The Hausdorff Metric and Geodesics in the Space of Compact Sets

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PSU Analysis Seminar
April 10, 2020

The Pompeiu-Hausdorff Metric

If p is a point and A is a set, the distance between p and A is defined by

$$d(p, A) = \inf\{d(p, a) : a \in A\}.$$

Definition

Given a metric space (X, d) and two subsets $A, B \subseteq X$, the *Pompeiu-Hausdorff distance* between A and B is defined by

$$h_d(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right\}.$$

Historical context: In 1905, Dimitrie Pompeiu (student of H. Poincaré) introduced the pseudo-metric

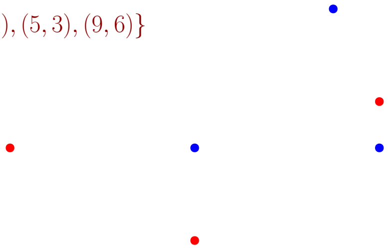
$$p_d(A, B) = \sup_{a\in A} d(a, B) + \sup_{b\in B} d(b, A).$$

Our definition is that popularized by Hausdorff in *Grundzüge der Mengenlehre* (1914).

The Pompeiu-Hausdorff Metric Example

$A = \{(5,5),(8,8),(9,5)\}$

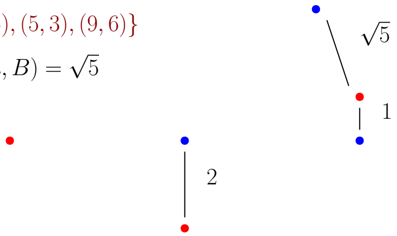
$B = \{(1,5),(5,3),(9,6)\}$



$A = \{(5,5),(8,8),(9,5)\}$

$B = \{(1,5),(5,3),(9,6)\}$

$\sup_{a \in A} d(a,B) = \sqrt{5}$



The Pompeiu-Hausdorff Metric Example

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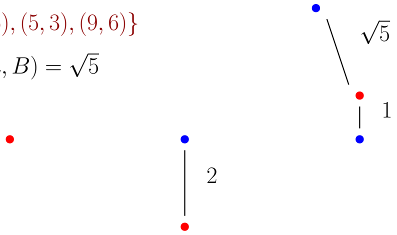
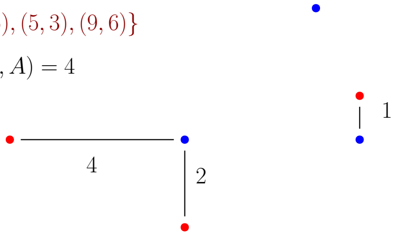


$A = \{(5,5), (8,8), (9,5)\}$

$B = \{(1,5), (5,3), (9,6)\}$

$\sup_{b \in B} d(b, A) = 4$



The Pompeiu-Hausdorff Metric contd.

For a metric space X, consider

$$\begin{aligned} B_X &= \{A \subseteq X : A \text{ is closed and bounded}, A \neq \varnothing\} \\ K_X &= \{A \subseteq X : A \text{ is compact}, A \neq \varnothing\} \\ F_X &= \{A \subseteq X : A \text{ is finite}, A \neq \varnothing\}. \end{aligned}$$

- (B_X, h_d), (K_X, h_d), and (F_X, h_d) are metric spaces.
- The metric h_d is known as the *Pompeiu-Hausdorff metric* (or commonly, the Hausdorff metric).
- We will primarily consider the space (K_X, h_d).

Theorem

If (X, d) is complete, then (K_X, h_d) is complete.

Paths and Intrinsic Metrics

Definition

A *path* is a continuous map from a closed interval into a space. For a path $\gamma : [a, b] \to X$, the length of γ is

$$L(\gamma) = \sup \left\{ \sum_{k=0}^{n-1} d\big(\gamma(t_k), \gamma(t_{k+1})\big) : a = t_0 < t_1 < \cdots < t_n = b \right\}.$$

Definition

The metric d is *intrinsic* if for all $x, y \in X$,
$d(x, y) = \inf\{L(\gamma) : \gamma$ connects x to $y\}$.

A metric space with an intrinsic metric is known as a *length space*.

Theorem (E.N. Sosov, 2001)

The metric space (B_X, h_d) is a length space if and only if (X, d) is a length space.

Geodesics

Definition

A *geodesic* is a path $\gamma : [0, T] \to X$ such that for all $s, t \in [0, T]$,

$$d\big(\gamma(s), \gamma(t)\big) = |t - s|.$$

The above definition is the same as what I have previously called a *unit-speed shortest path*. Notice that if $\gamma : [0, T] \to X$ is a geodesic, then $L(\gamma) = d\big(\gamma(0), \gamma(T)\big)$.

Definition

If a metric space is such that every pair of points can be connected by a geodesic, then the metric is said to be *strictly intrinsic* and the space is known as a *geodesic space*.

Theorem

Let (X, d) be a complete metric space.
(i) The metric d is intrinsic if and only if for every $x, y \in X$ and $\varepsilon > 0$, there is a $z \in X$ such that

$$d(x,z) = \frac{1}{2}d(x,y) \text{ and } d(x,z) + d(y,z) < d(x,y) + \varepsilon.$$

(ii) The metric d is strictly intrinsic if and only if for every $x, y \in X$, there is a $z \in X$ such that

$$d(x,z) = d(y,z) = \frac{1}{2}d(x,y).$$

Note that X must be complete. In $\mathbb{Q}$ there are always midpoints, but $(\mathbb{Q}, |\ |)$ is not a geodesic space.

Definition

For elements $x, y \in X$, let $\omega(x, y)$ be the set

$$\omega(x,y) = \big\{z \in X : 2\max\{d(x,z), d(y,z)\} \leq d(x,y)\big\}.$$

Lemma

If (X, d) is a complete metric space, then d is strictly intrinsic if and only if $\omega(x, y)$ is nonempty for every $x, y \in X$.

Other Metric Space Basics

Definition

A metric space is *proper* (or *boundedly compact*) if every closed and bounded subset is compact.

Theorem (generalized Heine-Borel)

A nonempty subset A of a metric space X is compact if and only if it is complete and totally bounded.

Corollary

If X is a proper metric space, then it is complete.

Theorem (E.N. Sosov, 2001)

Let (X, d) be a complete metric space. The metric space (B_X, h_d) is a length space if and only if (X, d) is a length space.

Proposition

Let (X, d) be a proper metric space. The space (K_X, h_d) is a geodesic space if and only if (X, d) is a geodesic space.

Given $x_0 \in X$ and $A \subseteq X$, the metric projection of x_0 onto A is the set

$$P_A(x_0) = A \cap \{z \in X : d(z, x_0) \leq d(x_0, A)\}.$$

Lemma

If (X, d) is a geodesic space, then for every compact set $A \subseteq X$ and point $x \in X$, the metric projection $P_A(x)$ is nonempty.

Proposition

Let (X, d) be a proper metric space. The space (K_X, h_d) is a geodesic space if and only if (X, d) is a geodesic space.

Let $A, B \in K_X$ be given. Define

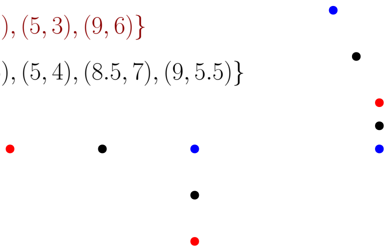
$$\mathcal{Z}_A = \bigcup \{\omega(a, u) : a \in A, u \in P_B(a)\}$$

and let $\mathcal{Z} = \mathcal{Z}_A \cup \mathcal{Z}_B$.

$A = \{(5,5), (8,8), (9,5)\}$

$B = \{(1,5), (5,3), (9,6)\}$

$\mathcal{Z} = \{(3,5), (5,4), (8.5,7), (9,5.5)\}$



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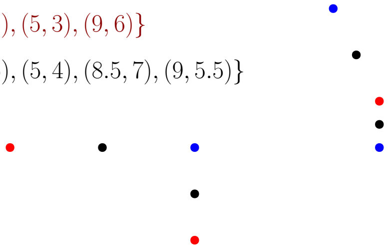

$$h_d(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\} = \max\{4, \sqrt{5}\} = 4$$

$$h_d(A,\mathcal{Z}) = \max\{\sup_{a\in A} d(a,\mathcal{Z}), \sup_{z\in \mathcal{Z}} d(z,A)\} = \max\{\sqrt{1.25}, 2\} = 2$$

$$h_d(B,\mathcal{Z}) = \max\{\sup_{b\in B} d(b,\mathcal{Z}), \sup_{z\in \mathcal{Z}} d(z,B)\} = \max\{2, 2\} = 2$$

Proposition

Let (X, d) be a proper metric space. The space (K_X, h_d) is a geodesic space if and only if (X, d) is a geodesic space.

'Proof'

- Show that $\mathcal{Z} \in K_X$
  - $\mathcal{Z}_A$ is bounded (so is $\mathcal{Z}_B$ by symmetry)
  - $\mathcal{Z}_A$ is closed (so is $\mathcal{Z}_B$ by symmetry)
- Show that $\mathcal{Z} \in \omega(A, B)$
  - $h_d(A, \mathcal{Z}) \leq \frac{1}{2} h_d(A, B)$
  - $h_d(\mathcal{Z}, B) \leq \frac{1}{2} h_d(A, B)$

Lemma

If (X, d) is a complete metric space, then d is strictly intrinsic if and only if $\omega(x, y)$ is nonempty for every $x, y \in X$.