A Survey of Wandering Domains in Complex Dynamics

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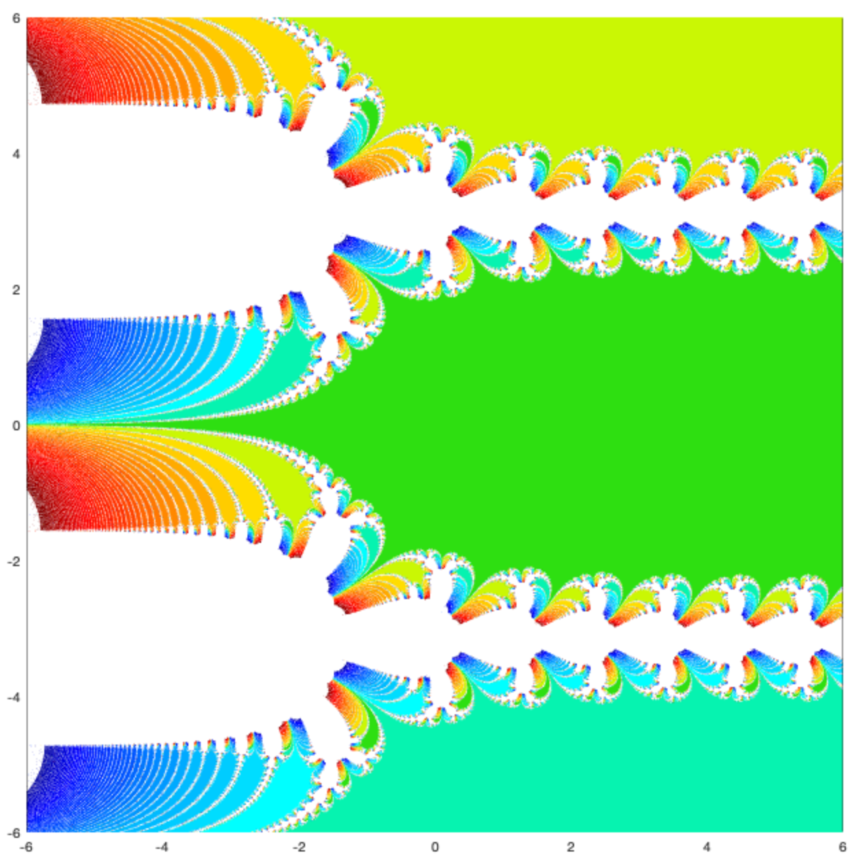


Figure 1: Basins of attraction for the fixed points of $g(z) = z - 1 + e^{-z}$.

Contents

1 Introduction

A central aspect of complex dynamics is the study of two complementary sets; the Fatou set and the Julia set. A formal definition of the Fatou and Julia sets will be given in Section 2. For now, we can think of the Fatou set is the collection of points where the dynamics are 'stable' and the Julia set is the collection of points where the dynamics are 'unstable'.

In this paper we will consider the following question: If f is a holomorphic function and U is a component of the Fatou set of f, is the iterated mapping $U \mapsto f(U)$ always eventually periodic?

In other words, we will explore the existence (or non-existence) of wandering domains; primarily focusing on the proof of the No Wandering Domains Theorem (see Theorem 3.1).

In order to construct our main results, we will first have do develop some theory concerning the iteration of holomorphic functions. As such, we will adopt the following notation:

$$f^{\circ n} = \underbrace{f \circ f \circ \cdots \circ f}_{n}$$

This is a common notation for iteration of function composition and avoids any possible confusion with f^n as the exponentiation of a function.

We will need to consider different Riemann surfaces, although the arguments will be restricted to the three simply connected surfaces;
(i) The complex plane $\mathbb{C}$,
(ii) the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and
(iii) the Riemann sphere or extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Definition 1.1: A **Riemann surface** is a one-complex-dimensional connected complex analytic manifold. We can think of a Riemann surface as a manifold $\mathscr{M}$ with a maximal set of charts $\{U_a, \phi_a\}_{a \in A}$ on $\mathscr{M}$ (i.e. $\{U_a\}_a$ is an open cover of $\mathscr{M}$ and $\phi_a : U_a \to \mathbb{C}$ is a homeomorphism onto an open subset of the complex plane) such that whenever $U_a \cap U_b$ is nonempty,

$$\phi_a \circ \phi_b^{-1} : \phi_b(U_a \cap U_b) \to \phi_a(U_a \cap U_b)$$

is holomorphic.

One significant aspect of the Fatou and Julia sets is that they decompose a Riemann surface into two complementary and completely invariant sets.

Definition 1.2 - For a function f, a set A is said to be
(i) **forward invariant** if $f(A) \subseteq A$;

(ii) **backward invariant** if $f^{-1}(A) \subseteq A$; and
(iii) **completely invariant** if A is both forward and backward invariant.

Finally, the following defines a wandering domain:

Definition 1.3 - Let Ω be a Riemann surface and $f : \Omega \to \Omega$ an analytic function. A component U of the Fatou set of f is
(i) **periodic** if for some $n \in \mathbb{N}$, $f^{\circ n}(U) = U$;
(ii) **eventually periodic** if for some $m \in \mathbb{N}$, $f^{\circ m}(U)$ is periodic; and
(iii) **wandering** if $f^{\circ n}(U) \cap f^{\circ m}(U) = \varnothing$ for all $n \neq m$.

Periodicity and eventually periodicity were well understood by Fatou and Julia, but the possibility of a wandering domain was left open. It wasn't until 1976 that Irvine Noel Baker showed in [6] an example of a holomorphic function with wandering domains. Then in 1985, Dennis Sullivan proved in [18] that for holomorphic functions on the Riemann sphere (i.e. for rational functions) no wandering domains exist.

While the No Wandering Domains Theorem is significant in and of itself, it is noted in [11] that the true significance of Sullivan's result is his application of quasiconformal maps in complex dynamics; a now standard technique.

In Section 3 we will prove Sullivan's result that every Fatou component is eventually periodic. Then in Section 4 we examine some of Baker's work and analyze two examples of transcendental entire functions of the complex plane which have wandering domains.

2 Preliminary Results

The goal of this section is the provide the majority of the background material which will be referenced in Sections 3 and 4. Any reader unfamiliar with complex dynamics should find the necessary definitions and theorems here for the results proven in the following sections. Other potentially useful definitions can be found in the Appendix.

2.1 The Riemann Sphere

In order to understand the structure of wandering domains, we must first understand the spaces in which we are working. This paper will explore analytic functions defined on Riemann surfaces; specifically, the the complex plane, $\mathbb{C}$, as well as the Riemann sphere, $\widehat{\mathbb{C}}$. It is assumed that the reader is familiar with the topology of the complex plane; however, that of the Riemann sphere will be briefly explored here. More information on the Riemann sphere can be found in [3], [8], and [16].

The Riemann sphere, also known as the extended complex plane, is the set $\mathbb{C} \cup \{\infty\}$. Topologically, the Riemann sphere is the one point compactification of the complex plane, i.e. all numbers of large modulus lie in a neighborhood of ∞ in the same way that all numbers with small modulus line in a neighborhood of 0. That is to say, z is near zero exactly when $1/z$ is near infinity and vice versa. The standard arithmetic operations of complex numbers are still valid in the extended complex plane, but there are some special considerations for the point ∞. For any finite nonzero z, we have

$$z + \infty = \infty, \quad z \cdot \infty = \infty, \quad \frac{z}{\infty} = 0, \quad \text{and } \frac{z}{0} = \infty$$

Note that similarly to the extended real numbers, $\infty - \infty$ and $0 \cdot \infty$ are undefined. The ability to divide by 0 and ∞ is part of what makes it convenient to consider functions on the Riemann sphere rather than the complex plane, as some functions with isolated singularities in $\mathbb{C}$ can be extended continuously to all of $\widehat{\mathbb{C}}$. An analytic function f with a pole at α can be continuously extended at α by defining $f(\alpha) = \infty$. However, this does not work for essential singularities.

We may regard the extended complex plane as a sphere since it is homeomorphic to the sphere S^2 via stereographic projection.[1] If π is the stereographic projection from the sphere (minus the north pole) to the plane, $\pi : S^2 \setminus (0,0,1) \to \mathbb{C}$, given by

$$\pi(x_1, x_2, x_3) = \frac{x_1 + ix_2}{1 - x_3}$$

by defining $\pi(0,0,1) = \infty$, we complete the spherical equivalence. We can now use this projection to embed the natural euclidian metric of $\mathbb{R}^3$ onto $\widehat{\mathbb{C}}$. This gives us the chord metric ρ defined by

$$\rho(z, w) = \frac{2|z - w|}{\sqrt{(1 + |z|^2)(1 + |w|^2)}}$$

for finite $z, w \in \mathbb{C}$ and

$$\rho(z, \infty) = \frac{2}{\sqrt{1 + |z|^2}}$$

for the point at infinity. The metric distance ρ represents the distance between any two points in $\widehat{\mathbb{C}}$ as the euclidian distance of the projection of those points in S^2. Some authors prefer to use the arc length, or spherical, metric ρ_a. This metric represents the distance between any two points as the arc length distance between their projections on S^2 (antipodal elements are now at a distance π). The chord metric and arc length metric are easily seen to be equivalent since

$$\rho(z, w) \leq \rho_a(z, w) \leq \frac{\pi}{2}\rho(z, w)$$

[1] see Appendix for explicit definition of S^2

for any $z, w \in \widehat{\mathbb{C}}$. For the remainder of the paper, whenever it is necessary to refer to a specific metric, I will be using the chord metric ρ.

It should be further noted that for any finite points on the sphere, we can find an equivalence between the euclidian metric and the chord metric. If $z, w \in \mathbb{C}$, $r = \min\{|z|, |w|\}$, and $R = \max\{|z|, |w|\}$ then

$$\frac{2}{1+R^2}|z-w| \leq \rho(z,w) \leq \frac{2}{1+r^2}|z-w|$$

So the open sets of $\mathbb{C}$ is precisely those of $\widehat{\mathbb{C}}$ which do not contain ∞.

We now wish to consider what makes a function holomorphic, or analytic, on $\widehat{\mathbb{C}}$. The metric ρ allows us to easily define open sets and continuity; however, it should be noted that not all holomorphic functions on the complex plane can be continuously extended to the Riemann sphere. For example, the exponential function e^z cannot be continuously extended to $\widehat{\mathbb{C}}$. The following definition is adapted from the standard definition for holomorphic on a Riemann surface, and gives the conditions for which a function is holomorphic on $\widehat{\mathbb{C}}$ using standard idea of being holomorphic on $\mathbb{C}$.[2]

Definition 2.1 - A continuous function $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is **holomorphic** at $z \in \widehat{\mathbb{C}}$ if one of the following hold:
(i) There is a neighborhood D of z such that $f : D \to \mathbb{C}$ is holomorphic in the usual sense.
(ii) If $z = \infty$ and $f(z) \neq \infty$, then the map $z \mapsto f(1/z)$ is holomorphic in a neighborhood of 0.
(iii) If $z \neq \infty$ and $f(z) = \infty$, then $z \mapsto 1/f(z)$ is holomorphic in a neighborhood of 0.
(iv) If $z = \infty$ and $f(z) = \infty$, then $z \mapsto 1/f(1/z)$ is holomorphic in a neighborhood of 0.

It turns out that every holomorphic function on $\widehat{\mathbb{C}}$ can be expressed as a rational function; that is, a function $R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, given by

$$R(z) = \frac{P(z)}{Q(z)}$$

where P and Q are polynomials with complex coefficients, Q is nonzero, and P and Q have no common factors. We define the **degree** of a rational function as the maximum of the degrees of its constituent polynomials; $\deg(R) = \max\{\deg(P), \deg(Q)\}$.

Finally, an important class of maps on the Riemann sphere are the Möbius transformations.

Definition 2.2 - A function $\tau : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ defined by

$$\tau(z) = \frac{az+b}{cz+d}, \quad a, b, c, d \in \mathbb{C}$$

such that $ad - bc \neq 0$ is called a **Möbius transformation** and is a conformal automorphism of $\widehat{\mathbb{C}}$. In fact, every conformal automorphism of $\widehat{\mathbb{C}}$ is a Möbius transformation. It will also be

[2] see Appendix for definition of holomorphic on $\mathbb{C}$ and holomorphic on an arbitrary Riemann surface

useful to note that the Möbius transformations which preserve the unit disk $\mathbb{D}$ are those of the form,

$$\tau(z) = \lambda \frac{z - \alpha}{\bar{\alpha} z - 1}$$

where $|\lambda| = 1$ and $|\alpha| < 1$.

When studying the dynamics of rational functions on $\widehat{\mathbb{C}}$, we are concerned primarily with topological properties, which are conjugation invariant. The dynamical properties of a rational function on the sphere are equivalent to the dynamics of the same rational function applied to a conformal automorphism of the sphere. As such, when studying holomorphic dynamics on the Riemann sphere, we are really considering an equivalence class of functions. If R and S are rational functions such that $S = \tau \circ R \circ \tau^{-1}$ for some Möbius transformation τ, then we say that R is conjugate to S. It follows easily that $S^{\circ n} = \tau \circ R^{\circ n} \circ \tau^{-1}$. Therefore, if R and S are conjugate, then we can study the dynamical properties of R by studying the dynamical properties of S and vice versa.

It is a simple exercise to show that a Möbius transformation, say τ, is holomorphic. The result follows directly from Definition 2.1, with $\tau(\infty) = a/c$ and $\tau(-d/c) = \infty$ (unless $c = 0$, then $\tau(\infty) = \infty$).

2.2 The Julia Set and the Fatou Set

In order to define the most familiar structures of complex dynamics, namely the Julia set and the Fatou set, we first need to establish some theory concerning normal families of functions. We begin with a preliminary definition of normality.

Definition 2.3 - A collection $\mathscr{F}$ of holomorphic functions, $f : X \to Y$ where X is a Riemann surface and Y is a compact Riemann surface, is called a **normal family** if every infinite sequence $\{f_n\}_{n=1}^{\infty} \subset \mathscr{F}$ contains a subsequence which converges locally uniformly. Alternatively, every sequence $\{f_n\}_{n=1}^{\infty} \subset \mathscr{F}$ contains a subsequence which converges uniformly on compact subsets of X.

Note that the above definition requires that the target space is compact. This is sufficient when working with functions on the Riemann sphere, but we will also be considering functions on the complex plane. As such, we also need to consider the idea of diverging locally uniformly.

Definition 2.4 - A sequence of functions $\{f_n\}_n$, $f_n : A \to B$ is said to **diverge locally uniformly** from B if for every compact $K_A \subset A$ and $K_B \subset B$, there is an n_0 such that $f_n(K_A) \cap K_B = \varnothing$ for all $n \geq n_0$.

It should be clear that if B is compact, then no sequence can diverge locally uniformly. Keeping Definition 2.3 in mind, we can define normal families of functions to include the case of noncompact target spaces.

Definition 2.5 - A collection $\mathscr{F}$ of holomorphic functions, $f : X \to Y$, where X and Y are Riemann surfaces, is called a **normal family** if every infinite sequence of functions in $\mathscr{F}$ contains a subsequence which either converges locally uniformly or diverges locally uniformly from Y.

In complex dynamics, we are particularly concerned with finding the cases when a sequences of functional iterates, $\{f^{\circ n}\}_{n=1}^{\infty}$, of a holomorphic function, forms a normal family. This brings us to the Fatou and Julia set.

Definition 2.6 - Let Ω be a Riemann surface and $f : \Omega \to \Omega$ be holomorphic. The **Fatou set** of f, denoted F_f, is the union of domains $D \subset \Omega$ such that the sequence of iterates $\{f^{\circ n}\}_{n=1}^{\infty}$ forms a normal family when restricted to D. The **Julia set** of f, denoted J_f, is the compliment of the Fatou set, $J_f = \Omega \backslash F_f$. A **Fatou component** is a maximal connected subset of F_f.

It is worth noting that the nomenclature for the Julia set was adopted much earlier than that of the Fatou set. Older texts may refer to the Fatou set as the set of normality or the stable set. Additionally, the Fatou set is always an open set, so every Fatou component is open.

For any given holomorphic function f, the Fatou set and Julia set are both completely invariant.[3] In general, the mapping $f : F_f \to F_f$ (or $f : J_f \to J_f$) may not be surjective, but it is for rational functions on the Riemann sphere. Additionally, either the Fatou set or the Julia set may be empty. For example, for any holomorphic function $f : \mathbb{D} \to \mathbb{D}$, where $\mathbb{D}$ is the unit disk, $F_f = \mathbb{D}$ and $J_f = \varnothing$ [15]. Conversely, for the functions $g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ given by $g(z) = (z-2)^2/z^2$ and the exponential function, $h : \mathbb{C} \to \mathbb{C}$, $h(z) = e^z$, we have $F_g = F_h = \varnothing$ (see [8] and [12] respectively). Cases like these may initially seem trivial, but to prove each requires a fair amount of insight.

Now although the necessary definitions have been presented, given any holomorphic function it is rarely an easy task to use the definition of normality to determine the components of the Fatou set or Julia set. Instead we often rely on theorems of Montel and Arzela-Ascoli.

Theorem 2.7 - Montel's Theorem - *Let Ω be a Riemann surface and let $\mathscr{F}$ be a collection of holomorphic maps $f : \Omega \to \widehat{\mathbb{C}}$ which omit three different values. That is, there are distinct points, $a, b, c \in \widehat{\mathbb{C}}$ such that $f(\Omega) \subset \widehat{\mathbb{C}} \setminus \{a, b, c\}$ for every $f \in \mathscr{F}$. Then $\mathscr{F}$ is a normal family.*

Definition 2.8: Let $\mathscr{F}$ be a family of functions $f : \mathscr{X}_0 \to \mathscr{X}_1$ where $(\mathscr{X}_0, d_0)$ and $(\mathscr{X}_1, d_1)$ are metric spaces. The family $\mathscr{F}$ is said to be **equicontinuous** on a domain D if for any $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x, y \in D$ and $f \in \mathscr{F}$, $d_0(x, y) < \delta$ implies $d_1\big(f(x), f(y)\big) < \varepsilon$.

[3] see Definition 1.2

Theorem 2.9 - Arzela-Ascoli Theorem - *A collection $\mathscr{F}$ of functions $f : \mathscr{X} \to \mathscr{X}$, where $\mathscr{X}$ is a metric space, is a normal family on a domain $D \subset \mathscr{X}$ if and only if: (i) $\mathscr{F}$ is equicontinuous on every compact set $K \subset D$ and (ii) for any $z \in D$, the set $\{f(z) : f \in \mathscr{F}\}$ is contained in a compact subset of $\mathscr{X}$.*

2.3 Fixed Points and Fatou Components

One method for determining parts of the Fatou and Julia sets is through fixed points. Recall that a point p is a fixed point of a function f if $f(p) = p$. Additionally, p is said to be *attracting* if $|f'(p)| < 1$, *repelling* if $|f'(p)| > 1$, and *neutral* or *indifferent* if $|f'(p)| = 1$. Any repelling fixed point lies in the Julia set and any attracting fixed point lies in the Fatou set. (For an exploration of neutral fixed points, see [15]). We will show here that any attracting fixed point of a holomorphic function lies in its own component of the Fatou set.

Definition 2.10 - If $f : X \to X$ is a continuous function with attracting fixed point p, the **basin of attraction** of p is the set

$$\left\{x \in X : \lim_{n\to\infty} f^{\circ n}(x) = p\right\}$$

The **immediate basin of attraction** of p is the maximal connected subset of the basin which contains the fixed point p; i.e. the component of the basin which contains p.

Lemma 2.11 - *Let $f : \Omega \to \Omega$ be a holomorphic function on a Riemann surface Ω with an attracting fixed point p. Then the basin of attraction of p is an open set.*

Proof - Let B_p be the basin of attraction of p. Since p is attracting, we can find a λ such that $|f'(p)| < \lambda < 1$ and fix $\delta > 0$ be such that

$$\frac{|f(z) - f(p)|}{|z - p|} < \lambda$$

whenever $|z - p| < \delta$. Let D be the basic open neighborhood $\{z \in \Omega : |z - p| < \delta\}$. Then for any $z \in D$,

$$|f(z) - p| = |f(z) - f(p)| \leq \lambda|z - p|$$

If follows that

$$|f^{\circ n}(z) - p| = |f^{\circ n}(z) - f^{\circ n}(p)| \leq \lambda^n|z - p|$$

As such, $f^{\circ n}(z) \to p$ for any $z \in D$, so B_p has nonempty interior.

Next, let z_0 be some element of B_p. Then $f^{\circ n}(z_0) \to p$. As such, we can find an $N \in \mathbb{N}$ such that $f^{\circ n}(z_0) \in D$ for all $n \geq N$. Therefore, $(f^{\circ N})^{-1}(D)$ is an open set containing z_0. Further, for each $z_D \in (f^{\circ N})^{-1}(D)$, $f^{\circ n}(z_D) \to p$, so $(f^{\circ N})^{-1}(D) \subseteq B_p$. Thus, for any $z \in B_p$, there is an open neighborhood of z contained in B_p, so B_p is open. □

Corollary 2.12 - *If p is an attracting fixed point of $f : \Omega \to \Omega$, then the immediate basin of attraction of p is a component of the Fatou set of f.*

Proof - Let f be given with attracting fixed point p, and let A_p be the immediate basin of attraction for p. First, we need to show that $A_p \subseteq F_f$, so we will show that $\{f^{\circ n}\}_n$ is a normal family on A_p. Let K be a compact subset of A_p and let $\varepsilon > 0$ be given. Since $K \subseteq A_p$, we know $f^{\circ n}|_K \to p$ pointwise; but it remains to show that the convergence is uniform. Let λ and δ be given as in Lemma 2.11, so that $D = \{z : |z - p| < \delta\}$ is a basic open neighborhood in which f is a contraction.

Note that $K \subseteq \bigcup_n (f^{\circ n})^{-1}(D)$, so $\{(f^{\circ n})^{-1}(D)\}_n$ is an open cover of K. As such, we can find a finite collection $\{n_1, \cdots, n_k\}$ such that

$$K \subseteq \bigcup_{\ell=1}^{k} (f^{\circ n_\ell})^{-1}(D)$$

Fix $N_0 = \max\{n_1, \cdots, n_k\}$. Then $f^{\circ n}(K) \subseteq D$ for all $n \geq N_0$. Recall that for all $z \in D$,

$$|f^{\circ n}(z) - f^{\circ n}(p)| = |f^{\circ n}(z) - p| < \lambda^n \delta.$$

Fix N_1 such that $\lambda^{N_1}\delta \leq \varepsilon$. Then for $N = N_0 + N_1$,

$$|f^{\circ n}(z) - p| < \varepsilon$$

for all $z \in K$ and $n \geq N$. Thus, $f^{\circ n}|_K \to p$ uniformly, so $\{f^{\circ n}\}_n$ is a normal family on A_p. Thus, $A_p \subseteq F_f$.

Now we know that A_p is contained in the Fatou set, so it remains only to show that A_p is a component of F_f. Suppose by way of contradiction that U is a component of F_f such that $A_p \subseteq U$ and $U \setminus A_p$ is nonempty. Since every component of F_f is connected and open, U must contain an element of the boundary of A_p, say $z_0 \in \partial A_p$. However, since $z_0 \in \partial A_p$, for any neighborhood D of z_0 such that $D \subseteq U$, $D \cap A_p$ is nonempty. As such, $f^{\circ n}(D \cap A_p) \to p$ uniformly, but $z_0 \notin A_p$ so $f^{\circ n}(z_0)$ cannot converge within the A_p neighborhood of p (i.e. $f^{\circ n}(z_0)$ cannot converge 'near' p). Thus, $\{f^{\circ n}|_D\}$ cannot contain a subsequence which converges to a continuous function. Hence, $\{f^{\circ n}|_D\}$ does not form a normal family, so z_0 is not an element of the Fatou set.

We conclude that A_p is a component of F_f. □

The above proof can easily be adapted to also show that the entire basin of attraction of an attracting fixed point is contained in the Fatou set.

2.4 Quasiconformal Maps

In order to complete the proof given is Section 3 of the No Wandering Domains Theorem, we will need some theory concerning quasiconformal homeomorphisms. This is by no means an introduction to quasiconformal maps, but rather a collection of the essential componentry which will be referenced later. While most of the propositions in this section will be presented without proof, the appropriate references will be cited; most can found in [2], [4], [8], and [14].

We will develop quasiconformal homeomorphisms as solutions to the Beltrami differential equation,

$$\frac{\partial f}{\partial \bar{z}} = \mu \frac{\partial f}{\partial z} \tag{2.1}$$

where $\mu \in L^\infty$ and

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \quad \text{and} \quad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

for $z = x + iy$. A precise derivation of $\partial f/\partial z$ and $\partial f/\partial \bar{z}$ is given in [16]; however, a purely formal derivation can be obtained by considering

$$x = \frac{z + \bar{z}}{2} \quad \text{and} \quad iy = \frac{z - \bar{z}}{2}$$

and observing that

$$\frac{\partial x}{\partial z} = \frac{1}{2}, \quad \frac{\partial x}{\partial \bar{z}} = \frac{1}{2}, \quad \frac{\partial y}{\partial z} = \frac{-i}{2}, \quad \text{and} \quad \frac{\partial y}{\partial \bar{z}} = \frac{i}{2}$$

Then substituting into the total derivative of f with respect to z and $\bar{z}$

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} \qquad \frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}}$$

yields the above differentials.

Note that for any analytic function g,

$$\frac{\partial g}{\partial \bar{z}} = 0 \quad \text{and} \quad \frac{\partial g}{\partial z} = g'(z)$$

Therefore, if f is conformal then f satisfies (2.1) when $\mu = 0$. The following definition collects all of the terminology to be used later.

Definition 2.13 - Let μ be a complex-valued Lebesgue measurable function on a domain Ω such that $||\mu||_\infty < 1$. Then μ is a **Beltrami coefficient**. If f satisfies the Beltrami differential equation for μ almost everywhere, then f is **quasiconformal** and μ is the **complex dilatation** of f. Such an f is said to be μ**-conformal**.

Roughly speaking, a quasiconformal map ϕ takes 'infinitesimal' ellipses of bounded eccentricity in the domain Ω and maps them to 'infinitesimal' circles in $\phi(\Omega)$. We are restricted to the condition that $||\mu||_\infty < 1$ so that orientation is preserved.

Our goal will be to construct quasiconformal homeomorphisms, say ϕ, which deform a rational function R enough so that $\phi \circ R \circ \phi^{-1}$ is a rational function different than R, but with the same degree as R. This is not a trivial application of quasiconformal maps and requires the following results.

Theorem 2.14 - [8] [14] *Let μ be a Beltrami coefficient on a domain Ω of $\widehat{\mathbb{C}}$. Then there exists a μ-conformal homeomorphism on Ω. Furthermore, if ϕ and ψ are two such maps then $\phi \circ \psi^{-1}$ is conformal.*

In addition to existence, we also want to know when quasiconformal maps are unique. Suppose that ϕ and ψ are both μ-conformal maps of $\widehat{\mathbb{C}}$. Since $\phi \circ \psi^{-1}$ is conformal when ϕ and ψ have the same complex dilatation, we know that ϕ and ψ are unique up to composition with a Möbius transformation (as the Möbius transformations are the only conformal automorphisms of the sphere). Further, there is only one Möbius transformation which takes the three distinct points $\{a_0, b_0, c_0\}$ in $\widehat{\mathbb{C}}$ to the distinct points $\{a_1, b_1, c_1\}$ in $\widehat{\mathbb{C}}$ (details can be found in [16]). This idea leads to the following theorem of Ahlfors and Bers which tells us that a μ-conformal map of $\widehat{\mathbb{C}}$ which fixes 0, 1, and ∞ is unique.

Theorem 2.15 - [4] *There exist unique μ-conformal homeomorphisms of the extended complex plane and the closed unit disk onto themselves with fixed points at 0, 1, ∞ and 0, 1 respectively.*

Theorem 2.14 leads to the idea that for each domain Ω and Beltrami coefficient μ, we can we can consider Ω to be a Riemann surface with a conformal structure induced by μ in the following manner:

We will let $\Omega[\mu]$ denote a structure which is similar to a Riemann surface, except that the coordinate charts $\{U_a, \phi_a\}_{a \in A}$ of $\Omega[\mu]$ are such that each ϕ_a is a quasiconformal homeomorphism with complex dilatation $\mu|_{U_a}$. Additionally, whenever $U_a \cap U_b$ is nonempty,

$$\phi_a \circ \phi_b^{-1} : \phi_b(U_a \cap U_b) \to \phi_a(U_a \cap U_b)$$

is conformal. This structure will be referred to as the μ-conformal structure on Ω. The standard conformal structure of Ω will either be denoted by $\Omega[0]$ or simply Ω when the conformal structure is clear.

Furthermore, for a map between conformal structures, say $f : \Omega[\mu] \to \Lambda[\eta]$, f is said to be analytic if for each chart $\{U, \phi\}$ on $\Omega[\mu]$ and $\{V, \psi\}$ on $\Lambda[\eta]$ with $U \cap f^{-1}(V)$ nonempty, the composition,

$$\psi \circ f \circ \phi^{-1} : \phi\big(U \cap f^{-1}(V)\big) \to \psi(V)$$

is analytic in the usual sense (as a map on a domain of $\mathbb{C}$).

The following lemma gives us the conditions under which maps between a μ-conformal structure on one domain to an η-conformal structure on another domain are analytic.

Lemma 2.16 - [8] *Suppose that μ and η are Beltrami coefficients on the domains U and V respectively and let f be an analytic map of U onto V. Then $f : U[\mu] \to V[\eta]$ is analytic if and only if $\eta\big(f(z)\big) = \mu(z)\big(f'(z)/\overline{f'(z)}\big)$ a.e. in U.*

The idea above also gives us the conditions for when we can conjugate a rational function by a quasiconformal homeomorphism and still have a rational function. This is central to the proof of Theorem 3.1.

Lemma 2.17 - [8] [10] *Suppose that R is a rational function and that ϕ is a μ-conformal map of $\widehat{\mathbb{C}}$ onto itself. Then $\phi \circ R \circ \phi^{-1}$ is rational if and only if $\mu\big(R(z)\big) = \mu(z)\big(R'(z)/\overline{R'(z)}\big)$ a.e. on $\widehat{\mathbb{C}}$. Further, the degree of $\phi \circ R \circ \phi^{-1}$ is the same as the degree of R.*

Finally, we need a few technical lemmas which will aid the main proof of Section 3. These lemmas work together to show when quasiconformal maps on a hyperbolic domain can (i) be extended to closure of the domain and (ii) behave as the identity on the boundary.

Lemma 2.18 - [2] *Every quasiconformal mapping of an open disk onto itself can be extended to a homeomorphism of the closed disks.*

Lemma 2.19 - [8] *For each $\varepsilon > 0$ there is a $\delta > 0$ such that if a Beltrami coefficient μ of $\widehat{\mathbb{C}}$ satisfies $||\mu|| < \delta$, then $\rho\big(\phi(z), z\big) < \varepsilon$ in $\widehat{\mathbb{C}}$ for every μ-conformal $\phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which fixes 0, 1, and ∞.*

Lemma 2.20 - [8] *Let ρ_h be the hyperbolic metric of a simply connected subdomain U of $\widehat{\mathbb{C}}$ and let μ be a Beltrami coefficient on $\widehat{\mathbb{C}}$. Fix δ_0 such that δ_0 satisfies Lemma 2.19 for $\varepsilon = 1/8$. If $||\mu|| < \delta_0$ then $\rho_h\big(\phi(z), z\big) < \log(2)$ in U for every μ-conformal map $\phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which satisfies $\phi(U) = U$ and $\phi(z) = z$ on ∂U.*

The utility of the above lemma is as follows:
Let U be a domain conformally equivalent to the disk, and let $h : \mathbb{D} \to U$ be such a conformal isomorphism. Then for any Beltrami coefficient on $\mathbb{D}$, we can use h to transfer μ to a Beltrami coefficient η on U as in Lemma 2.16. Suppose that ϕ is an η-conformal map of U onto itself and η and ϕ satisfy the conditions of Lemma 2.20.

Now define $f = h^{-1} \circ \phi \circ h$. Then f is a μ-conformal map of the disk onto itself. By Lemma 2.18, f extends to a homeomorphism of the closed disk. Additionally, since $\rho_h(\phi(z), z) < \log(2)$

and h acts as an isometry, $\rho_h(f(z), z)$ is bounded in $\mathbb{D}$. Therefore, we must have $f(z) = z$ on $\partial\mathbb{D}$. (Recall that for any fixed $z_0 \in \mathbb{D}$, $\rho_h(z_0, z) \to \infty$ as z approaches the boundary).

3 The No Wandering Domains Theorem

This section aims to prove the following theorem, known as Fatou's No Wandering Domains Conjecture.

Theorem 3.1 - [18] *Let $R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational function and U a component of F_R. Then the sequence of successive iterates, $U, R(U), R^{\circ 2}(U), R^{\circ 3}(U), \cdots$ is always eventually periodic.*

3.1 Background and Sullivan's Proof

The original proof of Fatou's no wandering domains conjecture is due to Dennis Sullivan in [18]. The main idea of the proof requires the use of quasiconformal perturbations of rational functions. The majority of the necessary theorems are in the preceding section, but more will be presented as the proof develops.

Sullivan's proof considers a rational function R with a wandering Fatou component W and shows that this is impossible. It is shown that two possible cases exist; either
(i) W is simply connected and $R : R^{\circ n}(W) \to R^{\circ n+1}(W)$ is injective or,
(ii) the direct limit as $n \to \infty$ of $R^{\circ n}(W)$ exists and has infinite topological type (meaning the maximal collection of disjoint simple closed geodesics in $R^{\circ\infty}(W)$ is infinite).

In either case, the basic principal is to use the wandering hypothesis to construct a space of quasiconformal homeomorphisms ϕ, of dimension strictly greater than that of the space of rational functions of the degree of R. We then consider a map $\phi \mapsto \phi \circ R \circ \phi^{-1}$ which takes each ϕ to a rational function of the same degree as R. Due to the conflicting dimensions, we must have a subspace of our quasiconformal homeomorphisms which are all mapped to the same rational function. This is shown to contradict how ϕ is defined.

Since the publication of Sullivan's result, the proof has been reworked to avoid all cases where the wandering domain is not simply connected (see Lemma 3.2). Besides Sullivan's own proof, proofs of Theorem 3.1 can be found in [8], [10], [11], [15], and [19].

The following proof will be split into three parts. First, we establish that any wandering Fatou component of a rational function must eventually be simply connected, which is the primary simplification of Sullivan's original proof. Then we will construct a space of compactly-supported quasiconformal homeomorphisms of $\mathbb{D}$ which will be used to contradict our wandering hypothesis. Finally, we will set up the quasiconformal structure of the wandering domain and create a composite map from our space of Beltrami coefficients to the space of rational functions of a fixed degree, which leads to the main contradiction.

3.2 Restricting to the Simply Connected Case

We first shorten Sullivan's original proof of Theorem 3.1 by restricting the argument to the simply connected case (see Section 9 of [18]). In [15] and [19], the lemma is attributed to I. Noel Baker. Nonetheless, I was unable to find a reference and am unsure if Prof. Baker published this result.

Proofs of the following lemma can be found in [8], [11], [15], and [19]. However, I believe the proof given below is unique in that we avoid: (i) using the fact that $R^{\circ n} : W \to R^{\circ n}(W)$ is a covering map and (ii) the argument that for any compact $K \subset W$, the diameter of $R^{\circ n}$ vanishes as $n \to \infty$.[4] As such, I consider the following proof of Lemma 3.2 a 'simplification' of the simplified proof; and I would like to acknowledge Prof. Joel Shapiro for providing the main idea.

Lemma 3.2 - *If $R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a rational function and W is a wandering domain of R, then $R^{\circ n}(W)$ is simply connected for all sufficiently large values of n.*

Proof - Let R and W be given and define $W_n = R^{\circ n}(W)$. Since W is wandering, either there is a single $k \in \mathbb{N}$ such that $\infty \in W_k$ or $\infty \notin W_n$ for any n. If the point at infinity is not in any W_n, conjugate R by a Möbius transformation τ such that the point at infinity is in some forward iterate of W.

As we are only concerned with large values of n, we can replace W by some W_k so that $\infty \in W$ and $\infty \notin W_n$ for all $n \geq 1$. Now, since W is open, we can find an open disk D, say $D = \{z \in \widehat{\mathbb{C}} : \rho(z, \infty) < \varepsilon\}$ for some $\varepsilon > 0$, such that $\infty \in D \subset W$. Then recalling that W is wandering,

$$W_n \subseteq \widehat{\mathbb{C}} \setminus D \implies W_n \subseteq \mathbb{C} \quad \text{whenever } n \geq 1$$

Based on our definition of D,

$$\widehat{\mathbb{C}} \setminus D = \{z \in \widehat{\mathbb{C}} : \rho(z, \infty) \geq \varepsilon\} = \{z \in \mathbb{C} : |z| \leq M\} \quad \text{for } M = \sqrt{\frac{4}{\varepsilon^2} - 1}$$

Therefore, $|R^{\circ n}(z)| \leq M$ for all $z \in W$ and $n \geq 1$.

Next, for $k \geq 1$, let γ_k be a simple closed loop in W_k and let B_k be the bounded component of

[4] In Appendix A.2 we give a more standard proof which closely follows that found in [8].

$\mathbb{C} \setminus \gamma_k$. By the Maximum Modulus Theorem,[5] for any $z \in B_k$ and $n \in \mathbb{N}$,

$$|R^{\circ n}(z)| \leq ||R^{\circ n}||_{\gamma_k} \leq M \qquad \left(||R^{\circ n}||_{\gamma_k} = \sup_{z \in \gamma_k} |R^{\circ n}(z)|\right)$$

Therefore, $R^{\circ n}(B_k) \subset \widehat{\mathbb{C}} \setminus D$ for all n. So by Montel's theorem, $\{R^{\circ n}|_{B_k}\}_{n=1}^{\infty}$ is a normal family. Thus, $B_k \subset W_k$, so W_n is simply connected for all sufficiently large n. □

3.3 A Space of Quasiconformal Homeomorphisms

Let d be the degree of R. The space of rational functions of degree d has real-dimension $4d+2$, so fix $N > 4d+2$. Divide the interval $[0, 2\pi]$ into $2N$ equal and consecutive (closed) intervals and call them

$$A_1, B_1, A_2, B_2, \cdots, A_N, B_N$$

For each A_k let ω_k be a C^∞ function on $[0, 2\pi]$ such that
(i) $\omega_k(x) > 0$ for all x in the interior of A_k,
(ii) $\omega_k(x) = 0$ otherwise, and
(iii) $|\omega_k'| < 1/2$ for all x and k.

Now let T be a vector in the cube $[0, \varepsilon_0]^N$, so $T = (t_1, t_2, \cdots, t_N)$, where $\varepsilon_0 > 0$ will be chosen later. Define $f_T : \mathbb{D} \to \mathbb{D}$ by

$$f_T(z) = z \exp\left(i \sum_{k=1}^{N} t_k \omega_k(\theta)\right) \quad \text{where } z = re^{i\theta} \tag{3.1}$$

Rephrasing $\partial f/\partial z$ and $\partial f/\partial \bar{z}$ in terms of polar coordinates yields,

$$\frac{\partial f}{\partial z} = \frac{e^{-i\theta}}{2}\left(\frac{\partial f}{\partial r} - \frac{i}{r}\frac{\partial f}{\partial \theta}\right) \quad \text{and} \quad \frac{\partial f}{\partial \bar{z}} = \frac{e^{i\theta}}{2}\left(\frac{\partial f}{\partial r} + \frac{i}{r}\frac{\partial f}{\partial \theta}\right)$$

As such, the complex dilatation, μ_T, of f_T can be computed by (2.1) (see pg. 9) in terms of the partial derivatives $\partial f_T/\partial r$ and $\partial f_T/\partial \theta$. We find that

$$\mu_T(z) = \frac{e^{2i\theta} \sum_{k=1}^{N} t_k \omega_k'(\theta)}{2 + \sum_{k=1}^{N} t_k \omega_k'(\theta)}$$

It follows that for any T,

$$|\mu_T(z)| \leq \frac{N\varepsilon_0}{2 - N\varepsilon_0}$$

By (3.1), $f_T(z) = z$ whenever the argument of z lies in one of the intervals B_k, and distinct

[5] see Appendix

values of T lead to distinct functions f_T. Further, given any $\varepsilon > 0$, ε_0 can be chosen sufficiently small so that $||\mu_T|| < \varepsilon$. Let $\mathscr{F}_\varepsilon$ and $\mathscr{M}_\varepsilon$ denote the space of quasiconformal homeomorphisms f_T and Beltrami coefficients μ_T, respectively,

$$\mathscr{F}_\varepsilon = \left\{f_T : T \in [0,\varepsilon]^N\right\} \quad \text{and} \quad \mathscr{M}_\varepsilon = \left\{\mu_T : T \in [0,\varepsilon]^N\right\}$$

We shall refer back to $\mathscr{F}_\varepsilon$ and $\mathscr{M}_\varepsilon$ in the following section.

3.4 Quasiconformal Deformations of a Rational Function

Continuing with the rational map R and wandering domain W, in light of Lemma 3.2 we may assume $W_n = R^{\circ n}(W)$ is simply-connected. If necessary, we can replace W by some W_k such that no forward iterate of W contains a critical point of R. This guarantees that $R : W_n \to W_{n+1}$ is a conformal isomorphism for all $n \in \mathbb{N}$.

By the Riemann Mapping Theorem,[6] W is conformally equivalent to $\mathbb{D}$. Let $h : \mathbb{D} \to W$ be such a conformal equivalence. Now for any Beltrami coefficient μ on $\mathbb{D}$, we can use h and μ to define a Beltrami coefficient η on W, by

$$\eta\big(h(z)\big) = \mu(z)\big(h'(z)/\overline{h'(z)}\big)$$

Then by Lemma 2.16, $h : \mathbb{D}[\mu] \to W[\eta]$ is analytic.

We can extend the Beltrami coefficient η to all of $\widehat{\mathbb{C}}$ in the following manner. Let $GO(W)$ be the grand orbit of W,

$$GO(W) = \bigcup_{w \in W} GO(w,R) = \bigcup_{w \in W} \{z \in \widehat{\mathbb{C}} : R^{\circ n}(z) = R^{\circ m}(w) \text{ for any } n, m \geq 0\}$$

Note that $GO(W)$ and $\widehat{\mathbb{C}} \setminus GO(W)$ are disjoint and each completely invariant under R. On the forward images of W, define η such that $\eta\big(R(z)\big) = \eta(z)\big(R'(z)/\overline{R'(z)}\big)$. Then by Lemma 2.16, the map

$$R : W_n[\eta] \to W_{n+1}[\eta]$$

is analytic for all $n \geq 0$. Similarly, on the successive inverse images of W, define η such that $\eta\big(R(z)\big) = \eta(z)\big(R'(z)/\overline{R'(z)}\big)$ a.e. (so we can ignore points for which $R'(z) = 0$) for any component U of $R^{\circ -n}(W)$. Again by Lemma 2.16, the map

$$R : U[\eta] \to R(U)[\eta]$$

[6] see Appendix

is analytic. Thus we have defined η throughout $GO(W)$. Finally, for any element of the compliment, $z \in \widehat{\mathbb{C}} \setminus GO(W)$, define $\eta(z) = 0$.

It is important to note that since R is a rational function, the set of points where $R'(z) = 0$ or ∞ is measure zero, so we have $|R'(z)/\overline{R'(z)}| = 1$ almost everywhere. As such, $||\eta||_\infty = ||\mu||_\infty$.

Now, solving the Beltrami equation

$$\frac{\partial \phi}{\partial \bar{z}} = \eta \frac{\partial \phi}{\partial z}$$

yields an η-conformal map of $\widehat{\mathbb{C}}$ onto itself. Note that by the way η is defined, Lemma 2.17 tells us that $\phi \circ R \circ \phi^{-1}$ is a rational map with the same degree as R.

Combining ϕ with a Möbius transformation if necessary, we can assume that ϕ fixes the points 0, 1, and ∞. So by Theorem 2.15, ϕ is uniquely determined by η. We can now form the composite map

$$\mu \mapsto \eta \mapsto \phi \mapsto \phi \circ R \circ \phi^{-1} \tag{3.2}$$

which maps a Beltrami coefficient on $\mathbb{D}$ to a rational function with the same degree as R.

Since the space of Beltrami coefficients on $\mathbb{D}$ is infinite dimensional, while the space of rational functions of a fixed degree d is finite, we can use (3.2) to map a large subspace Beltrami coefficients onto a single rational map. This will be the essential component which produces the contradiction.

Lemma 3.3 - *Suppose that $\delta > 0$. Then we can find $\varepsilon > 0$ such that for each $t \in [0,1]$ there is a Beltrami coefficient μ_t in $\mathscr{M}_\varepsilon$ such that $||\mu_t|| < \delta$ and (3.2) maps each μ_t to the same rational function S. Further, this construction can be made so that for each z, the map $t \mapsto \phi_t(z)$ is continuous on $[0,1]$, where $t \mapsto \mu_t \mapsto \eta_t \mapsto \phi_t$.*

The proof of Lemma 3.3 is omitted here. An outline of the proof can be found in [8] and relies on results from [4]. We assume that the lemma holds and continue to the main result.

It follows from Lemma 3.3 that for all $t \in [0,1]$, conjugating R by ϕ_t yields the same rational function as conjugation by ϕ_0. In other words, there is some rational function S with the same degree as R such that

$$\phi_t \circ R \circ \phi_t^{-1} = \phi_0 \circ R \circ \phi_0^{-1} = S$$

Now define $\Phi_t = \phi_0^{-1} \circ \phi_t$.

Corollary 3.4 - *If Φ_t is defined as above then the following hold:*
(i) $\Phi_0(z) = z$ for $z \in \widehat{\mathbb{C}}$.
(ii) For each $t \in [0,1]$, Φ_t commutes with R.
(iii) For each $t \in [0,1]$, $\Phi_t : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a homeomorphism.
(iv) For each $z \in \widehat{\mathbb{C}}$, the map $t \mapsto \Phi_t(z)$ is continuous on $[0,1]$.

Proof - For (i): by definition of Φ_0,

$$\Phi_0(z) = \phi_0^{-1} \circ \phi_0(z) = z$$

For (ii): since $\phi_t \circ R \circ \phi_t^{-1} = \phi_0 \circ R \circ \phi_0^{-1}$, we have

$$\begin{aligned}\Phi_t \circ R &= \phi_0^{-1} \circ \phi_t \circ R \\ &= \phi_0^{-1} \circ \phi_t \circ R \circ \phi_t^{-1} \circ \phi_t \\ &= \phi_0^{-1} \circ \phi_0 \circ R \circ \phi_0^{-1} \circ \phi_t \\ &= R \circ \Phi_t\end{aligned}$$

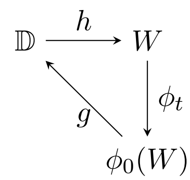
For (iii): since ϕ_0 and ϕ_t are both homeomorphisms, so is the composition $\phi_0^{-1} \circ \phi_t$.
And for (iv): by Lemma 3.3, the map $t \mapsto \phi_t$ is continuous, so the map $t \mapsto \phi_0^{-1} \circ \phi_t$ is also continuous. □

Lemma 3.5 - *For each $t \in [0,1]$, Φ_t is the identity on J_R. Further, Φ_t maps each component of F_R onto itself.*

Proof - For each positive integer k, let P_k be the set of fixed points of $R^{\circ k}$ and let $P = \bigcup_k P_k$, so that P is the set of all periodic points of R. Since R is rational (and also not the identity function), the set P_k is finite. By Corollary 3.4(ii), Φ_t commutes with R; so for any $z \in P_k$, $\Phi_t(z) \in P_k$. By Corollary 3.4(iv), $t \mapsto \Phi_t(z)$ is continuous map of $[0,1]$ into the discrete set P_k, so $\Phi_t(z)$ does not depend on the value of t and we have $\Phi_t(z) = \Phi_0(z)$ on P_k. It follows from Corollary 3.4(i) that $\Phi_t(z) = z$ for all $t \in [0,1]$ and $z \in P_k$. Thus, $\Phi_t(z) = z$ for any $z \in P$. By Corollary 3.4(iii), Φ_t is a homeomorphism of $\widehat{\mathbb{C}}$, so $\Phi_t(z) = z$ on the closure of P. Since the Julia set, J_R, is contained in the closure of periodic points of R, $\Phi_t(z) = z$ for any $z \in J_R$.

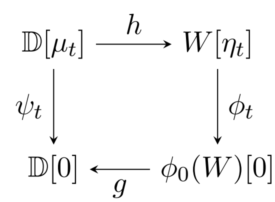
Now since $\Phi_t(J_R) = J_R$, we have $\Phi_t(F_R) = F_R$. Let z_0 be an element of a component U of F_R. By Corollary 3.4(i) and (iv), the mapping $t \mapsto \Phi_t(z_0)$ takes the interval $[0,1]$ to a connected subset of F_R which includes z_0. As such, $\Phi_t(z_0) \in U$ for each t. Hence, $\Phi_t(U) \subseteq U$ for each component U of F_R. □

Given Lemma 3.5, Φ_t maps W onto itself, so the quasiconformal homeomorphism ϕ_t must map W onto $\phi_0(W)$ regardless of the value of t. Since ϕ_t is a homeomorphism, $\phi_0(W)$ is also simply-connected. Therefore, there is a conformal equivalence between $\phi_0(W)$ and $\mathbb{D}$, call it g. Recalling our earlier conformal equivalence, $h : \mathbb{D} \to W$, the relationship between $\mathbb{D}$, W, and $\phi_0(W)$ can be summarized by the diagram below.



By our construction of the Beltrami coefficient η, $||\eta||_\infty = ||\mu||_\infty$. Therefore, by taking δ sufficiently small, we can ensure the complex dilatation of Φ_t is less than δ_0 as given in the hypothesis of Lemma 2.20. By Lemma 3.5, $\Phi_t(W) = W$ and $\Phi_t(z) = z$ for $z \in \partial W \subseteq J_R$, so all of the conditions of Lemma 2.20 are satisfied. As such, the map $h^{-1} \circ \Phi_t \circ h$ of $\mathbb{D}$ onto itself extends to the identity on $\partial\mathbb{D}$.[7]

Now if we consider the conformal structure of each domain, then each of the maps h, ϕ_t, and g are analytic (in the sense of Lemma 2.16). Therefore, we can define the μ_t-conformal function of $\mathbb{D}$ by $\psi_t = g \circ \phi_t \circ h$. Adjusting our previous diagram to consider the conformal structure of each domain yields the following.



Note that $\psi_0^{-1} \circ \psi_t = h^{-1} \circ \Phi_t \circ h$. Therefore, on $\partial\mathbb{D}$, $\psi_0^{-1} \circ \psi_t$ is the identity function, so $\psi_0 = \psi_t$.

Now let f_t be the element of $\mathscr{F}_\varepsilon$ that is a μ_t-conformal map of $\mathbb{D}$ onto itself. Recalling that any two quasiconformal homeomorphisms with the same complex dilatation are unique up to composition with a conformal map, we have $\psi_t = \tau_t \circ f_t$ for some Möbius automorphism τ_t of $\mathbb{D}$. Since $\psi_0 = \psi_t$ we must have $\tau_t \circ f_t = \tau_0 \circ f_0$ on $\partial\mathbb{D}$ for any $t \in [0, 1]$.

Recall that for any $f \in \mathscr{F}_\varepsilon$, $f(z) = z$ on the interior of the intervals B_k of $\partial\mathbb{D}$. Therefore, if $\tau_t \circ f_t = \tau_0 \circ f_0$, then $\tau_t = \tau_0$ on B_k. But Möbius transformations which fix three points are unique, so $\tau_t = \tau_0$. Therefore, we must have $f_t = f_0$. However, each function in $\mathscr{F}_\varepsilon$ is unique by construction, so this is impossible.

Thus completes the proof of Theorem 3.1. □

[7] This is explained in the text following Lemma 2.20.

4 Examples of Wandering Domains

4.1 An Example from Herman

In [7], I.N. Baker credits M. Herman for the following example. Consider the function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) = z - 1 + e^{-z} + 2\pi i$$

Then f is transcendental entire on $\mathbb{C}$. We will show that f has wandering domains.

Begin by considering the function $g(z) = z - 1 + e^{-z}$. It may be of interest to note that the function g is generated by applying Newton's method to the function defined by $z \mapsto e^z - 1$. As such, $z_k = 2\pi k i$ is a fixed point of g for each $k \in \mathbb{Z}$. Further, $g'(z_k) = 0$ for each z_k, so each fixed point is super-attracting. For each k, let A_k be the immediate basin of attraction for the fixed point z_k. By Corollary 2.12, each A_k is a component of F_g.

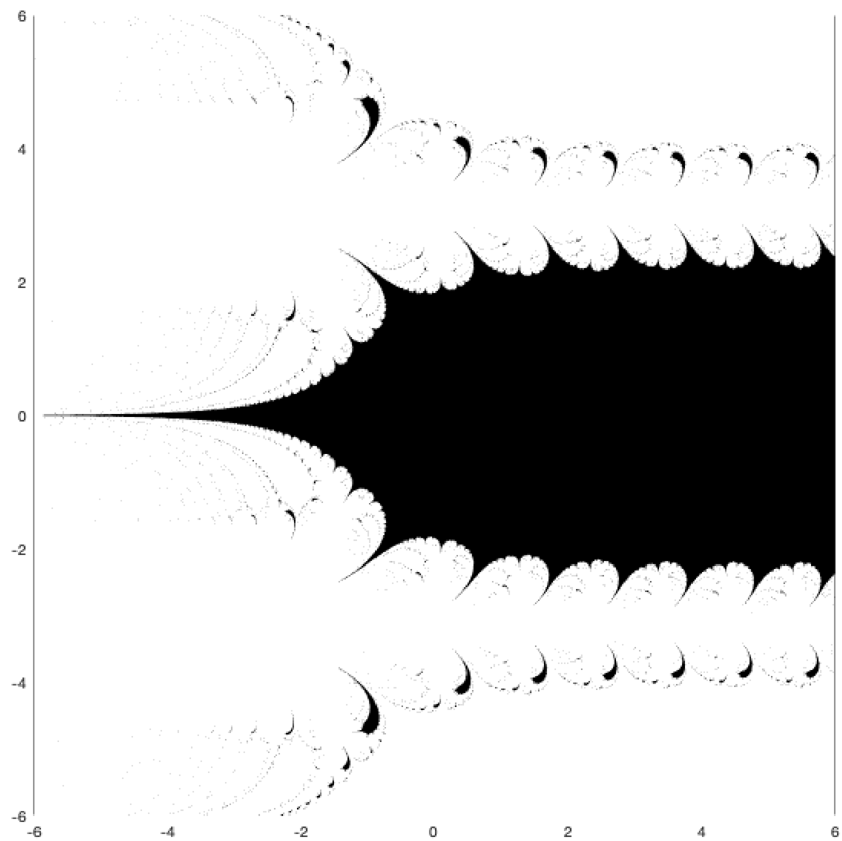


Figure 2: Basin of attraction (in black) of the fixed point 0 for $g(z) = z - 1 + e^{-z}$. (Compare to Figure 1 on the title page)

Additionally, observe that

$$g(z+2\pi i) = z + 2\pi i - 1 + e^{-z-2\pi i} = z - 1 + e^{-z} + 2\pi i = g(z) + 2\pi i$$

As such, $g^{\circ n}(z+2\pi i) = g^{\circ n}(z) + 2\pi i$. So if $g^{\circ n}(z) \to z_k$ then $g^{\circ n}(z+2\pi i) \to z_{k+1}$. Therefore, $A_{k+1} = A_k + 2\pi i$.

Now we can use the following lemma from [7] to establish that each A_k is also a component of the Fatou set of f.

Lemma 4.1 - [7] *Suppose f and g are entire, f commutes with g, and $f = g + c$ where c is some constant. Then $J_f = J_g$.*

It should be clear that f and g are both entire, and $f = g + c$ where $c = 2\pi i$. It remains only to check that f and g commute. Observe that

$$\begin{aligned} f \circ g(z) &= (z - 1 + e^{-z}) - 1 + e^{-z+1-e^{-z}} + 2\pi i \\ &= (z - 1 + e^{-z} + 2\pi i) - 1 + (e^{-2\pi i})e^{-(z-1+e^{-z})} \\ &= (z - 1 + e^{-z} + 2\pi i) - 1 + e^{-(z-1+e^{-z}+2\pi i)} \\ &= g \circ f(z) \end{aligned}$$

So by Lemma 4.1, $J_f = J_g$. It follows immediately that $F_f = F_g$, so each A_k described previously is a component of F_f. Recalling that $A_k + 2\pi i = A_{k+1}$, observe that

$$f(A_k) = g(A_k) + 2\pi i = A_{k+1}$$

Thus, each A_k is a wandering component of F_f.

This example in not unique in its construction. In fact, many functions of the form $f(z) = z + p(z)$ where $p(z)$ is a periodic function can be shown to have wandering domains. For example, the function $f(z) = z + \sin(z) + 2\pi$ has wandering domains and can be analyzed in a similar manner. (Let $g(z) = z + \sin(z)$).

4.2 An Example from Baker

The following example is given predominantly for its historical context, as this is the first published example of a wandering domain for an entire function. As opposed to the above example which has simply connected wandering domains, this function exhibits multiply connected wandering domains. Complete details can be found in [5] and [6].

Fix $\alpha_1 > 4e$. Define the sequence $\{\alpha_n\}_n$ of real numbers inductively by

$$\alpha_{n+1} = \frac{\alpha_n^2}{4e}\left(1+\frac{\alpha_n}{\alpha_1}\right)\left(1+\frac{\alpha_n}{\alpha_2}\right)\cdots\left(1+\frac{\alpha_n}{\alpha_n}\right) \tag{4.1}$$

Then $4e < \alpha_1 < \alpha_2 < \cdots$. Now let $b : \mathbb{C} \to \mathbb{C}$ be given by

$$b(z) = \frac{z^2}{4e}\prod_{n=1}^{\infty}\left(1+\frac{z}{\alpha_n}\right) \tag{4.2}$$

The function b satisfies the following properties:
(i) For any z such that $|z| = 1$, $|b(z)| < 1/4$;
(ii) $\alpha_{n+1} < b(\alpha_n) < 2\alpha_{n+1}$;
(iii) $b(\sqrt{\alpha_n}) < \sqrt{\alpha_{n+1}}$; and
(iv) $\alpha_{n+1}^2 < b(\alpha_n^2)$

Further, if A_n is the annulus

$$A_n = \left\{z \in \mathbb{C} : \alpha_n^2 < |z| < \sqrt{\alpha_{n+1}}\ \right\}$$

then there is an $N \in \mathbb{N}$ such that $b(A_n) \subseteq A_{n+1}$ for all $n \geq N$. It follows that if we fix some $m \geq N$, then for any compact subset K of A_m, $b^{\circ n}(K) \to \infty$, i.e. $\{b^{\circ n}|_K\}_n$ diverges locally uniformly from $\mathbb{C}$. Thus, $\{b^{\circ n}\}_n$ forms a normal family on A_m for $m \geq N$.

Theorem 4.2 - *For each $n \geq N$, each A_n lies in a unique component U_n of F_b and each U_n is a wandering domain.*

Proof - By way of contradiction, suppose there exist $n_0, n_1 \geq N$ such that A_{n_0} and A_{n_1} lie in the same component U of F_b. Without loss of generality, assume $n_1 > n_0$ so that $n_1 = n_0 + k$ for some positive integer k. Since A_{n_0} and A_{n_1} lie in the same component U, we can find a path γ such that $\gamma(0) \in A_{n_0}$ and $\gamma(1) \in A_{n+1}$. By the way the annuli A_n are defined, γ connects A_{n_0} to A_{n_0+1} and A_{n_0+1} to A_{n_0+2} etc. all the way to A_{n_0+k}. Therefore, we can assume that $n_1 = n_0 + 1$. Further, since F_b is completely invariant and $b(A_n) \subseteq A_{n+1}$ whenever $n \geq N$, $b^{\circ k}(\gamma)$ is a path in F_b which connects A_{n_0+k} to $A + n_0 + k + 1$. Thus, each A_m is in the same component U for all $m \geq n_0$ so U must be multiply connected and unbounded.

Therefore, in order to prove the theorem, it is enough to show that A_n and A_{n+2} cannot be joined in F_b when n is sufficiently large. By (4.1), there is an M such that for all $n \geq M$, $4\alpha_n^2 < \sqrt{\alpha_{n+1}}$. Fix $n > \max\{N, M\}$ and let γ be a path in F_b such that

$$\gamma(0) = 2a_n^2 \in A_n \quad \text{and} \quad \gamma(1) = \frac{1}{2}\sqrt{\alpha_{n+3}} \in A_{n+2}$$

Since F_b is open and γ lies in a path connected component, we can find a simply connected open subset, say U_γ, of F_b which contains γ. Let $\psi : \mathbb{D} \to U_\gamma$ be a conformal isomorphism such that $\psi(0) = \gamma(0)$ and $\phi(u) = \gamma(1)$ for some nonzero $u \in \mathbb{D}$.

Now, by our construction of U_γ, $b^{\circ n}|_{U_\gamma}$ diverges locally uniformly from $\mathbb{C}$. It then follows from property (i) above that $|b^{\circ n}(z)| > 1$ for all $z \in U_\gamma$. Now consider the maps $f_n = b^{\circ n} \circ \psi^{-1}$. Then each f_n is a map from $\mathbb{D}$ to $\mathbb{C}$ with $|f_n(z)| > 1$ for all $n \in \mathbb{N}$ and $z \in \mathbb{D}$. As such, we can apply Schottky's Theorem to get

$$|b^{\circ n}\big(\gamma(1)\big)| = |f_n(u)| \leq \exp\left(\frac{1+|u|}{1-|u|}\left(7 + \log\big(\max\{1, |f_n(0)|\}\big)\right)\right)$$

Recalling that $f_n(0) = b^{\circ n}\big(\gamma(0)\big) > 1$, set

$$q = \exp\left(\frac{1+r}{1-r}\right) \quad \text{and} \quad K = \exp(7q)$$

Then we get the inequality

$$0 < |b^{\circ n}\big(\gamma(1)\big)| = |f_n(u)| \leq K|f_n(0)|^q = K|b^{\circ n}\big(\gamma(0)\big)|^q$$

which is independent of n. Recalling the values of $\gamma(0)$ and $\gamma(1)$ the above inequality becomes

$$0 < \left|b^{\circ n}\left(\frac{1}{2}\sqrt{\alpha_{n+3}}\right)\right| \leq K\left|b^{\circ n}\big(2a_n^2\big)\right|^q \tag{4.3}$$

However, since we have fixed n sufficiently large that $4\alpha_n^2 < \sqrt{\alpha_{n+1}}$, we have

$$2\alpha_n^2 < \sqrt{\alpha_{n+1}} < \alpha_{n+1}$$

Additionally, by (4.2), b is increasing on the real axis for $z > \alpha_1$, and by property (ii), $b(\alpha_n) < 2\alpha_{n+1}$. Therefore, for $k > 1$,

$$b^{\circ k}\big(2\alpha_n^2\big) < b^{\circ k}\big(\alpha_{n+1}\big) = b^{\circ k-1}\big(b(\alpha_{n+1})\big) < b^{\circ k-1}\big(2\alpha_{n+2}\big) < b^{\circ k-1}\left(\frac{1}{2}\sqrt{\alpha_{n+3}}\right)$$

Further, by (4.2) we can find a sufficiently large $x \in \mathbb{R}$ such that $b(x) > Kx^q$. Therefore for sufficiently large k,

$$\left|b^{\circ k}\left(\frac{1}{2}\sqrt{\alpha_{n+3}}\right)\right| = \left|b\left(b^{\circ k-1}\left(\frac{1}{2}\sqrt{\alpha_{n+3}}\right)\right)\right| > \left|b\left(b^{\circ k}\big(2\alpha_n^2\big)\right)\right| > K\left|b^{\circ k}\big(2a_n^2\big)\right|^q$$

which contradicts (4.3) and thus completes the proof. □

A Appendix

A.1 Additional Theorems and Definitions

A collection of definitions and theorems which may be helpful to the reader but were not included in the main text.

Maximum Modulus Theorem - *Suppose* Ω *is a connected open subset of* $\mathbb{C}$*,* f *is holomorphic on* Ω*, and* f *is continuous on* $\overline{\Omega}$*. Then*

$$|f(z)| \leq ||f||_{\partial\Omega}$$

for all $z \in \Omega$*, where* $||\ ||_{\partial\Omega}$ *is the supremum norm on the boundary of* Ω*.*

Riemann Mapping Theorem - *Every simply connected region* Ω *in the plane, other than the whole plane itself, is conformally equivalent to the open unit disk* $\mathbb{D}$*.*

Rouché's Theorem - *Let* γ *be a closed curve in a simply connected domain* Ω *such that the winding number is* 1 *or* 0 *for any* $z \in \Omega$ *(*$z \notin \gamma$*). Suppose that* f *and* g *are analytic in* Ω *and satisfy the inequality* $|f(z) - g(z)| < |f(z)|$ *for* $z \in \gamma$*. Then* f *and* g *have the same number of zeros enclosed by* γ*.*

Schottky's Theorem - *If* f *is analytic on* $\mathbb{D}$ *and* f *omits the values* 0 *and* 1*, then*

$$\log|f(z)| < \frac{1+r}{1-r}\left(7 + \log\big(\max\{1, |f(0)|\}\big)\right)$$

for $|z| \leq r < 1$*.*

$\mathbb{D}$ denotes the unit disk in $\mathbb{C}$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$

S^2 denotes the sphere in $\mathbb{R}^3$, $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$

Critical Point: For a map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, a point $z_0 \in \widehat{C}$ is a critical point if for every neighborhood of z_0, f fails to be injective.

Conformal: A holomorphic function $f : \Omega \to \mathbb{C}$ is conformal if $f'(z) \neq 0$ for all $z \in \Omega$.

Conformally Equivalent: Two domains Ω_0 and Ω_1 are conformally equivalent if there exists a bijective holomorphic function from Ω_0 onto Ω_1. Note that a holomorphic bijection must be conformal on its domain.

Covering Map: Let $\phi : X \to Y$ be a continuous function such that each $y \in Y$ has an open neighborhood V such that $\phi^{-1}(V)$ can be decomposed into a family $\{U_\alpha\}$ of pairwise disjoint open subsets of X in such a way that that the restriction of ϕ to each U_α is a homeomorphism from U_α to V. Then f is a covering map and X is a covering space of Y.

Degree of a Rational Function: The degree of a rational function $f = P/Q$ is the maximum of the degrees of its constituent polynomials P and Q. (See also *Rational Function*)

Grand Orbit: Let $f : S \to S$ be a function on a Riemann surface S. For any $z_0 \in S$ the grand orbit of z_0 is denoted $GO(z_0, f)$ and is given by

$$GO(z_0, f) = \{z \in S : f^{\circ n}(z) = f^{\circ m}(z_0) \text{ for any } n, m \geq 0\}$$

As such, z_0 and z_1 have the same grand orbit if $f^{\circ m}(z_0) = f^{\circ n}(z_1)$ for some $m \geq 0$ and $n \geq 0$.

Holomorphic Function: Also known as an *analytic function*; a holomorphic function is a complex-valued function which is differentiable in a neighborhood of every point in its domain.

Holomorphic for an Arbitrary Riemann Surface (Definition I.1.5 of [13]): A continuous mapping $f : M \to N$ between Riemann surfaces is called holomorphic or analytic if for every chart $\{U, \phi\}$ on M and $\{V, \psi\}$ on N with $U \cap f^{-1}(V) \neq \varnothing$, the map

$$\psi \circ f \circ \phi^{-1} : \phi\big(U \cap f^{-1}(V)\big) \to \psi(V)$$

is holomorphic as a map from $\mathbb{C}$ to $\mathbb{C}$. If f is also bijective, then f is conformal.

Order: The order of an entire function f is given by

$$\operatorname{ord}(f) = \limsup_{z \to \infty} \frac{\ln\big(\ln|f(z)|\big)}{\ln|z|}$$

As an example, the function $f(z) = \exp(z^d)$ has order d.

Rational Function: A rational function is a function $R : \mathbb{C} \to \mathbb{C}$ given by

$$R(z) = \frac{P(z)}{Q(z)}$$

where P and Q are polynomials with complex coefficients. It should be further noted that Q is nonzero and P and Q have no common factors.

A.2 Original Simply Connected Proof

The following is a proof of Lemma 3.2. The proof found here is essentially the same as those found in [8], [11], [15], and [19]. This proof will be split into two lemmas, as it is presented in [8], as I feel this better captures the argument concerning the vanishing compact sets.

Recall that the diameter of a set $A \subseteq \widehat{\mathbb{C}}$ is given by

$$d(A) = \sup\{\rho(a,b) : a, b \in A\}$$

where ρ is the chord metric described in Section 2.

Lemma A.1 - *Suppose that W is a wandering domain of a rational function R. Then for any compact subset K of W, the diameter of the set $R^{\circ n}(K)$ vanishes as $n \to \infty$.* [8]

Proof - Let R be given and let W be a wandering component of F_f. For the sake of contradiction, suppose there is some $K \subset W$, $\varepsilon > 0$, and subsequence of $\{R^{\circ n}(K)\}$ such that

$$d\big(R^{\circ n_j}(K)\big) \geq \varepsilon$$

for all j. Since $\{R^{\circ n}\}$ is a normal family on W, our subsequence $\{R^{\circ n_j}\}$ contains a subsequence which converges locally uniformly on W to a holomorphic function, say h. For convenience, we will relabel our original subsequence to this subsequence so that $R^{\circ n_j} \to h$ locally uniformly on W and $d\big(R^{\circ n_j}(K)\big) \geq \varepsilon$ for all j.

Now the limit function h is either constant or non-constant on W. If h is constant, say $h(z) = \alpha$ for all $z \in W$, then since K is compact and $\{f^{\circ n_j}\}$ converges locally uniformly, there is a $j_0 \in \mathbb{N}$ such that $\rho(R^{\circ n_j}(z), \alpha) < \varepsilon/3$ for all $z \in K$ and $j \geq j_0$. This contradicts $d\big(R^{\circ n_j}(K)\big) \geq \varepsilon$, so h must be non-constant.

Since h is non-constant, choose $c \in W$ such that $h'(c) \neq 0$. Let C be a circle centered at c such that the interior of the circle, say D, is contained in W and $h(z) \neq g(c)$ for all $z \in C$. Since h is analytic and W is open, such a circle will exist. Then there is some $j_1 \in \mathbb{N}$ such that for any $z \in C$ and $j \geq j_1$,

$$|R^{\circ n_j}(z) - h(c) - \big(h(z) - h(c)\big)| = |R^{\circ n_j}(z) - h(z)| < \inf_{w \in C} |h(w) - h(c)| \leq |h(z) - h(c)|$$

Therefore, by Rouché's theorem, there is some $\zeta \in D$ such that $R^{\circ n_j}(\zeta) = h(c)$ whenever $j \geq j_1$. This contradicts that W is wandering and completes the proof. □

The above lemma allows us to prove the main lemma, that a wandering domain must eventually be simply-connected.

Lemma A.2 - *If W is a wandering domain of a rational function R, then $R^{\circ n}(W)$ is simply connected for all sufficiently large values of n.*

Proof - Let R be a rational map and suppose W is a wandering component of F_R. Define $W_n = R^{\circ n}(W)$. If necessary, replace W by W_k for some k so that no W_n contains a critical point of R. This is allowable as W is wandering and R has at most finitely many critical points, so there must be some k such that W_n contains no critical points for all $n \geq k$. Having made this adjustment, the map

$$R^{\circ n} : W \to W_n$$

is a covering map for every n. To verify this, note that $R^{\circ n} : W \to W_n$ is surjective by definition. Also, since there are no critical values in W or any W_n, the mapping is locally injective. Finally, note that since R is a rational function, it is continuously invertible (an open mapping) away from its critical points.

Next, let γ be a simple closed loop in W and set $\gamma_n = f^{\circ n}(\gamma)$. Recall from Lemma A.1, for any compact $K \subset W$, $d\big(f^{\circ n}(K)\big) \to 0$ as $n \to \infty$. As such, $d(\gamma_n) \to 0$.

Let B_n be the union of the bounded components of $\mathbb{C} \setminus \gamma_n$. Then we must have $d(B_n) \to 0$ as well. Then for n sufficiently large, $\partial R(B_n) \subseteq \gamma_{n+1}$ so $R(B_n) \subseteq \gamma_{n+1}$. Then by Montel's theorem, $B_n \subseteq F_R$. As such, γ_n is null homotopic in W_n. Since $R^{\circ n} : W \to W_n$ is a covering map, we can lift this property to W. $\square$

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