Complex Dynamics
Part III: Quasiconformal Mappings

Logan S. Fox

Portland State University

PSU Analysis Seminar
February 5, 2021

- Part I: Entire Functions and Wandering Domains
- Part II: Rational Functions on the Riemann Sphere
- **Part III: Quasiconformal Mappings**
- Part IV: Sullivan's Proof of the No Wandering Domains Conjecture

Some references:

- John Milnor, *Dynamics in One Complex Variable*. arXiv 9201272[math.DS]
- Nathan Mercer and Rich Stankewitz, *Quasiconformal Mappings in the Plane with an Application to Quasiconformal Surgery* Find it Here
- Lars Alfors, *Lectures on Quasiconformal Mapings*. Find it here.
- Some notes of my own

The Cauchy-Riemann Equations

(we will follow Rudin's introduction from *Real and Complex Analysis*)

Let f be a complex-valued function and suppose $f(0) = 0$. When z is near 0,

$$f(z) = \alpha x + \beta y + \varepsilon(z)z, \quad z = x + iy$$

where $\alpha, \beta \in \mathbb{C}$ are the partial derivatives w.r.t. x and y, and $\varepsilon(z) \to 0$ as $z \to 0$.

Observing that $2x = z + \bar{z}$ and $2iy = z - \bar{z}$,

$$f(z) = \frac{\alpha - i\beta}{2} z + \frac{\alpha + i\beta}{2} \bar{z} + \varepsilon(z)z.$$

Introduce the differential operators ∂ and $\bar{\partial}$,

$$\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \bar{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

$$\frac{f(z)}{z} = \partial f(0) + \bar{\partial} f(0)\frac{\bar{z}}{z} + \varepsilon(z)$$

z real $\implies \frac{\bar{z}}{z} = 1$

z pure imaginary $\implies \frac{\bar{z}}{z} = -1$

Thus, the *derivative* of f exists precisely when $\bar{\partial} f = 0$. And so $f'(0) = \partial f(0)$.

$$\partial f = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \qquad \bar{\partial} f = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

f is **holomorphic** if it is differentiable at every point in its domain.

f is holomorphic at $z \iff \bar{\partial} f(z) = 0$.

If $f(x+iy) = u(x,y) + iv(x,y)$, where $u, v : \mathbb{R}^2 \to \mathbb{R}$, then $\bar{\partial} f = 0$ is equivalent to

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

the **Cauchy-Riemann equations**.

f is **conformal** if it is holomorphic and a homeomorphism.

Ex: Every conformal map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a Möbius transformation,

$$f(z) = \frac{az+b}{cz+d} \quad ac - bd \neq 0$$

A **quasiconformal** map is 'almost' conformal in the following quantifiable sense.

Consider the **Beltrami equation**: $\bar{\partial} f = \mu \partial f$.

If f satisfies $\bar{\partial} f = \mu \partial f$ a.e.
μ is Lebesgue meas. with $\|\mu\|_\infty < 1$,
then f is quasiconformal with complex dilitation μ.

Let f be given by

$$f(x+iy) = u(x,y) + iv(x,y)$$

Letting $u_x = \partial u/\partial x$, $u_y = \partial u/\partial y$, etc. define the Jacobian matrix

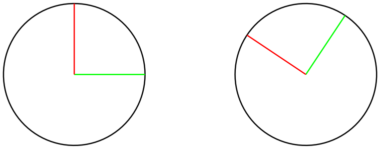
$$J_f = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}.$$

Note $\det(J_f) = |\partial f|^2 - |\bar{\partial} f|^2$.

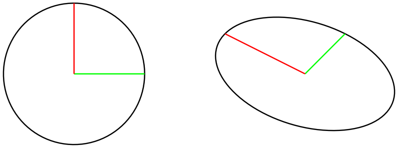
By the Cauchy-Riemann equations, if f is conformal then

$$J_f = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Conformal maps preserve angles:

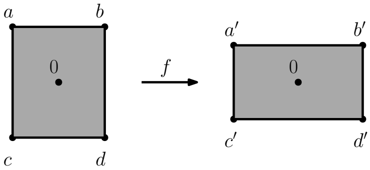


Quasiconformal maps are sense-preserving:

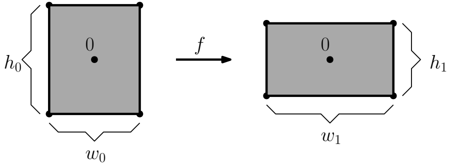


$$\|\mu\|_\infty < 1 \iff |\partial f| > |\bar{\partial} f| \iff \det(J_f) > 0$$

Grötzsch's Problem



Find the 'closest-to-conformal' f satisfying $f(a) = a'$, $f(b) = b'$, etc.



One possibility:

$$f(x+iy) = \frac{w_1}{w_0}x + i\frac{h_1}{h_0}y$$

$$\partial f = \frac{1}{2}\left(\frac{w_1}{w_0} + \frac{h_1}{h_0}\right) \qquad \bar{\partial} f = \frac{1}{2}\left(\frac{w_1}{w_0} - \frac{h_1}{h_0}\right)$$

f satisfies $\bar{\partial} f = \mu_f \partial f$ with

$$\mu_f = \frac{\bar{\partial} f}{\partial f} = \frac{w_1 h_0 - h_1 w_0}{w_1 h_0 + h_1 w_0}.$$

Grötzsch's question: Is there a quasiconformal map g with $\|\mu_g\|_\infty < \|\mu_f\|_\infty$?
Answer: No.

Existence and Uniqueness

Let μ be such that $\|\mu\|_\infty < 1$ on a domain D.

(i) There exists a quasiconformal map f with complex dilatation μ on D.

(ii) If f and g are two maps with complex dilatation μ, then $g \circ f^{-1}$ is conformal. I.e. $\mu_{g \circ f^{-1}} = 0$.

(iii) If g is conformal, then

$$\mu_{g \circ f} = \mu_f \quad \text{and} \quad \mu_{f \circ g} = (\mu_f \circ g)\frac{\overline{g'}}{g'}.$$

Measurable Riemann Mapping Theorem

Theorem

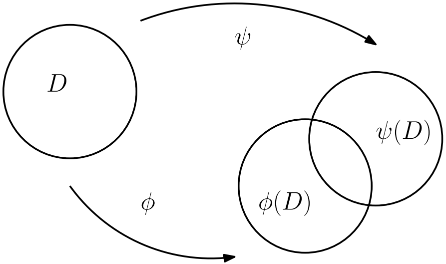
For any $\mu \in L^\infty(\mathbb{C})$ with $\|\mu\|_\infty < 1$, there is a unique quasiconformal map f with complex dilatation μ, which is normalized to fix 0, 1, and ∞.

See Chapter 5 of Lars Alfors' *Lectures on Quasiconformal Mapings*. Find it here.

Quasiconformal Riemann Surfaces

$\mu_f = \mu_g \iff g \circ f^{-1}$ is conformal.

Let D be a domain in $\mathbb{C}$.
Let μ be a Beltrami coefficient.
Consider all of the quasiconformal embeddings $\phi : D \to \mathbb{C}$ satisfying $\bar{\partial}\phi = \mu\partial\phi$.



Given that $\phi \circ \psi^{-1} : D \to D$ is conformal, we can treat this like a Riemann surface.

Define $D[\mu]$ as a Riemann surface with μ-conformal structure:

$$\text{atlas}(D[\mu]) = \{\phi : \bar{\partial}\phi = \mu\partial\phi\}$$

A map $f : D[\mu] \to E[\nu]$ is holomorphic if $\phi_\nu \circ f \circ \phi_\mu^{-1}$ is holomorphic in the usual sense.

$$\begin{array}{ccc} D[\mu] & \xrightarrow{f} & E[\nu] \\ \downarrow{\scriptstyle \phi_\mu} & & \downarrow{\scriptstyle \phi_\nu} \\ \mathbb{C} & \longrightarrow & \mathbb{C} \end{array}$$

Pull-back of Conformal Structure

$$\begin{array}{ccc} D[\mu] & \xrightarrow{f} & E[\nu] \\ \downarrow{\scriptstyle \phi_\mu} & & \downarrow{\scriptstyle \phi_\nu} \\ \mathbb{C} & \longrightarrow & \mathbb{C} \end{array}$$

Lemma

$f : D[\mu] \to E[\nu]$ is holomorphic if and only if

$$\mu(z) = \frac{\overline{f'(z)}}{f'(z)} \nu(f(z)).$$

$$\implies \|\mu\|_\infty = \|\nu\|_\infty$$

Let $f : D \to E$ be surjective holomorphic.

Given ν, can we find μ such that $f : D[\mu] \to E[\nu]$ is holomorphic?

Define μ as the pull-back of ν, given by the previous lemma.