and the invariance of ν. Thus $\int f\,d\mu = \int f\,d\nu$ for every $f \in C(G)$, so $\mu = \nu$ by the uniqueness part of the Riesz Representation Theorem.

EXISTENCE: We aim to apply the Markov-Kakutani Theorem to produce the desired invariant measure; our task is to find the appropriate topological vector space V, compact subset K, and family of affine maps $\mathscr{A}$ to which to apply the theorem.

For any compact group G, abelian or not, let $C(G)$ denote the space of continuous real-valued functions on G (it's a Banach space when endowed with the supremum (= "maximum") norm, but we won't be needing this fact). Each element γ of G defines a linear transformation L_γ on $C(G)$ by

$$[L_\gamma f](\alpha) = f(\gamma\alpha) \qquad (f \in C(G), \alpha \in G)\ .$$

For each pair γ, δ of elements of G we see that $L_\gamma L_\delta = L_{\delta\gamma}$. From this it follows that each operator L_γ is invertible, with inverse $L_{\gamma^{-1}}$. The map $\gamma \to L_\gamma$ is therefore an "anti-homomorphism" (called the *left regular representation of* G) taking G into the group of linear transformations taking $C(G)$ onto itself. In our case, where G is abelian, the collection $\{L_\gamma : \gamma \in G\}$ is an abelian group of linear transformations on $C(G)$.

However, in order to apply the Markov-Kakutani theorem to produce the desired measure, we can't stay within the comfortable confines of $C(G)$; it's necessary to work instead in its *algebraic dual space* $C(G)^\sharp$, the space of *all* linear functionals on $C(G)$. This is a real vector space, and it's naturally endowed with a Hausdorff vector topology, namely *the topology of pointwise convergence on* $C(G)$. This is the topology $C(G)^\sharp$ inherits as a subset of the product space $\mathbb{R}^{C(G)}$ (= all functions $C(G) \to \mathbb{R}$). In this topology a net of functions on $C(G)$ converges if and only if it converges at each point of $C(G)$. The restriction of this product topology to $C(G)^\sharp$ is called the *weak-star* topology.

Let K denote the collection of "means" on $C(G)$, i.e. linear functionals Λ on $C(G)$ that are "positive" in the sense that $\Lambda(f) \geq 0$ whenever $f \in C(G)$ takes only non-negative values, and which take the value 1 on the constant function 1.[10] Such a functional respects pointwise inequalities in the sense that if f and g are in $C(G)$ and $f(x) \leq g(x)$ for every $x \in G$, then $\Lambda(f) \leq \Lambda(g)$ (proof: $g - f \geq 0$ on G, so $\Lambda(g - f) \geq 0$; now use linearity.).

Now each $f \in C(G)$, being continuous on a compact set, is bounded, and $|f|$ attains its supremum on G. Let's denote this supremum by $\|f\|$, and note that, since $-\|f\| \leq f \leq \|f\|$ on G, we have for each $\Lambda \in K$,

$$-\|f\| \;=\; -\|f\|\Lambda(1) \;=\; \Lambda(-\|f\|) \;\leq\; \Lambda(f) \;\leq\; \Lambda(\|f\|) \;=\; \|f\|\Lambda(1) \;=\; \|f\| \; .$$

For $f \in C(G)$ let I_f denote the closed real interval $[-\|f\|, \|f\|]$. The above calculation shows that each $\Lambda \in K$ belongs to the product space $\prod_{f \in C(G)} I_f$, which is compact by the Tychonoff theorem. The weak-star topology on $C(G)^\sharp$ coincides on K with this product topology (both of them being just the topology of pointwise convergence on $C(G)$), so K is a relatively weak-star compact subset of $C(G)^\sharp$. It's easy to check that K is weak-star closed in $C(G)^\sharp$ (partial proof: positivity is preserved by pointwise convergence), hence weak-star compact, and that it's convex.

Summary: We now possess two-thirds of the ingredients necessary for an application of the Markov-Kakutani theorem:

- The linear topological space $V := C(G)^\sharp$, taken in its weak-star topology, and
- The compact convex subset K of V consisting of all means on $C(G)$.

It remains only to determine the commuting family $\mathscr{A}$ of continuous affine maps of K. For this, recall that for a vector space X, any linear transformation $T\colon X \to X$, induces on the dual space $X^\sharp$ the *adjoint transformation* $T^\sharp\colon X^\sharp \to X^\sharp$ by means of the formula

$$T^\sharp\Lambda = \Lambda \circ T \qquad (\Lambda \in X^\sharp).$$

One checks easily that, if $X^\sharp$ is given its weak-star topology (the topology of pointwise convergence on X), then

- $T^\sharp$ is a continuous linear transformation on $X^\sharp$, and
- If S is another linear transformation on X, then $(ST)^\sharp = T^\sharp S^\sharp$.

[10]Perhaps "probability functional" would be a more descriptive term, but this does not seem to have caught on.

Returning to the case at hand, $X = C(G)$, set

$$\mathscr{A} := \{L_\gamma^\sharp : \gamma \in G\}$$

Note that the mapping $\gamma \to L_\gamma^\sharp$ is—even if the group G were not abelian—a homomorphism (called the *left-regular representation of G*) taking G into the collection of continuous linear operators on $C(G)^\sharp$. Since we're assuming G is abelian, it follows that $\mathscr{A}$ is also an abelian group of continuous linear mappings taking $C(G)^\sharp$ into itself.

One checks easily that each map $L_\gamma^\sharp$ takes the weak-star compact subset K of means on $C(G)$ into itself, hence the triple $(V = C(G)^\sharp, K, \mathscr{A})$ satisfies the hypotheses of the Markov-Kakutani theorem, so there exists $\Lambda \in K$ fixed by all the maps $L_\gamma^\sharp$, i.e. for which $L_\gamma^\sharp\Lambda = \Lambda$ for each $\gamma \in G$.

The Riesz Representation Theorem (see, for example, [30, Theorem 2.14, pp. 40–47]) provides a positive measure μ on the Borel subsets of G such that

$$\Lambda(f) \;=\; \int_G f\,d\mu \qquad (f \in C(G)).$$

Since $\Lambda(1) = 1$ we have $\mu(G) = 1$, i.e. μ is a probability measure on G. Finally, for every $\gamma \in G$ and $f \in C(G)$:

$$\int_G f\,d\mu\gamma^{-1} \;=\; \int_G f(\gamma\delta)d\mu(\delta) \;=\; [\Lambda \circ L_\gamma](f) \;=\; [\mathrm{L}_\gamma^*\Lambda](f) \;=\; \Lambda(f) \;=\; \int_G f\,d\mu$$

where, in the first equality, $\mu\gamma^{-1}$ is the measure on Borel subsets of G taking E to $\mu(\gamma^{-1}E)$, the equality itself following from the change of variable formula for measures. The uniqueness part of the Riesz Representation Theorem now guarantees that $\mu\gamma^{-1} = \mu$, i.e. that $\mu(\gamma^{-1}E) = \mu(E)$ for every $\gamma \in G$, which (upon replacing γ by γ^{-1}) is what we wanted to prove. □

6.3. **Further invariant "measures."** The argument used to prove Theorem 6.1 can, with very little modification, be adapted to give the existence of invariant objects in different settings. For example, for G any abelian group let's define a "finitely additive probability measure" (henceforth: a "FAPM") on G to be a function $\mu : 2^G \to [0, 1]$ that is finitely additive, and for which $\mu(G) = 1$. We'll call such a function "G-invariant" if $\mu(gE) = \mu(E)$ for every subset E of G. (Here and henceforth, 2^S denotes the collection of all subsets of the set S. Note also that our notion of "G-invariant" is really "left G-invariant:" more on this later.) Our previous argument yields:

Theorem 6.2. *If G is any abelian group then there exists a finitely additive G-invariant probability measure on* 2^G.

Proof. Let $B(G)$ denote the collection of bounded, real-valued functions on G. Now repeat the proof of Theorem 6.1, replacing $C(G)$ by $B(G)$. The result is a mean Λ on $B(G)$ that is G-invariant in the sense that $L_\gamma^\sharp \Lambda = \Lambda$ for every $\gamma \in G$. The finitely additive G-invariant measure we seek is then given by

$$\mu(E) = \Lambda(\chi_E) \qquad (E \subset G),$$

where χ_E denotes the characteristic function of E (taking value 1 on E and 0 off E). □

For another generalization, let's observe that in the proofs of either of the previous two theorems the crucial role was played by the operators L_γ which identified a group element γ with its action on the whole group. Suppose, instead, that our group G is simply a group of transformations taking a set S into itself (so necessarily each element of G is a bijection of S onto itself). If, in the proof of Theorem 6.2, we define, for each $\gamma \in G$ the operator L_γ on $B(S)$ (the space of all bounded real-valued functions on S) by:

$$L_\gamma f(s) := f(\gamma(s)) \qquad (f \in B(S), s \in S),$$

then, just as in the previous two proofs, L_γ is a linear transformation taking $B(S)$ onto itself, and the family of adjoint operators $\mathscr{A} := \{L_\gamma^\sharp : \gamma \in G\}$ is a group of linear transformations taking the dual space $B(S)^\sharp$ into itself, each of which is continuous in the weak-star topology of $B(S)^\sharp$. Thus if we assume G is abelian, let K denote the set of means on $B(S)$ and take $V = B(S)^\sharp$ in its weak-star topology (pointwise convergence on V), then once again the triple $(V, K, \mathscr{A})$ satisfies the hypotheses of the Markov-Kakutani Theorem, which rewards us with an element of K that is $L_\gamma^\sharp$-invariant for every $\gamma \in G$. From this G-invariant mean arises, just as in the proof of Theorem 6.2, a G-invariant FAPM.[11]

If, in addition, S is a compact Hausdorff space, then in the proof of Theorem 6.1 we can replace $C(G)$ by $C(S)$, after which the proof of that theorem goes through word-for-word to produce a G-invariant Borel probability measure on S. To summarize:

Theorem 6.3. *Suppose G is an abelian group of transformations taking a set S into itself. Then:*

(a) *There exists a G-invariant finitely additive probability measure on 2^S.*

[11] This was originally proved, via a different argument, by Banach—see [3, Ch. II, pp. 30–31].

(b) *If, in addition, S is a compact, Hausdorff, topological space on which each element of G is continuous, then there exists on the Borel sets of S a G-invariant (countably additive) probability measure.*

6.4. **Application: Paradoxical decompositions.** One of the most intriguing results in all of mathematics was discovered in the 1920's by Banach and Tarski [4]. One version of this "Banach-Tarski Paradox" asserts that the closed unit ball $\mathbb{B}^3$ of $\mathbb{R}^3$ can be decomposed "paradoxically" into finitely many pairwise disjoint sets $\{A_1, A_2, \ldots, A_n, B_1, B_2, \ldots, B_m\}$ which can then be reassembled via rotations $\{\rho_1, \rho_2, \ldots, \rho_n, \sigma_1, \sigma_2, \ldots, \sigma_m\}$ into *two* copies of the unit ball:

$$\mathbb{B}^3 = \bigcup_{j=1}^{n} \rho_j(A_j) = \bigcup_{k=1}^{m} \sigma_k(B_k) \,. \tag{18}$$

The work of the previous section shows that nothing like this can happen for either the unit circle $\mathbb{T}$, or the closed unit disc $\mathbb{B}^2$. For the circle, Theorem 6.2 (with $G = \mathbb{T}$) shows that there exists a finitely additive probability measure μ on $2^{\mathbb{T}}$ that is invariant under rotations of the circle (i.e., under the group operation: complex multiplication of points on the circle). If there were a rotational paradoxical decomposition for $\mathbb{T}$ like the one given by (18) (with $\mathbb{T}$ on the left-hand side instead of $\mathbb{B}^3$), then we'd have, using the rotation-invariance of μ in the second line below:

$$\begin{aligned}
1 &= \mu(\mathbb{T}) = \mu\left(\bigcup_{j=1}^{n} A_j \cup \bigcup_{k=1}^{m} B_k\right) \\
&= \sum_{j=1}^{n} \mu(A_j) + \sum_{k=1}^{m} \mu(B_k) \\
&= \sum_{j=1}^{n} \mu(\rho_j(A_j)) + \sum_{k=1}^{m} \mu(\sigma_k(B_k)) \\
&= \mu\left(\bigcup_{j=1}^{n} \rho_j(A_j)\right) + \mu\left(\bigcup_{k=1}^{m} \sigma_k(B_k)\right) \\
&= \mu(\mathbb{T}) + \mu(\mathbb{T}) \\
&= 2
\end{aligned}$$

a contradiction!

Similarly, Theorem 6.3, with S the closed unit disc $\mathbb{B}^2$ and G the group of rotations of $\mathbb{B}^2$ about the origin, shows that there exists a rotation-invariant

FAPM on $\mathbb{B}^2$, and this renders impossible a rotation-induced paradoxical decomposition of $\mathbb{B}^2$.

The fact that there *is* a paradoxical decomposition for $\mathbb{B}^3$ shows that there does *not* exist on $2^{\mathbb{B}^3}$ a rotation-invariant FAPM. Note that the rotation-invariance of Lebesgue measure on $\mathbb{B}^3$ shows (not surprisingly) that the sets involved in the paradoxical decomposition of $\mathbb{B}^3$ cannot be Lebesgue measurable.

Before moving on, let's give an elementary example of a paradoxical decomposition. This is the *free group group F_2 on two generators;* its elements are "reduced words" of the form $x_1 x_2 \cdots x_n$ for $n \in \mathbb{N}$ where each x_j comes from the set of symbols $\{a, a^{-1}, b, b^{-1}\}$, subject only to the restriction that no symbol occurs next to its "inverse." Multiplication in F_2 is defined to be concatanation of words, followed by "reduction" (e.g. $aba^{-1} \cdot abba = abbba = ab^3a$). Upon allowing the "empty word" to belong to F_2 we obtain a group, which one can visualize as the fundamental group of a figure-eight.

Theorem. *F_2 is paradoxical.*

Proof. Here "paradoxical" is taken with respect to the mappings of left multiplication by group elements. We are going to find disjoint subsets A_1, A_2, B_2, B_2 of F_2, and left-multiplication mappings $\varphi_1, \varphi_2, \psi_1, \psi_2$ on F_2 such that

$$F_2 \ = \ \varphi_1(A_1) \cup \varphi_2(A_2) \ = \ \psi_1(B_1) \cup \psi_2(B_2)\,. \tag{19}$$

To this end, for $x \in \{a, a^{-1}, b, b^{-1}\}$ let $W(x)$ denote the set of reduced words that begin with x. For example, a and $ab^{-1}ab^2$ belong to $W(a)$, while b and $a^{-1}ba^2b^{-1}$ do not. Let

$$A_1 = W(a), \quad A_2 = W(a^{-1}), \quad B_1 = W(b), \quad \text{and} \quad B_2 = W(b^{-1})\,.$$

Clearly these are pairwise disjoint subsets of F_2. For each $x \in F_2$ define

$$\varphi_1(x) = \psi_1(x) = e \cdot x \ (= x), \quad \varphi_2(x) = a \cdot x \quad and \quad \psi_2(x) = b \cdot x\,.$$

One checks easily that the sets and maps defined above satisfy (19), and so yield the desired paradoxical decomposition of F_2. □

Simple as it seems, F_2 is, in fact, the foundation stone for the Banach-Tarski Paradox (see, e.g., [32, Chapter 1] or [34, Chapter 3]).

6.5. **Extending invariant measures.** Can (normalized) Lebesgue measure on the unit circle $\mathbb{T}$ be extended to a rotation-invariant probability measure on $2^{\mathbb{T}}$? The usual construction of a non-measurable subset of $\mathbb{T}$ shows that this is impossible, unless we give up on countable additivity. Indeed, that construction, which involves choosing (via the Axiom of Choice) one element from each coset of $\mathbb{T}$ modulo its subgroup of rational points, results in a subset NM whose rotates, via the rational subgroup, form a countable, pairwise disjoint family of sets whose union is $\mathbb{T}$. If Lebesgue measure on $\mathbb{T}$ could be extended to a rotation-invariant (countably additive!) measure on all subsets of $\mathbb{T}$, then that measure would have to assign the same weight to each rational rotation of NM, and the sum of the weights of all those rotates would have to add up to 1; a contradiction.

By Theorem 6.2 we know that there is a rotation-invariant FAPM on $2^{\mathbb{T}}$; what we *don't* know is whether or not such a set function can coincide on the Borel sets with Lebesgue measure. We'll see shortly that, even in the setting of compact abelian groups, such an extension is always possible. Our starting point is the following "real" version of the fundamental extension theorem for linear functionals (see, for example, [3, Ch. II, pp. 27–29], [19, Theorem 3.4, page 21] or [31, Theorem 3.2, pp. 57–58]).

The Hahn-Banach Theorem. *Suppose V is a vector space over the real field and $p : V \to \mathbb{R}$ is a* gauge function *on V, i.e.,*

- $p(u+v) \le p(u) + p(v)$ *for all $u, v \in V$, and*
- $p(av) = ap(v)$ *for every $a \in \mathbb{R}$ with $a \ge 0$ and $v \in V$.*

If W is a linear subspace of V on which there is a linear functional λ with $\lambda(w) \le p(w)$ for all $w \in W$, then there exists a linear extension Λ of λ to V such that

$$\Lambda(v) \le p(v) \quad \text{for all } v \in V.$$

Now consider the problem of extending Lebesgue measure m on the unit circle $\mathbb{T}$ to a finitely additive measure μ on $2^{\mathbb{T}}$. Lebesgue measure induces, via integration, a rotation-invariant mean λ on $C(\mathbb{T})$, and $C(\mathbb{T})$ is a linear subspace of $B(\mathbb{T})$, the space of all bounded real-valued functions on $\mathbb{T}$. In order to make the desired extension of m it will be enough to extend λ to a rotation-invariant mean on $B(\mathbb{T})$. Once again the Markov-Kakutani theorem comes to our rescue! (For the rest of this section we'll be following [13, §3.3 and §3.4].)

The "Invariant" Hahn-Banach Theorem. *Suppose V is a vector space and G is a commutative family (not necessarily a group) of linear transformations $V \to V$. Suppose W is a linear subspace of V that is taken into itself by* *every transformation in G, and that p is a gauge function on V for which*

$$p(\gamma(v)) \le p(v) \text{ for every } v \in V \text{ and } \gamma \in G.$$

Suppose λ is a G-invariant functional on W that is dominated by p, i.e.,

$$\lambda \circ \gamma = \lambda \text{ for all } \gamma \in G \quad \text{and} \quad \lambda(v) \le p(v) \text{ for all } v \in W.$$

Then λ has a G-invariant linear extension to V that is dominated on V by p.

Proof. Let $V^\sharp$ denote the algebraic dual of V (i.e. the space of *all* linear functionals on V) endowed with the weak-star topology—the topology of pointwise convergence on V. Let K be the collection of all linear extensions of λ to V that are dominated on V by p. Clearly K is a convex subset of $V^\sharp$ that is, by the Hahn-Banach Theorem (!!) non-empty.

Claim: *K is weak-star compact in $V^\sharp$.*

To prove the claim, note that if $\Lambda \in K$ then for every $x \in V$ we have, in addition to the defining property $\lambda(x) \le p(x)$, also $-\lambda(x) = \lambda(-x) \le p(-x)$. Thus

$$(20) \qquad -p(-x) \;\le\; \lambda(x) \;\le\; p(x) \qquad (x \in V)$$

i.e. $\lambda(x) \in I_x := [-p(-x), p(x)]$. Thus K is a subset of the compact product space $P = \prod_{x \in V} I_x$. Now convergence (of nets) in P is just pointwise convergence on the index set V, hence the topology of P agrees, on K, with the weak star topology. It's easy to see that K is closed in P, hence it is weak-star compact in $V^\sharp$.

Finally, since each $\gamma \in G$ is a linear map $V \to V$, it has an *adjoint* $\gamma^\sharp\colon V^\sharp \to V^\sharp$, defined by

$$\gamma^\sharp(\Lambda) = \Lambda \circ \gamma \qquad (\Lambda \in V^\sharp).$$

Let $\mathscr{A} := \{\gamma^\sharp\colon \gamma \in G\}$. One checks easily that $\mathscr{A}$ is an abelian group of weak-star continuous linear maps on $V^\sharp$ each of which, thanks to the "G-subinvariance" of the gauge function p ($p \circ \gamma \le p$ for every $\gamma \in G$), takes K into itself. Thus the triple $(V^\sharp, K, \mathscr{A})$ satisfies the hypotheses of the Markov-Kakutani theorem, so there exists $\Lambda \in K$ fixed by $\mathscr{A}$, i.e., $\Lambda \circ \gamma = \gamma^\sharp(\Lambda) = \Lambda$ for every $\gamma \in G$. This functional Λ is the desired G-invariant extension of our original one λ. □

Here, stated in great generality, is our application to measures. To bring it down to earth, think of μ as normalized Lebesgue measure on the Borel subsets of the unit circle, and G the group of rotations of the circle.

Corollary. *Let S is a compact topological space upon which acts a commutative family G of continuous mappings Suppose μ is a (countably additive) probability measure on the Borel subsets $\mathscr{B}$ of S, and that μ is G-*invariant*, i.e. $\mu(g(E)) = \mu(E)$ for every $g \in G$ and $E \in \mathscr{B}$. Then μ extends to a G-invariant, finitely additive probability measure on 2^S.*

Proof. Let λ be the mean on $C(S)$ defined by integration against μ. Let $B(S)$ denote the space of all bounded real-valued functions on S, and for $g \in G$ let L_g denote the linear map on $B(S)$ defined by

$$L_g f(s) = f(g(s)) \qquad (f \in B(S), s \in S).$$

By the change of variable formula of measure theory, λ is invariant for each of the mappings L_g, in the sense that $\lambda \circ L_g = \lambda$ for each $g \in G$. On $B(S)$ define the gauge function p by

$$p(f) := \sup_{s \in S} f(s) \qquad (f \in B(S)).$$

Clearly p is L_G-invariant ($p \circ L_g = p$ for every $g \in G$), and $\lambda \leq p$ on $C(S)$.

The Invariant Hahn-Banach Theorem now supplies an extension of λ to a linear functional Λ on $B(S)$ that's also dominated by p, and is invariant for each mapping L_g ($g \in G$). Upon applying inequality (20) to our gauge function $p(\cdot) = \sup_S(\cdot)$, we see that

$$\inf_{s \in S} f(s) \ \leq \ \Lambda(f) \ \leq \ \sup_{s \in S} f(s) \qquad (f \in B(S)),$$

so if $f \geq 0$ on S the $\Lambda(f) \geq 0$, i.e., Λ is a positive linear functional on $B(S)$. Since $\Lambda)(1) = \lambda(1) = 1$, the functional Λ is a G-invariant mean on $B(S)$.

The desired extension, $\tilde{\mu}$ of μ now emerges from a familiar formula:

$$\tilde{\mu}(E) := \Lambda(\chi_E) \qquad (E \subset S),$$

with the desired G-invariance following from the L_g-invariance of Λ for each $g \in G$. □

Our final application of the invariant Hahn-Banach theorem involves an attempt to create a "limit" for every bounded real sequence. We'll use the notation ℓ^∞ for the space of all such sequences. The result below is due (with a different proof) to Banach [3, Ch. II, p. 34].

Corollary: Banach Limits. *There exists a positive, translation-invariant linear functional Λ on ℓ^∞ such that*

$$\liminf_{n\to\infty} f(n) \le \Lambda(f) \le \limsup_{n\to\infty} f(n) \qquad (f \in \ell^\infty).$$

Proof. Let c denote the space of real sequences $f : \mathbb{N} \to \mathbb{R}$ for which $\lambda(f) := lim_{n\to\infty} f(n)$ exists (in $\mathbb{R}$). For $f \in \ell^\infty$ let $p(f) = \limsup_{n\to\infty} f(n)$. Then p is a gauge function on ℓ^∞, and $\lambda \le p$ on c. For $k \in \mathbb{N}$ define the "translation map" T_k on ℓ^∞ by

$$T_k f(n) := f(n+k) \qquad (f \in \ell^\infty, n \in \mathbb{N}).$$

Then each map T_k is a linear transformation that takes c into itself; furthermore, both λ and p are invariant for each $T_k \in G$, and the maps in G commute under composition with each other. Thus the Invariant Hahn-Banach Theorem applies, and produces a G-invariant extension Λ of λ to ℓ^∞, with $\Lambda \le p$ on ℓ^∞. By inequality (20) we have

$$\liminf_{n\to\infty} f(n) \;=\; -p(-f) \;\le\; \Lambda(f) \le p(f) \;=\; \limsup_{n\to\infty} f(n) \qquad (f \in \ell^\infty)$$

as desired. □

The functional Λ produced above is called a *Banach limit*; instead of writing $\Lambda(f)$ the notational convention for these is: $\operatorname*{LIM}_{n\to\infty} f(n)$. Let's note—still following Banach [3, Remarques, §3, p. 231]—that each Banach limit provides a translation-invariant FAPM on $2^{\mathbb{N}}$, defined by

$$\mu(E) \;:=\; \operatorname*{LIM}_{n\to\infty} \chi_E(n) \qquad (E \subset \mathbb{N}),$$

and that, since $\mu(\{n\}) = 0$ for every $n \in \mathbb{N}$, our FAPM μ is most assuredly *not* countably additive.

7. Fixed-point theorems for non-commuting maps

In the last section we showed that, on the collection of all subsets of the closed unit disc $\mathbb{B}^2$ of $\mathbb{R}^2$, there exists a finitely additive, rotation-invariant probability measure, and consequently—in stark contrast to what the Banach-Tarski theorem says about $\mathbb{B}^3$—that $\mathbb{B}^2$ does not have a paradoxical decomposition with respect to rotations. What made this possible was the combination of the Markov-Kakutani Fixed Point Theorem and the commutativity of the group of rotations of $\mathbb{B}^2$. Note that the rotation group of $\mathbb{B}^3$ does *not* share this commutativity (take, for an example, a pair of 45° rotations about different orthogonal axes; for more on this rotation group, see §7.3).

Let's ask another question: Does $\mathbb{B}^2$ have a paradoxical decomposition with respect to *all* of its isometries? Now, in addition to rotations, we allow reflections in a line through the origin, and it's easy to see that this renders the larger group non-commutative (see below for more details).

In this section we'll prove a version of the Markov-Kakutani theorem that applies to collections of affine continuous maps that are "almost" commutative, and we'll show that the new result applies to the full isometry group of $\mathbb{B}^2$. For completeness: we begin with a characterization of the isometries of $\mathbb{B}^n$, and then, with the goal in mind of making precise the notion of "almost commutativity," review some elementary group theory.

7.1. **The isometries of $\mathbb{B}^n$.** By an *isometry* of a subset of a metric space we mean a mapping of the set into itself that preserves distances. The most obvious examples of isometries of the closed unit ball $\mathbb{B}^n$ of $\mathbb{R}^n$ are the ones induced by *orthogonal matrices*, i.e. the maps $v \to Av$ where A is an $n \times n$ real matrix whose transpose is its inverse: $A^tA = AA^t = I$. In fact, these are the *only* isometries of $\mathbb{B}^n$. In what follows, $O(n)$ denotes the group of orthogonal $n \times n$ matrices. We'll regard elements of $\mathbb{R}^n$ as column matrices, and we'll use the notation $\langle u, v\rangle$ for the inner product in $\mathbb{R}^n$.

Theorem. *Suppose $T : \mathbb{B}^n \to \mathbb{B}^n$ is an isometry. Then there is a unique $A \in O(n)$ such that $T(v) = Av$ for every $v \in \mathbb{B}^n$.*

In particular, every such isometry maps the ball *onto* itself, and extends to a linear transformation of the ambient Euclidean space.

Proof. Let's first assume that T is an isometry taking $\mathbb{B}^n$ into $\mathbb{R}^n$.

(a) *Suppose, in addition, that $T(0) = 0$. Then $\langle T(u), T(v)\rangle = \langle u, v\rangle$ for every pair u, v of vectors in $\mathbb{B}^n$.*

This follows immediately from the relationship between norms of differences and inner products. For $u, v \in \mathbb{B}^n$:

$$\|u - v\|^2 = \|u\|^2 - 2\langle u, v\rangle + \|v\|^2$$

and

$$\begin{aligned} \|T(u) - T(v)\|^2 &= \|T(u)\|^2 - 2\langle T(u), T(v)\rangle + \|T(v)\|^2 \\ &= \|u\|^2 - 2\langle T(u), T(v)\rangle + \|v\|^2 \end{aligned}$$

where the second equality arises from the fact that the distance from the vector 0 to v is the same as that from $0 = T(0)$ to Tv. Similarly the distance from u to v is the same as that from Tu to Tv, so the left-hand sides of both of the equations above are equal, hence so are the right-hand sides, and this yields the desired identity.

(b) *If T is an isometry taking $\mathbb{B}^n$ into $\mathbb{R}^n$ with $T(0) = 0$, then there exists $A \in O(n)$ for which $T(v) = Av$ for every $v \in \mathbb{B}^n$.*

Let $\{e_1, e_2, \ldots e_n\}$ denote the standard orthonormal basis for $\mathbb{R}^n$. Since each of these vectors belongs to $\mathbb{B}^n$ we can apply T to them, and, since T preserves inner products, the result is another orthonormal basis $\{f_1, f_2, \ldots f_n\}$ for $\mathbb{R}^n$. Let A be the matrix that has, as its j-th column, the coefficients of f_j with respect to the original basis $\{e_j\}$. Then $A \in O(n)$, and $T(e_j) = Ae_j$, hence for every $v \in \mathbb{B}^n$:

$$T(v) \;=\; \sum_{j=1}^{n}\langle T(v), f_j\rangle f_j \;=\; \sum_{j=1}^{n}\langle T(v), T(e_j)\rangle T(e_j) \;=\; \sum_{j=1}^{n}\langle v, e_j\rangle Ae_j \;=\; Av$$

as desired.

(c) It follows from part (b) that any isometry T of $\mathbb{B}^n$ into $\mathbb{R}^n$ has the form $T(v) = Av + T(0)$ for some orthogonal matrix A. Thus T takes $\mathbb{B}^n$ onto the closed ball of radius 1 centered at $T(0)$, so if $T(\mathbb{B}^n) = \mathbb{B}^n$ then $T(0) = 0$, hence T must be effected by multiplication by an orthogonal matrix. That this matrix is unique is clear, since—as we saw in part (b)—its columns are just the images of the standard basis vectors for $\mathbb{R}^n$ under the action of T. This completes our characterization of isometries of $\mathbb{B}^n$. □

Before proceeding, let's note that:

If $n \geq 2$ then the group $O(n)$ is not commutative.

Indeed, here are two matrices in $O(2)$ that do not commute:

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

the first of which induces rotation through an angle of 45 degrees, while the second induces reflection about the horizontal axis. To get an example in $O(n)$

for $n > 2$ just put each of the above matrices in the upper right hand corner of an $n \times n$ matrix, and fill in the remaining entries with zeros.

7.2. **A little group theory.** Suppose G is a group and H a subgroup (notation: $H < G$). To say H is a *normal* subgroup of G (notation: $H \lhd G$) means that $gH = Hg$ for any $g \in G$. For subsets A and B of G we'll write AB for the collection of all products ab with $a \in A$ and $b \in B$.

Proposition. *Suppose $H < G$. Then $H \lhd G$ if and only if the collection of cosets of G mod H forms a group under the multiplication inherited from G, in which case $(g_1H)(g_2H) = g_1g_2H$ for all $g_1, g_2 \in G$.*

Proof. If $H \lhd G$ and $g_1, g_2 \in G$, then

$$(g_1H)(g_2H) = g_1(Hg_2)H = g_1g_2HH = g_1g_2H$$

as desired. From this it follows that for each $g \in G$ we have:

$$(gH)(g^{-1}H) = gg^{-1}H = eH = H$$

(here e denotes the identity element of G) and similarly $(g^{-1}H)(gH) = H$. Thus the set of cosets of G mod H (notation: G/H) is a group under the inherited multiplication.

Conversely, if the set of cosets G/H is a group under the obvious multiplication, then for any $g \in G$:

$$gHg^{-1} \subset gHg^{-1}H = gg^{-1}H = eH = H$$

so $gH = Hg$, as desired. □

Corollary. *If G is a group and Φ is a homomorphism of G onto a group G', then* $\ker \Phi$ *is a normal subgroup of G.*

Proof. $G/\ker \Phi$ is, in the operation inherited from G, a group (that is isomorphic to G'). Thus $\ker \Phi \lhd G$ by the above Corollary. □

Returning to the setting of the last section, note that, since the matrices A in $O(n)$ are characterized by the equation $AA^t = A^tA = I$, the determinant of each such matrix is either 1 or -1. Since the determinant function is multiplicative, it effects a homomorphism of $O(n)$ onto G_2, the multiplicative group $\{1, -1\}$, which is commutative. The kernel of this homomorphism, those orthogonal matrices with determinant 1, is called the "special orthogonal group" of $n \times n$ matrices, denoted $SO(n)$. By our previous corollary, *$SO(n)$ is a normal subgroup of $O(n)$*, and since G_2 is commutative, so is the quotient group $O(2)/SO(2)$.

Thus, in contrast to the 3-dimensional case, SO(2) is commutative, hence for the chain of normal subgroups $\{I_2\} \triangleleft SO(2) \triangleleft O(2)$ (where I_2 is the 2×2 identity matrix), the quotient of each subgroup by the previous one is commutative. This is the "almost-commutativity" alluded to at the beginning of this section. Here it is in more generality:

Proposition. *Suppose G is a group and H a normal subgroup. Then G/H is commutative if and only if every pair g_1, g_2 of elements of G has a "commutator" $h \in H$ such that $g_1 g_2 = g_2 g_1 h$.*

Proof. This is just a rephrasing of the condition: $g_1 g_2 H = g_2 g_1 H$. □

7.3. **On the geometric structure of $O(n)$** (optional)**.** The following material, while illuminating, is not needed for the sequel.

The case $n = 2$ is special:

Proposition. *If $A \in O(2)$ has determinant 1 (i.e. if $A \in SO(2)$) then it induces on $\mathbb{R}^2$ a rotation about the origin. If $A \in O(2)$ has determinant -1, then it induces on $\mathbb{R}^2$ a reflection in a line through the origin.*

Proof. Each $A \in O(2)$ takes the unit vectors (e_1, e_2) respectively along the horizontal and vertical axes to an orthogonal pair (u, v) of unit vectors, where u is the rotate of e_1 through some angle θ, and v is either the rotate of e_2 through that angle—in which case the determinant of A is 1 and A is the mapping of "rotation by θ"—or v is the negative of that vector. In this latter case $\det A = -1$, and A effects the mapping of reflection in the line through the origin parallel to u. □

Proposition. *If $A \in SO(3)$ then the map $x \to Ax$ is a rotation of $\mathbb{R}^3$, with center at the origin.*

What we're saying here is that for each $A \in SO(3)$ the associated linear transformation fixes a line through the origin (the axis of rotation), and, in the plane through the origin orthogonal to that line, acts as a rotation. This isn't obvious. Clearly the product of two matrices in $SO(3)$ also belongs to $SO(3)$ (multiplicativity of the determinant), but it's not so clear that the result of composing two rotations about different axes has to fix a line through the origin.

Proof. Suppose $A \in SO(3)$. To find the axis of rotation we need to show that $Av = v$ for some non-zero vector $v \in \mathbb{R}^3$, i.e. that 1 is an eigenvale of A, i.e. that $\det(A - I) = 0$. For this, note that since $AA^t = I$ we have

$$(A - I)A^t = AA^t - A^t = I - A^t = -(A - I)^t$$

hence, since $\det A = \det A^t = 1$:

$$\begin{aligned} \det(A - I) &= \det(A - I)\det(A^t) = \det[(A - I)A^t] \\ &= \det[-(A - I)^t] = (-1)^3 \det(A - I)^t \\ &= -\det(A - I) \end{aligned}$$

so $\det(A - I) = 0$, as desired.

Let $v_1 \in \mathbb{R}^3$ be the non-zero vector promised by the last paragraph: $Av_1 = v_1$. Let (v_2, v_3) be an orthonormal basis for the subspace E of $\mathbb{R}^3$ orthogonal to v_1. Then (v_1, v_2, v_3) is an orthonormal basis for $\mathbb{R}^3$, relative to which the matrix of the transformation $x \to Ax$ has block diagonal form $\operatorname{diag}(1, B)$, where B is a 2×2 orthogonal matrix. Thus A and B have the same determinant, so $\det B = 1$, i.e. $B \in SO(2)$, so by the previous proposition B induces on E either the identity map or a rotation (about the origin). □

This last proposition (or more accurately, the statement that each rotation in $\mathbb{R}^3$ about the origin has a fixed axis), was first proved by Euler in 1775-6; it's called "Euler's rotation theorem." For a lively article that gives much more detail about this result, see [27]. The results above on $SO(2)$ and $SO(3)$ generalize to higher dimensions, but now reflections can be present. For $O(n)$ the full story is this (see e.g. [6, Theorem 10.12, page 152]):

> *For $A \in O(n)$ there exists an orthonormal basis for $\mathbb{R}^n$ relative to which the transformation $x \to Ax$ has block diagonal matrix $(I_p, -I_q, B_1, \ldots, B_r)$ where the I's are identity matrices of orders p and q respectively, the B's are 2×2 orthogonal matrices, and $p + q + 2r = n$.*

7.4. **First generalization of Markov-Kakutani.** Here, at last, is a generalization of the Markov-Kakutani theorem that applies to isometries of $\mathbb{B}^2$.

Theorem. *Suppose K is a compact, convex subset of a (Hausdorff) topological vector space, and $\mathscr{A}$ is a family of continuous affine maps $K \to K$ that contains a commutative subfamily $\mathscr{C}$ such that: for every pair A_1, A_2 of mappings in $\mathscr{A}$ there exists $C \in \mathscr{C}$ such that $A_1A_2 = A_2A_1C$. Then $\mathscr{A}$ has a common fixed point in K.*

Proof. We know from the original Markov-Kakutani theorem (which we can regard as the case $\mathscr{C} = \{\text{identity element of } G\}$ of the theorem we're trying to prove) that K_0, the set of common fixed points for $\mathscr{C}$ in K, is not empty. It's easy to see from the continuity and affineness of the mappings in $\mathscr{C}$ (or, for that matter, from the proof of the Markov-Kakutani Theorem), that K_0 is also compact and convex.

Claim: *Each map $A \in \mathscr{A}$ takes K_0 into itself.*

To see why this is true, take $p \in K_0$ and $A \in \mathscr{A}$. We need to show that $A(p) \in K_0$, i.e. that $A(p)$ is fixed by every $C \in \mathscr{C}$. For each such map C we know that there exists a map $C_1 \in \mathscr{C}$ such that $C \circ A = A \circ C \circ C_1$, hence

$$C(A(p)) = A(C(C_1(p))) = A(C(p)) = A(p)$$

so $A(p)$, as desired, is fixed by C.

Let $\mathscr{A}_0$ denote the collection of restrictions of maps in $\mathscr{A}$ to K_0. By the above claim, this is a collection of continuous, affine maps taking K_0 into itself.

Claim: *$\mathscr{A}_0$ is commutative.*

Indeed, suppose A_1 and A_2 belong to $\mathscr{A}$. There exists $C \in \mathscr{C}$ such that $A_1 \circ A_2 = A_2 \circ A_1 \circ C$, so for $p \in K_0$ (hence a fixed point for C) we have

$$A_1(A_2(p)) = A_2(A_1(C(p))) = A_2(A_1(p))$$

i.e., $A_1 \circ A_2 = A_2 \circ A_1$ on K_0, as desired.

It now follows from the original Markov-Kakutani theorem that $\mathscr{A}_0$ has a fixed point in K_0; clearly every such fixed point is also one for $\mathscr{A}$. □

The proof that, for the closed unit disc $\mathbb{B}^2$, there is a finitely additive, rotation-invariant probability measure on $2^{\mathbb{B}^2}$ now goes through *verbatim* to show that such a measure can be taken to be *isometry*-invariant. In particular,

> *The closed unit disc cannot be paradoxically decomposed by isometries.*

If we view the isometry group of the closed unit disc (a.k.a the orthogonal group $O(2)$) as acting on itself by left composition, then it, too, supports—on its collection of all subsets—an invariant FAPM, and so has no paradoxical decomposition. Recall that by contrast there exist groups that *do* have a paradoxical decompositions, and therefore do *not* support FAPM's, an example being the free group F_2 on two generators discussed in §6.4.

Application: *Invariant extension of Lebesgue measure on* $\mathbb{R}$. Lebesgue measure m on the Borel subsets of the real line is invariant under translations, and "scales properly" under dilations. More precisely, for each pair (r, t) of real numbers, and each Borel subset E of $\mathbb{R}$, we have $m(rE + t) = |r|m(E)$.

Theorem. *There is an extension of Lebesgue measure to a finitely additive measure μ on all subsets of $\mathbb{R}$, such that for each $E \subset \mathbb{R}$:*

$$\mu(rE + t) = |r|\mu(E) \qquad (r, t \in \mathbb{R}) \tag{21}$$

and

$$m_*(E) \le \mu(E) \le m^*(E) \tag{22}$$

where $m_*(E)$ and $m^*(E)$ denote, respectively, the inner and outer measures of E.

Proof. Let G denote the group of maps $\gamma_{r,t} \colon x \to rx + t$ on $\mathbb{R}$, where r and t run through all real numbers. Although not commutative, G is generated by two commutative subgroups: R, the group of dilations $x \to rx$ for $r \in \mathbb{R}$, and T, the group of translations $x \to x + t$ for $t \in \mathbb{R}$. The map that takes $g_{r,t}$ to the dilation $g_{r,0}$ is a homomorphism taking G onto R with kernel T, hence we have the chain of normal subgroups

$$\{I\} \lhd T \lhd G \quad \text{with} \quad G/T = R \quad \text{commutative.} \tag{23}$$

Let V be the collection of real-valued functions f on $\mathbb{R}$ such that the upper integral of $|f|$ over $\mathbb{R}$ is finite:

$$p(f) := \int^* |f| := \inf \left\{ \int |s| : s \in \mathscr{S}, |f| \le s \right\} < \infty$$

where $\mathscr{S}$ denotes the collection of Borel-measurable, simple, integrable functions on $\mathbb{R}$.

For $\gamma \in G$ define L_g on V by

$$(L_\gamma f)(x) = f(\gamma(x)) \qquad (f \in V, x \in \mathbb{R}),$$

Then each of these maps L_g is linear on V, and—by the change of variable theorem for Lebesgue integrals—the functional p is a gauge on V that is invariant for L_g:

$$p(R_\gamma f) = p(f) \qquad (f \in V, \gamma \in G).$$

Let W denote the subspace of V consisting of continuous, real-valued functions on $\mathbb{R}$ with compact support, and on W let λ be the linear functional of integration with respect to Lebesgue measure m. Then λ, too, is G-invariant. By (23) G is "almost commutative" so our generalization of the Markov-Kakutani theorem gives a corresponding generalization of the invariant Hahn-Banach theorem, and provides a G-invariant linear functional Λ on V that extends λ and is dominated on V by p.

To get the desired finitely additive measure: if E is a subset of R with finite outer measure then its characteristic function χ_E is in V (its upper integral is precisely $m^*(E)$), so we can set $\mu(E) = \Lambda(\chi_E)$ The G-invariance of Λ translates into property (21) for μ, while the fact that $\Lambda \leq p$ on V shows us that

$$\int_* f = -p(-f) \leq \Lambda(f) \leq p(f) = \int^* f \qquad (f \in V)$$

where on the right we see the *lower integral* of f, i.e. the supremum of the integrals of integrable simple functions that are $\leq f$ at each point of $\mathbb{R}$. In particular, for $f = \chi_E$ with $m^*(E) < \infty$ we obtain (22). Now extend μ to *all* subsets of $\mathbb{R}$ by defining $\mu(E) = \infty$ whenever $m^*(E) = \infty$, and check that, with the usual conventions involving arithmetic with ∞, the result is still a finitely additive measure that preserves the desired properties. □

7.5. **Markov-Kakutani: further generalization.** The argument used to prove the Markov-Kakutani generalization of the previous section is just the first step in an induction that yields something far more general. Let us say that a family $\mathscr{A}$ of selfmaps of some set "almost-commutes" (cf. [7, page 285] and [13, Theorem 3.2.1, pp. 155–156]) if there a chain of subfamilies

$$\text{(24)} \qquad \{\text{Identity map}\} = \mathscr{A}_0 \subset \mathscr{A}_1 \subset \mathscr{A}_2 \subset \cdots \subset \mathscr{A}_\text{n} = \mathscr{A}$$

such that for each $k \in \mathbb{N}$ and each pair A_1, A_2 of maps in $\mathscr{A}_k$ there exists a map $C \in \mathscr{A}_{k-1}$ such that $A_1 \circ A_2 = A_2 \circ A_1 \circ C$.

Theorem. *Suppose K is a compact, convex subset of a Hausdorff topological vector space, and $\mathscr{A}$ is an almost-commuting family of continuous, affine maps $K \to K$, then $\mathscr{A}$ has a common fixed point in K.*

Proof. This is an induction on the index n in (24). Note that (24) dictates that the family of maps $\mathscr{A}_1$ is commutative, so the case $n = 1$ of our theorem is just the original Markov-Kakutani theorem. The case $n = 2$ is, of course, our previous generalization of that theorem, and its proof gives the present generalization. To see how this goes, suppose the result is true for all chains (24) of length n. We desire to prove it for all chains of length $n + 1$. By our induction hypothesis, the family $\mathscr{A}_n$ has common fixed points in K, so the set K_n of all these fixed points is nonempty. As in the proof of the $n = 2$ case, K_n is also compact and convex, and just as in that proof, we can show that $\mathscr{A}_{n+1}$ takes K_n into itself, and acts commutatively thereon. Here, for the record, are the details. Suppose $p \in K_n$ and $A \in \mathscr{A}_{n+1}$. Then for each $C \in \mathscr{A}_n$ there exists $C_1 \in \mathscr{A}_n$ such that $C \circ A = A \circ C \circ C_1$, so:

$$C(A(p)) = A(C(C_1(p))) = A(C(p)) = A(p)$$

so $A(p)$ is a common fixed point for $\mathscr{A}_n$, i.e., $A(p) \in K_n$. Thus each member of $\mathscr{A}_{n+1}$ takes K_n into itself.

As for commutativity, suppose A_1 and A_2 belong to $\mathscr{A}_{n+1}$, and choose $C \in \mathscr{A}_n$ such that $A_1 \circ A_2 = A_2 \circ A_1 \circ C$. Then for each $p \in K_n$ we have, since $C(p) = p$:

$$A_1(A_2(p)) = A_2(A_1(C(p))) = A_2(A_1(p))$$

i.e., $A_1 \circ A_2 = A_2 \circ A_1$ on K_n. Thus $\mathscr{A}_n$ commutes on K_n, and so, by the original Markov-Kakutani Theorem, has a fixed point therein. □

7.6. **A little more group theory: solvability and amenability.** A group G (with identity element e) is called *solvable* if there is a chain of normal subgroups

$$\{e\} \ = \ G_0 \ \lhd \ G_1 \ \lhd \ G_2 \ \lhd \ \cdots \ \lhd \ G_n \ = \ G$$

with each quotient group G_k/G_{k-1} commutative ($k = 1, 2, \ldots, n$). Thus commutative groups are solvable ($n = 1$), and, as we saw in §7.2, so is $O(2)$ ($n = 2$). The last proposition of §7.2 shows that each solvable group is "almost commutative" in the sense of the last section, so the generalization given there of the Markov-Kakutani theorem yields:

Corollary. *Suppose K is a compact convex subset of a Hausdorff linear topological space, and $\mathscr{G}$ is a solvable group of continuous, affine selfmaps of K. Then $\mathscr{G}$ has a common fixed point.*

The earlier applications of the Markov-Kakutani theorem to providing probability measures (finitely or countably additive) that are invariant under commutative transformation groups extend immediately to "almost-commutative families" of transformations, and in particular to solvable groups. Here is one important special case:

Corollary. *If G is a solvable group then 2^G has a finitely additive G-invariant probability measure.*

In the 1920's von Neumann [26] proved this result in the course of studying groups that supported such FAPM's. He called such groups "measurable;" the currently preferred term, "amenable," was coined by M. M. Day in the late 1950's [8] . For the record:

Definition. *To say a group G is* amenable *means that there is a finitely additive G-invariant probability measure on 2^G.*

Positive linear functionals associated with FAPM's are called "means," hence the term "amenable," which is reputed to have been something of a pun on the part of Day[12].

Since we are now dealing with groups that need not be commutative, the notion of "invariant" for means and their associated finitely additive probability measures needs to be explored further. The problem is that now there are really two kinds of invariance: *left*- and *right*-, and hence two possible notions of amenability. More precisely, for a group G recall the space $B(G)$ of bounded real-valued functions on G, and from §6.5 (more precisely: our proof of the invariant Hahn-Banach theorem) the linear transformation $L_g\colon B(G) \to B(G)$ defined, for each $g \in G$ by

$$L_g f(x) = f(gx) \qquad (f \in B(G), x \in G).$$

For $g \in G$ let R_g denote the corresponding "right-hand" operator

$$R_g f(x) = f(xg) \qquad (f \in B(G), x \in G).$$

The means we've been talking about for $B(G)$ should really be called "left-invariant" means. Recall that these are positive linear functionals λ on $B(G)$ with $\lambda(1) = 1$ such that $L_g^\sharp \lambda = \lambda$ (i.e. $\lambda(g \circ L_g = \lambda(g))$ for every $g \in G$). However, in the non-commutative case there's an equally valid notion of "right-invariant" mean, namely: a positive linear functional ρ on $B(G)$ with $\rho(1) = 1$ and $R_g^\sharp \rho = \rho$.

[12] See [32, page 34], for example.

It turns out that whenever there's a left-invariant mean there's a right-invariant one, and even a "left-right-invariant one." In particular, there are *not* separate notions of left amenable, right amenable, and left-right amenable—it's all just "amenable." The arguments below, due to M. M. Day [7, Lemma 7, page 285], establish this.

To see that every left-invariant mean has a right-invariant cousin, define the linear transformation $J\colon B(G) \to B(G)$ by

$$Jf(x) = f(x^{-1}) \qquad (f \in B(G), x \in G)$$

and note that

$$R_g J = J L_{g^{-1}} \qquad \text{and} \qquad L_g J = J R_{g^{-1}}\,.$$

Thus if λ is a left-invariant mean for $B(G)$ then the calculation below shows that $\rho := J^\sharp \lambda$ is right invariant. For every $g \in G$:

$$R_g^\sharp \rho \;=\; R_g^\sharp J^\sharp \lambda \;=\; (JR_g)^\sharp \lambda \;=\; (L_{g^{-1}}J)^\sharp \lambda \;=\; J^\sharp L_{g^{-1}}^\sharp \lambda \;=\; J^\sharp \lambda \;=\; \rho$$

where the middle equality above comes from the first identity of the previous display, and the next-to-last one from the left-invariance of λ. It's easy to check that ρ is a positive linear functional on $B(G)$, and clearly $\rho(1) = 1$. Thus ρ is a right-invariant mean.

It's now easy to see that once we have a left-invariant mean on $B(G)$ then we actually have a *bi-invariant* one, i.e. a mean ν on $B(G)$ with $L_g^\sharp \nu = R_g^\sharp \nu = \nu$ for every $g \in G$. Indeed, letting λ be a left-invariant mean and ρ a right-invariant one, the definition below does the trick:

$$\nu(F) := \lambda(\widetilde{F}) \quad \text{where} \quad \widetilde{F}(g) := \rho(L_g F) \qquad (F \in B(G), g \in G).$$

One checks easily that ν is a mean on $B(G)$. As for its bi-invariance, a little calculation shows that for each $g \in G$ and $F \in B(G)$:

$$\widetilde{R_g F} = \widetilde{F} \quad \text{and} \quad \widetilde{L_g F} = L_g \widetilde{F}$$

from which we see that

$$\nu(R_g F) = \lambda(\widetilde{R_g F}) = \lambda(\widetilde{F}) = \nu(F)\,, \quad \text{and} \quad \nu(L_g F) = \lambda(\widetilde{L_g F}) = \lambda(L_g \widetilde{F}) = \nu(F)$$

as desired.

Finite groups are amenable (for the invariant measure, just assign, to each element, mass equal to "one over the order of the group"), and our work with the Markov-Kakutani theorem and its generalizations shows that the same is true of commutative groups and solvable groups. In the next section we'll show that the amenable groups are precisely the ones for which the conclusion of the Markov-Kakutani theorem holds!

For a simple example of a group that is *not* amenable, recall from §6.4 the free group F_2 on two generators. We observed in that section that F_2 has a paradoxical decomposition relative to left-multiplication by its own elements, hence it does not support, on its collection of all subsets, an invariant FAPM. Therefore F_2 it is not amenable.

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