Quantum Nash Equilibrium Notes Part I

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Introduction

A paper of great importance in the quantum (quantized) game literature is Landsburg's paper *Nash Equilibrium in Quantum Games* [4]. The goal of these notes is to introduce the reader to the relevant game theory to understand the paper. Throughout these notes we will generally assume that in the classical game players only have two strategies. This is done for readability.

Classical Material

Classical Games

A two-player game [6] [2] [3] is function which maps the Cartesian product of the strategy space of player 1, which we will denote S, and the strategy space of player 2,which we will denote T, to some outcome space. We will consider the outcome space $\mathbb{R}^2$. G is often called

$$G : S \times T \longrightarrow \mathbb{R}^2$$

Figure 1: A two-player game.

the payoff function. In general we have more information. We can write

$$G(s,t) = (u_1(s,t), u_2(s,t)), \quad \forall (s,t) \in S \times T$$

Here u_1 is the utility function of player 1 and u_2 is the utility function of player 2. These functions take strategies as their input and output a number that represents a players preferences. Higher utility means that a player likes that outcome more.

Often times we will represent a game with it's *Normal Form*, that is we will look at it as a bi-matrix where the labels of the columns and rows are the strategies and the cells are pairs of numbers which represent the utility function evaluated at those strategies. For a game where each player only has two strategies a bi-matrix would be the following.

		II	
		t_1	t_2
I	s_1	$u_2(s_1,t_1)$ $u_1(s_1,t_1)$	$u_2(s_1,t_2)$ $u_1(s_1,t_2)$
	s_2	$u_2(s_2,t_1)$ $u_1(s_2,t_1)$	$u_2(s_2,t_2)$ $u_1(s_2,t_2)$

Figure 2: Bi-Matrix for Two-Player Two Game with Two Strategies

When we analyze games we often look for strategies that have special properties, one of the most studied strategy types is *Nash Equilibriums* [5]. A Nash Equilibrium is a strategic pair where each strategy is a best reply to the other. Formally in a two-player game the strategic pair $(s^\star, t^\star) \in S \times T$ is a Nash Equilibrium if the following two equations hold

$$u_1(s^\star, t^\star) \geq u_1(s, t^\star), \quad \forall s \in S \tag{1}$$

$$u_2(s^\star, t^\star) \geq u_2(s^\star, t), \quad \forall t \in T \tag{2}$$

The obvious question to ask is does every game have a Nash Equilibrium. The following is the *Normal Form* of a game which has no Nash Equilibrium. This game is called Matching Pennies.

		II	
		t_1	t_2
I	s_1	-1 1	1 -1
	s_2	1 -1	-1 1

Figure 3: Matching Pennies

The question then becomes is it possible to "faithfully" extend this game in such a way that we can still find the original game within the new game and are able to identify Nash Equilibriums in this new game. The answer is yes, and there are a variety ways of doing this. The most common extension is to look at the so called mixed game.

Mixed Games

The formal definition [3] of a mixed game is as follows. The strategy space of the mixed game of G, which we will denote G^{mix}, for player 1 is the set of probability distributions over S, which we will denote $\Delta(S)$, similarly the strategy space for player 2 is the set of probability distributions over T, which we will denote $\Delta(T)$. If S and T are finite $\Delta(S)$ and $\Delta(T)$ are nothing but convex linear combinations of elements of S and T. Once we have the distributions over the strategy space we can obtain a distribution over $S \times T$. Since the distributions are independent the joint distribution is simply the product of the two marginal distributions. Using this joint distribution we obtain a probability distribution over the image of G, which we denote $\Delta(\text{Im } G)$. From there we take expectation to calculate the outcome.

This is a "proper" extension of the game G because we have a embedding e_1 and e_2 that take pure strategies for player 1 and player 2 respectively and map them to the distribution where you play the given strategy with probability 1 and all other strategies with probability 0. This means $G^{\text{mix}} \circ e_1 \times e_2(s,t) = G(s,t)$. The commutative diagram below, modified and used with permission of Professor Bleiler, summarizes how to extend a game to a mixed game properly.

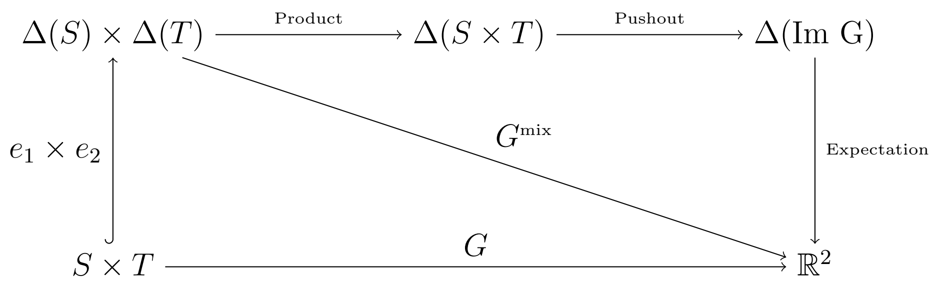

Figure 4: Extension for G^{mix}

We have faithfully extended the game, but does this new game G^{mix} have any Nash Equilibrium? This question was answered by Nash [5] in his seminal paper. He proved that in any n-player game where each player's strategy space is finite there will always be a Nash Equilibrium in the mixed game. Now the question becomes how do we find the equilibrium in this new game. We will show how to solve this in the general case and then will conclude with an examples. In order to solve a 2×2 game you must solve the following equations

$$pu_2(s_1, t_1) + (1-p)u_2(s_2, t_1) = pu_2(s_1, t_2) + (1-p)u_2(s_2, t_2) \quad (3)$$

$$qu_1(s_1, t_1) + (1-q)u_1(s_1, t_2) = qu(s_2, t_1) + (1-q)u(s_2, t_2) \quad (4)$$

In the game of Matching Pennies this set of equations becomes

$$p(-1) + (1-p)1 = p(1) + (1-p)(-1)$$

$$q(1) + (1-q)(-1) = q(-1) + (1-q)(1)$$

Which has the obvious solution $p = q = 0.5$.

One thing to note about mixed strategies is that there exists distributions over $\Delta(S \times T)$ that are unachievable through mixed strategies. This is one of the motivations behind extending a game to a game of classical mediated communication.

Classical Mediated Communication

Games of classical mediated communication [1] are another way to properly extend a game. In classical mediated communication players have a referee mediate and they communicate their strategies to the referee. In the two player case before the game begins the players agree on a distribution over the image of G, call this distribution ρ. The referee will perform a random act that follows the distributions over the image of G and he will then tell each player what to play. Players then communicate the strategy they play to the referee. Players in this new game, which we will call G_ρ^{com}, have the following strategy spaces.

$$S_\rho^{\text{com}} = \{s_1', s_2', c, d\} \tag{5}$$

$$T_\rho^{\text{com}} = \{t_1', t_2', c, d\} \tag{6}$$

If player 1 plays strategy s_1' they will play s_1 regardless of what the referee tells him to play, if they play s_2' they will play s_2 regardless of what the referee tells them, if they play c they will play the strategy the referee tells them, and if they play d they will play the opposite strategy the referee tells them to. T_ρ^{com} is defined in the same way.

The following commutative diagram, where f_1 and f_2 are defined in the obvious way such that the diagram commutes, is a visual way of looking at extending G to G_ρ^{com}. A correlated equilibrium [2] is when (c, c) is a Nash Equilibrium in G_ρ^{com}.

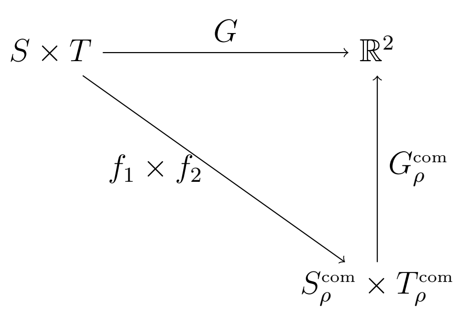

Figure 5: Extension to G_ρ^{com}

Below is the normal form of the game of chicken.

		II	
		t_1	t_2
I	s_1	2 2	3 0
	s_2	0 3	-1 -1

Figure 6: Chicken

This game has two Nash Equilibrium in pure strategy, (s_1, t_2) and (s_2, t_1). Now we will see, with the proper distribution, that this game has a correlated equilibrium. The distribution we will use has (s_1, t_1) played with probability $\frac{1}{3}$, (s_1, t_2) played with probability $\frac{1}{3}$, and (s_2, t_1) played with probability $\frac{1}{3}$. The bi-matrix for this game is below

		II			
		t'_1	t'_2	c	d
I	s'_1	2 2	3 0	$7/3$ $4/3$	$8/3$ $2/3$
	s'_2	0 3	-1 -1	$-1/3$ $5/3$	$-2/3$ $1/3$
	c	$4/3$ $7/3$	$5/3$ $-1/3$	$5/3$ $5/3$	$4/3$ $1/3$
	d	$2/3$ $8/3$	$1/3$ $-2/3$	$1/3$ $4/3$	$2/3$ $2/3$

Figure 7: Classical Mediated Chicken

Checking best responses we see that this game has a Nash Equilibrium at (c, c) so the game of chicken has a correlated equilibrium with the previously noted distribution.

I \ II	t_1'	t_2'	c	d
s_1'	2 2	$\boxed{3}$ $\boxed{0}$	$7/3$ $4/3$	$8/3$ $\boxed{2/3}$
s_2'	$\boxed{0}$ $\boxed{3}$	-1 -1	$-1/3$ $\boxed{5/3}$	$-2/3$ $1/3$
c	$4/3$ $7/3$	$\boxed{5/3}$ $-1/3$	$\boxed{5/3}$ $\boxed{5/3}$	$4/3$ $1/3$
d	$\boxed{2/3}$ $8/3$	$1/3$ $-2/3$	$1/3$ $4/3$	$\boxed{2/3}$ $\boxed{2/3}$

Figure 8: Classical Mediated Chicken with Best Responses

Conclusion

These notes should prepare the reader for the game theory required to understand Landsburg's paper. The only other required background is a small understanding of quantum mechanics.

References

[1] Robert J. Aumann. Subjectivity and correlation in randomized strategies. *Journal of Mathematical Economics*, 1(1):67–96, 1974.

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[5] John F Nash. Equilibrium points in n-person games. *Proceedings of the national academy of sciences*, 36(1):48–49, 1950.

[6] John Von Neumann and Oskar Morgenstern. *Theory of games and economic behavior (commemorative edition)*. Princeton university press, 2007.