Definitions 1-3 are from [3].

Definition 1 - A sequence of functions $\{f_n\}_n$, $f_n : A \to B$ is said to **diverge locally uniformly** from B if for every compact $K_A \subset A$ and $K_B \subset B$, there is an n_0 such that $f_n(K_A) \cap K_B = \varnothing$ for all $n \geq n_0$.

Definition 2 - A collection $\mathscr{F}$ of holomorphic functions, $f : X \to Y$, where X and Y are Riemann surfaces, is called a **normal family** if every infinite sequence of functions in $\mathscr{F}$ contains a subsequence which either converges locally uniformly (i.e. converges uniformly on compact sets) or diverges locally uniformly from Y.

Definition 3 - Let Ω be a Riemann surface and $f : \Omega \to \Omega$ be holomorphic. The **Fatou set** of f, denoted F_f, is the union of domains $D \subset \Omega$ such that the sequence of iterates $\{f^{\circ n}\}_{n=1}^{\infty}$ forms a normal family when restricted to D. The **Julia set** of f, denoted J_f, is the compliment of the Fatou set, $J_f = \Omega \setminus F_f$. A **Fatou component** is a maximal connected subset of F_f.

One method for determining parts of the Fatou and Julia sets is through fixed points. Recall that a point p is a fixed point of a function f if $f(p) = p$. Additionally, p is said to be *attracting* if $|f'(p)| < 1$, *repelling* if $|f'(p)| > 1$, and *neutral* or *indifferent* if $|f'(p)| = 1$. Any repelling fixed point lies in the Julia set and any attracting fixed point lies in the Fatou set. (For an exploration of neutral fixed points, see [3]). We will show here that any attracting fixed point of a holomorphic function lies in its own component of the Fatou set.

Definition 4 - If $f : X \to X$ is a continuous function with attracting fixed point p, the **basin of attraction** of p is the set

$$\left\{x \in X : \lim_{n\to\infty} f^{\circ n}(x) = p\right\}$$

The **immediate basin of attraction** of p is the maximal connected subset of the basin which contains the fixed point p; i.e. the component of the basin which contains p.

Lemma 1 - *Let $f : \Omega \to \Omega$ be a holomorphic function on a Riemann surface Ω with an attracting fixed point p. Then the basin of attraction of p is an open set.*

Proof - Let B_p be the basin of attraction of p. Since p is attracting, we can find a λ such that $|f'(p)| < \lambda < 1$ and fix $\delta > 0$ be such that

$$\frac{|f(z) - f(p)|}{|z - p|} < \lambda$$

whenever $|z - p| < \delta$. Let D be the basic open neighborhood $\{z \in \Omega : |z - p| < \delta\}$. Then for any $z \in D$,

$$|f(z) - p| = |f(z) - f(p)| \leq \lambda|z - p|$$

If follows that

$$|f^{\circ n}(z) - p| = |f^{\circ n}(z) - f^{\circ n}(p)| \leq \lambda^n|z - p|$$

As such, $f^{\circ n}(z) \to p$ for any $z \in D$, so B_p has nonempty interior.

Next, let z_0 be some element of B_p. Then $f^{\circ n}(z_0) \to p$. As such, we can find an $N \in \mathbb{N}$ such that $f^{\circ n}(z_0) \in D$ for all $n \geq N$. Therefore, $(f^{\circ N})^{-1}(D)$ is an open set containing z_0. Further, for each $z_D \in (f^{\circ N})^{-1}(D)$, $f^{\circ n}(z_D) \to p$, so $(f^{\circ N})^{-1}(D) \subseteq B_p$. So for any $z \in B_p$, there is an open neighborhood of z contained in B_p. Thus, B_p is open. □

Lemma 2 - *If p is an attracting fixed point of $f : \Omega \to \Omega$, then the immediate basin of attraction of p is a component of the Fatou set of f.*

Proof - Let f be given with attracting fixed point p, and let A_p be the immediate basin of attraction for p. First, we need to show that $\{f^{\circ n}\}_n$ is a normal family on A_p, so let K be a compact subset of A_p and let $\varepsilon > 0$ be given. Since $K \subseteq A_p$, we know $f^{\circ n}|_K \to p$ pointwise; it remains only to show that the convergence is uniform. Let λ and δ be given as in Lemma 2.10, so that $D = \{z : |z - p| < \delta\}$ is a basic open neighborhood in which f is a contraction.

Note that $K \subseteq \bigcup_n (f^{\circ n})^{-1}(D)$, so $\{(f^{\circ n})^{-1}(D)\}_n$ is an open cover of K. As such, we can find a finite collection $\{n_1, \cdots, n_k\}$ such that

$$K \subseteq \bigcup_{\ell=1}^{k} (f^{\circ n_\ell})^{-1}(D)$$

Fix $N_0 = \max\{n_1, \cdots, n_k\}$. Then $f^{\circ n}(K) \subseteq D$ for all $n \geq N_0$. Recall that for all $z \in D$,

$$|f^{\circ n}(z) - f^{\circ n}(p)| = |f^{\circ n}(z) - p| < \lambda^n \delta.$$

Fix N_1 such that $\lambda^{N_1}\delta \leq \varepsilon$. Then for $N = \max\{N_0, N_1\}$,

$$|f^{\circ n}(z) - p| < \varepsilon$$

for all $z \in K$ and $n \geq N$. Thus, $f^{\circ n}|_K \to p$ uniformly, so $\{f^{\circ n}\}_n$ is a normal family on A_p. Thus, $A_p \subseteq F_f$.

Now suppose by way of contradiction that A_p is strictly contained in some component of F_f, say U. Since every component of F_f is connected and open, U must contain an element of the boundary of A_p, say $\bar{z} \in \partial A_p$. However, if $\bar{z} \in \partial A_p$, for any neighborhood D of $\bar{z}$ such that $D \subseteq U$, $D \cap A_p$ is nonempty. As such, $f^{\circ n}(D \cap A_p) \to p$ but for any $z \in D \cap \partial A_p$, $\{f^{\circ n}(z)\}_n$ cannot have a subsequence which converges within the A_p neighborhood of p. Thus, $\{f^{\circ n}|_D\}_n$ cannot contain a subsequence which converges to a continuous function, so is not a normal family. It follows that no component of F_f strictly contains A_p.

Hence, A_p is a component of F_f. □

An Example

In [2], I.N. Baker credits M. Herman for the following example. Consider the function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) = z - 1 + e^{-z} + 2\pi i$$

Then f is transcendental entire on $\mathbb{C}$. We will show that f has wandering domains.

Definition 5 - Let Ω be a Riemann surface and $f : \Omega \to \Omega$ an analytic function. A component U of the Fatou set of f is

(i) **periodic** if for some $n \in \mathbb{N}$, $f^{\circ n}(U) = U$;

(ii) **eventually periodic** if for some $m \in \mathbb{N}$, $f^{\circ m}(U)$ is periodic; and

(iii) **wandering** if $f^{\circ n}(U) \cap f^{\circ m}(U) = \varnothing$ for all $n \neq m$.

Begin by considering the function $g(z) = z - 1 + e^{-z}$. It may be of interest to note that the function g is generated by applying Newton's method to the function defined by $z \mapsto e^z - 1$. As such, $z_k = 2\pi k i$ is a fixed point of g for each $k \in \mathbb{Z}$. Further, $g'(z_k) = 0$ for each z_k, so each fixed point is super-attracting. For each k, let A_k be the immediate basin of attraction for the fixed point z_k. By Lemma 2, each A_k is a component of F_g.

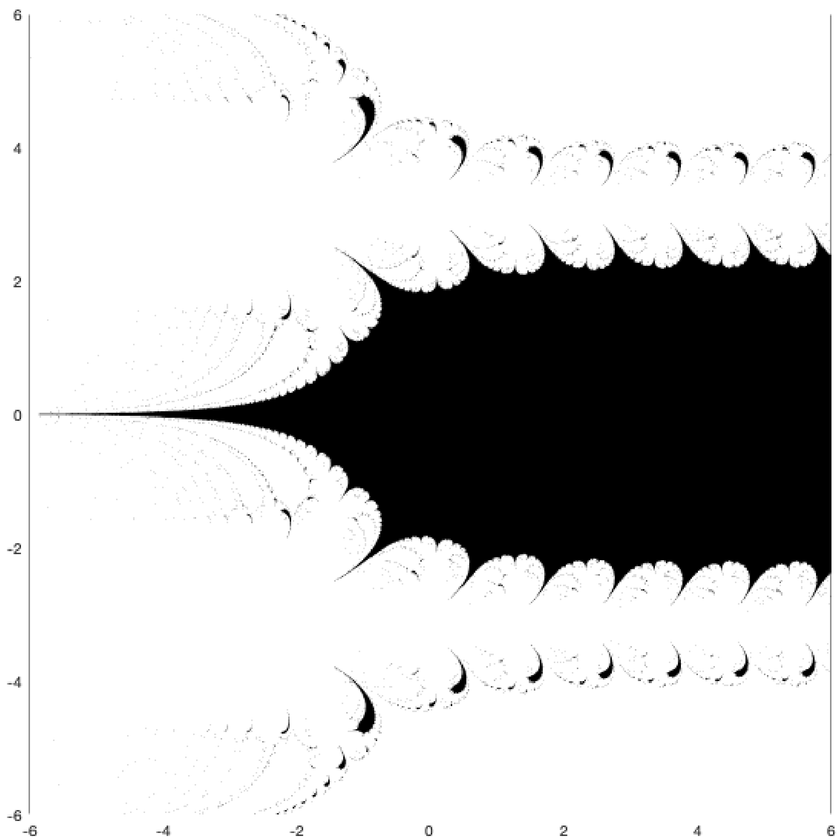

Figure 1: Basin of attraction for the fixed point 0 of $g(z) = z - 1 + e^{-z}$.

Additionally, observe that

$$g(z + 2\pi i) = z + 2\pi i - 1 + e^{-z-2\pi i} = z - 1 + e^{-z} + 2\pi i = g(z) + 2\pi i$$

As such, $g^{\circ n}(z + 2\pi i) = g^{\circ n}(z) + 2\pi i$. So if $g^{\circ n}(z) \to z_k$ then $g^{\circ n}(z + 2\pi i) \to z_{k+1}$. Therefore, $A_{k+1} = A_k + 2\pi i$.

Lemma 3 - *Suppose f and g are entire, f commutes with g, and $f = g + c$ where c is some constant. Then $J_f = J_g$.*

The proof of Lemma 3 is omitted here, but can be found in [2]. The principal idea is to show that $f(F_g) \subseteq F_g$ and $f(J_g) \subseteq J_g$. It follows immediately that $J_f = J_g$ and $F_f = F_g$. A general argument for this specific function can be found in [5].

It should be clear that f and g are both entire, and $f = g + c$ where $c = 2\pi i$. In order to apply the lemma, it remains only to check that f and g commute. Observe that

$$\begin{aligned} f \circ g(z) &= (z - 1 + e^{-z}) - 1 + e^{-z+1-e^{-z}} + 2\pi i \\ &= (z - 1 + e^{-z} + 2\pi i) - 1 + (e^{-2\pi i})e^{-(z-1+e^{-z})} \\ &= (z - 1 + e^{-z} + 2\pi i) - 1 + e^{-(z-1+e^{-z}+2\pi i)} \\ &= g \circ f(z) \end{aligned}$$

So by Lemma 3, $J_f = J_g$. It follows immediately that $F_f = F_g$, so each A_k described previously is a component of F_f. Recalling that $A_k + 2\pi i = A_{k+1}$, observe that

$$f(A_k) = g(A_k) + 2\pi i = A_{k+1}$$

Thus, each A_k is a wandering component of F_f.

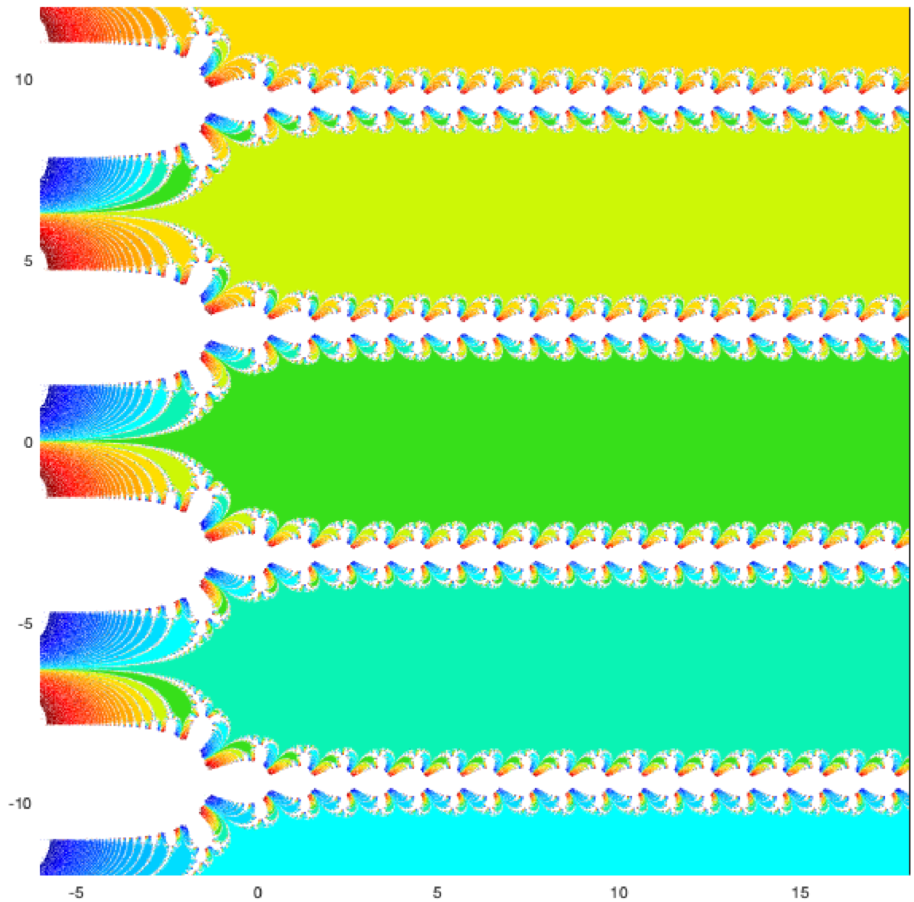

Figure 2: Basins of attraction for the fixed points of $g(z) = z - 1 + e^{-z}$. For the function $f(z) = g(z) + 2\pi i$, the basins wander vertically from blue to red.

It should be noted that the above example is not unique in its construction. As another example, the function $f(z) = z + \sin(z) + 2\pi$ has wandering domains and can be analyzed in a similar manner. (Let $g(z) = z + \sin(z)$)

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