The Analysis Seminar
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The Invariant Subspace Problem

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Abstract: Notes for my lectures in the PSU Analysis Seminar during the winter and spring terms 2013–14, with special emphasis on the 1972 results of Victor Lomonosov.

Contents

1. Introduction

Notation and terminology. In all that follows:

- X is a Banach space over the field $\mathbb{C}$ of complex numbers. Unless otherwise stated we'll assume X has dimension > 1.
- $\mathscr{L}(X)$ is the collection of all operators on X, where "operator" means "continuous linear transformation."

A linear subspace M of X will be said to be:

- "Nontrivial" if it is closed in X and is neither the whole space nor the zero-subspace,
- "Invariant for $T \in \mathscr{L}(X)$" if $T(M) \subset M$,
- "Hyperinvariant for $T \in \mathscr{L}(X)$" if it's invariant for every operator on X that commutes with T, and more generally:
- "Invariant for $\mathscr{T} \subset \mathscr{L}(X)$" if it's invariant for every $T \in \mathscr{T}$.

The Invariant Subspace Problem. The most general statement of this fascinating problem is:

Must every $T \in \mathscr{L}(X)$ have a nontrivial invariant subspace?

The answer is "Yes" if:

- *X is finite dimensional.* In this case X is linearly homeomorphic to $\mathbb{C}^N$ for some $N > 1$. Suppose $T \in \mathscr{L}(X)$. Then T has an eigenvalue—call it λ. The corresponding eigensubspace $\ker(T - \lambda I)$ is hyperinvariant for T, and assuming the operator is not a scalar multiple of the identity, is nontrivial.
- *X is nonseparable.* In this case, fix $x \in X \backslash \{0\}$, and let M denote the closure of the linear span of the T-orbit of x; that is,

  $$M = \overline{\operatorname{span}}\{T^n x : n = 0, 1, 2, \ldots\}.$$

  M is easily seen to be invariant for T, and to be a separable subspace of X, so $M \neq X$. Since M contains x, it's $\neq \{0\}$ so it's a nontrivial invariant subspace for T.
- *T is compact.* This is nontrivial. It was first proved for Hilbert space by von Neumann in the 1920's. His proof, which was never published, was picked up in the 1950's by Aronszajn and Smith [2] who simplified the argument and generalized it to the Banach space setting.
- *$p(T)$ is compact for some polynomial p.* This result was proved in the 1960's by Bernstein and Robinson [5], who used nonstandard analysis" (see [11] for a translation of

[5] into "standard" analysis). Further research by other authors produced incremental improvements, none of which settled the question of whether or not every compact operator on separable Hilbert space had a *hyperinvariant* subspace.

A *compact operator* is one for which the image of the closed unit ball (and therefore of every bounded set) is relatively compact. Every operator on a finite dimensional Banach space is compact (a consequence of the the Bolzano-Weierstrass theorem and the fact that every such space is linearly homeomorphic to a Euclidean space). By contrast, for any infinite dimensional Banach space the closed unit ball is *not* compact (see, e.g., [21, Theorem 1.22, page 17] for this result in a more general context), hence the identity operator on such a space is not compact.

The answer is, in general, "No!"

- The first counter-example was produced by Per Enflo in the 1970's, but the Banach space he constructed was very complicated, and his paper [9] did not appear until the late '80's.
- In the interim Charles Read [17, 18] gave a more accessible counterexample; in particular he showed that there exists an operator R on the sequence space ℓ^1 that has no nontrivial invariant subspace. Read's operator R is still the object of study; we'll come back to this below.

The problem is *open* for:

- *Separable Hilbert space* (this is what's usually termed "The Invariant Subspace Problem"), and more generally
- *Separable reflexive Banach spaces,* and even more generally
- *Adjoint operators:* If the dual space X^* of X is separable and $T \in \mathscr{L}(X)$, must $T^* : X^\star \to X^*$ have a nontrivial invariant subspace?

Read's operator R was considered to be a prime candidate for a counterexample to this last question, but in 1998 Vladimir Troitsky [23] showed that R is not the adjoint of any operator on any predual of ℓ^1 (there are many such preduals; their study is an interesting subarea of Banach space theory.)

Finally: in any of the above settings where the Invariant Subspace Problem is open, the corresponding "Hyperinvariant Subspace Problem" (whose formulation I leave to you) is also open, and appears to be open even for nonseparable Hilbert spaces.

2. The "Big Lomonosov"

Does every non-zero compact operator on infinite dimensional Hilbert space have a *hyperinvariant subspace?* This problem attracted a lot of interest during the 1960's, and remained open until 1972 when an unknown young Russian mathematician named Victor Lomonosov stunned the operator-theory world with a beautiful two-page paper [12] that actually proved a lot more.

Lomonosov led off his paper with a positive answer to the question posed above:

Lomonosov's "Little" Theorem. *If X is an infinite dimensional Banach space and $T \in \mathscr{L}(X)$ commutes with a non-zero compact operator on X, then T has a nontrivial invariant subspace.*

Then, almost in passing, Lomonosov noted that his argument actually proves something much stronger:

Lomonosov's "Big' Theorem. *Suppose X is an infinite dimensional Banach space and $T \in \mathscr{L}(X)$, not a scalar multiple of the identity operator, commutes with a nonzero compact operator. Then T has a nontrivial hyperinvariant subspace.*

Scalar multiples of the identity operator must be excluded from any hyperinvariant subspace theorem since *every* operator on X commutes with such a "scalar" operator. It's an easy exercise to show that no nontrivial subspace of X is invariant for the collection $\mathscr{L}(X)$ of all operators on X (Hint: each nontrivial subspace of X is non-invariant for some rank-one operator).

The key to Lomonosov's results is the following lemma, which he did not state explicitly, but which nevertheless constitutes the major part of his argument.

Lomonosov's Lemma. *Suppose X is an infinite dimensional Banach space, K is a nonzero compact operator on X, and $\mathscr{A}$ is a subalgebra of $\mathscr{L}(X)$ that has no nontrivial invariant subspace. Then there exists an operator $A \in \mathscr{A}$ such that KA has eigenvalue 1.*

By a "subalgebra of $\mathscr{L}(X)$" I mean a vector subspace of $\mathscr{L}(X)$ that is closed under operator multiplication (a.k.a. "composition"). In Lomonosov's Lemma it is not required that the compact operator K belong to the algebra $\mathscr{A}$. (Note that the "eigenvalue one" conclusion is not special; if A_0K has a nonzero eigenvalue λ for some $A_0 \in \mathscr{A}$ then $A = A_0/\lambda \in \mathscr{A}$ and AK has eigenvalue 1.)

Lomonosov's Lemma $\implies$ "The Big Lomonosov Theorem." We're given that X is an infinite dimensional Banach space and $T \in \mathscr{L}(X)$, not a scalar multiple of the identity operator, commutes with a non-zero compact $K \in \mathscr{L}(X)$. We want to show that T has a hyperinvariant subspace. To this end, let $\mathscr{A}$ be the "commutant of T," i.e. the collection of all operators on X that commute with T. $\mathscr{A}$ is a subalgebra of $\mathscr{L}(X)$; the theorem we wish to prove asserts that it has an invariant subspace.

Suppose it does not.

Then Lomonosov's Lemma promises an operator $A \in \mathscr{A}$ such that KA has eigenvalue 1, so the corresponding eigenspace

$$M := \ker(KA - I)$$

is a closed subspace $\neq 0$ that is hyperinvariant for KA. Since KA is compact (the algebra of compact operators is easily seen to be a two-sided ideal in $\mathscr{L}(X)$), M is finite dimensional, so $\neq X$. *Conclusion:* M is a nontrivial hyperinvariant subspace for KA.

Now our object of interest, the operator T, commutes with K (hypothesis) and with A (definition of $\mathscr{A}$), so it commutes with KA and therefore has M as a nontrivial invariant subspace (hyperinvariance of M for KA). Since $\dim M < \infty$, the restriction of T to M, and therefore T itself, has an eigenvalue. Call this eigenvalue λ and consider the subspace

$$E := \ker(T - \lambda I).$$

As before, E is $\neq 0$, closed, and hyperinvariant for T, i.e. invariant for $\mathscr{A}$. That $E \neq X$ is a consequence of our assumption that T is not a scalar multiple of the identity operator. Thus E is a nontrivial invariant subspace for $\mathscr{A}$. But this contradicts our assumption that no such subspace exists. Therefore such a subspace *does* exist, i.e. $\mathscr{A}$ has a nontrivial invariant subspace. □

Proof of Lomonosov's Lemma. We're given a subalgebra $\mathscr{A}$ of $\mathscr{L}(X)$ that has no nontrivial invariant subspace.

CLAIM: For every $x \in X\backslash\{0\}$ the vector subspace

$$\mathscr{A}x := \{Ax : A \in \mathscr{A}\}$$

of X is dense in X.

Proof of CLAIM. If this were not the case, then the closure of $\mathscr{A}x$ would be a nontrivial invariant subspace for $\mathscr{A}$. (Note that it's crucial here that $\mathscr{A}$ be an algebra, i.e. a vector space, which guarantees that $\mathscr{A}x$, and therefore its closure, is a linear subspace of X, and closed under multiplication, which guarantees that $\mathscr{A}x$, and therefore its closure, is invariant for every operator in $\mathscr{A}$.) This density can be rephrased as follows:

For every nonempty open subset V of X the collection of open sets

$$\{A^{-1}(V) : A \in \mathscr{A}\}$$

covers $X\backslash\{0\}$.

Now for the heart of the argument. We may assume without loss of generality that $\|K\| = 1$. Fix $x_0 \in X$ such that $\|Kx_0\| > 1$ (take any vector not in the kernel of K and multiply it by an appropriately large scalar), so that also $\|x_0\| > 1$. Let B be the closed ball in X of radius one, centered at x_0, and let C denote the closure of $K(B)$. Then C is a nonvoid closed convex subset of X that, because of the compactness of the operator K, is compact.

Claim: $0 \notin C$.

Proof of Claim. Since K has norm 1 it does not increase distances between points. Thus each point of K(B) lies within one unit of $K(x_0)$, and so C, the closure of $K(B)$ has the same property, i.e. it lies in the closed disc of radius 1 about $K(x_0)$. Since $\|K(x_0)\| > 1$, the origin of X lies outside this disc, and so outside of C.

In the best of worlds we'd hope to find an "operator hero" $A \in \mathscr{A}$ that takes C into B, in which case KA would be a continuous mapping of C into itself, and so would, by the Schauder Fixed-Point Theorem ([21, Theorem 5.28, pp. 143–144], [22]), have a fixed point that lies in C, and therefore is different from 0. In other words, KA would have an eigenvector for the eigenvalue 1.

All this is too much to hope for, but we *can* do much the same thing "locally." To see what this means, let B° denote the interior of B, i.e. the *open* ball of radius 1 centered at x_0. Then, as we saw more generally near the bottom of page 5, the collection of sets $\{A^{-1}(B^\circ) : A \in \mathscr{A}\}$ is an open cover of $X\backslash\{0\}$, hence of the compact convex set C. This open cover of C therefore has a finite subcover $\mathscr{U} = \{U_j = A_j^{-1}(B^\circ) : 1 \leq j \leq N\}$. These operators $A_j \in \mathscr{A}$ are our "local" heroes; each one takes $U_j \cap C$ into B°, hence into B.

To make the transition from "local" to "global" we start by gluing the operators A_j together into a continuous function that takes C into B. For "glue" we'll use a partition of unity $\{p_1, p_2, \ldots, p_N\}$ on C that is subordinate to $\mathscr{U}$; that is:

- Each p_j is continuous on K, with values in the closed real interval $[0,1]$.
- Each p_j is supported in U_j (i.e. it's $\equiv 0$ in $K\backslash U_j$).
- $\sum_{j=1}^{N} p_j(y) = 1$ for all $y \in C$.

(For metric spaces the existence of such partitions of unity is easy to prove; see Appendix A below.)

On with the proof: Define $\Phi : C \to X$ by

$$\Phi(y) := \sum_{j=1}^{N} p_j(y) A_j y \qquad (y \in C). \tag{2.1}$$

Then Φ, being a sum of continuous functions is continuous. Moreover,

$$\Phi(C) \subset B \text{ (!!)}$$

To see why, fix $y \in K$ and note that $p_j(y) = 0$ unless $y \in U_j$, and for these indices j we know that $A_j y \in A_j(U_j) = B^\circ$. Thus the right-hand side of (2.1) is a convex sum of vectors in B°, and so belongs to B°, and therefore to B.

Conclusion: Φ is a continuous map taking C into B, so $K \circ \Phi$ takes C continuously into itself, and therefore by the Schauder Theorem, has a fixed point y_0 in C (at the risk of belaboring the point: $y_0 \neq 0$).

So Φ is a nonlinear version of the operator A about which we fantasized several paragraphs ago. From Φ it's easy to construct an operator A which, while it doesn't necessarily take C into B, still does the job. Let

$$A := \sum_{j=1}^{N} p_j(y_0) A_j$$

Then A, being a linear combination of operators in $\mathscr{A}$, also belongs to $\mathscr{A}$, and $Ay_0 = \Phi(y_0)$. Thus $(KA)y_0 = (K \circ \Phi)(y_0) = y_0$. □

3. "Little Lomonosov" without fixed points

Soon after Lomonosov published his seminal paper [12], the operator theorist Larry Wallen of the University of Hawai'i asked his colleague Mike Hilden (a topologist) if, in Lomonosov's argument, it might be possible to replace the Schauder fixed-point theorem by the more elementary Banach Contraction-Mapping Principle. Hilden rose to the bait, and to the surprise of everyone, managed to remove fixed-point theorems entirely from the proof of Lomonosov's "little theorem."

Hilden's proof of Lomonosov's Little Theorem. We're given an infinite dimensional Banach space X and a non-zero compact operator K on X. We want to find a nontrivial subspace of X that is invariant for every operator on X that commutes with K.

(a) *Enough to assume K has no eigenvalue.* For if $\lambda \in \mathbb{C}$ is an eigenvalue of K then, as we've seen several times before, the eigenspace

$$M := \ker(K - \lambda I)$$

is a nontrivial hyperinvariant subspace for K (it's $\neq X$ because it's finite dimensional, while X is not).

(b) *Some operator theory.* The Riesz Theory of Compact Operators (see e.g. [21, Theorem 4.25, pp. 108–109]) asserts that the *spectrum* of the compact operator K (the set of complex numbers λ for which $K - \lambda I$ is not invertible) consists only of zero (since K is not invertible) and possibly eigenvalues. Since we're assuming K has no eigenvalues, its spectrum is therefore the singleton $\{0\}$. Now for any $T \in \mathscr{L}(X)$ the *spectral radius* (the supremum the moduli of the complex numbers in the spectrum) is $\lim_{n\to\infty} \|T^n\|^{1/n}$ (see e.g. [21, Theorem 10.13, page 253] for this result in the more general context of Banach algebras). Thus for our eigenvalue-free compact operator K,

$$\lim_{n\to\infty} \|K^n\|^{1/n} = 0. \tag{3.1}$$

(c) *Brief review of Lomonosov's argument.* Let $\mathscr{A}$ be the commutant of K; our goal is to find a nontrivial invariant subspace for $\mathscr{A}$, and as before we'll proceed by contradiction. Assume $\mathscr{A}$ has no nontrivial invariant subspace. Suppose $\|K\| = 1$, and fix $x_0 \in X$ such that $\|Kx_0\| > 1$, noting that this implies that also $\|x_0\| > 1$. Let C be the closure of $K(B)$, so C is nonvoid, compact, convex, and $0 \notin C$. The assumption that $\mathscr{A}$ has no nontrivial invariant subspace implies that $\mathscr{A}x$ is dense in X for every $x \in X\backslash\{0\}$, and this implies that the collection of open sets $\{A^{-1}(B^\circ) : A \in \mathscr{A}\}$ covers $X\backslash\{0\}$, and therefore C. Therefore there is a finite subset $\{A_1, A_2, \ldots A_N\}$ of operators in $\mathscr{A}$ such that the open sets

$$U_j := A_j^{-1}(B^\circ) \qquad (j = 1, 2, \ldots N)$$

cover C.

At this point the Lomonosov argument introduced a partition of unity to "glue the operators A_j together" into a continuous mapping Φ that took C into B, whereupon the map $K \circ \Phi$ took C continuously into itself, and so had a fixed point in C.

(d) *Hilden's "ping-pong" argument.* Instead of gluing operators together into something nonlinear, we imagine a ping-pong match with K on the "B-side" of the table and the team of A_j's on the "C-side." Initially the ball sits at the point $x_0 \in B$, and K sends it over to $y_0 := Kx_0 \in C$. Now y_0 lies in (at least) one of the sets U_j, say in U_{j_1}. Thus player

A_{j_1} sends y_0 back into B, after which K sends the resulting point to $y_1 = KA_{j_1}y_0 \in C$. Repeat the process over and over to obtain a sequence

$$y_n = KA_{j_n}y_{n-1} = KA_{j_n}KA_{j_{n-1}}KA_{j_{n-2}} \cdots KA_{j_1}Kx_0 \in C \qquad (n = 1, 2, \ldots) \tag{3.2}$$

Recall that all the "A-operators" on the right-hand side of (3.2) lie in $\mathscr{A}$, i.e. *they commute with* K, hence

$$y_n = K^n A_{j_n}A_{j_{n-1}}A_{j_{n-2}} \cdots A_{j_1} \qquad (n = 1, 2, \ldots). \tag{3.3}$$

Upon setting $M = \max\{\|A_j\| : 1 \le j \le N\}$ we see from (3) that $\|y_n\|^{1/n} \le M\|K^n\|^{1/n}$ which, by (3.1) above, $\to 0$.

Consequence: The sequence (y_n), consisting of vectors in the compact set C, converges to zero, so $0 \in C$. But this contradicts the fact that C has been carefully constructed to *not* contain the zero-vector. Thus our assumption that $\mathscr{A}$ lacks nontrivial invariant subspaces has led to a contradiction. □

Because of (3.1) above, operators with spectrum $\{0\}$ are called *quasinilpotent*. For a while there was hope that Hilden's argument might be modified to show that every quasinilpotent operator has a nontrivial hyperinvariant—or at least an invariant—subspace. This hope lasted until 1997, when it was was shot down, at least for the sequence space ℓ^1, by Read in [20], where he exhibited a quasinilpotent operator on ℓ^1 having no nontrivial invariant subspace.

4. The Transitive Algebra Problem

Transitivity: what and why? A subalgebra $\mathscr{A}$ of $\mathscr{L}(X)$ is said to be "transitive" if it has no nontrivial invariant subspace. The terminology comes from the fact, well known to us from the proof of Lomonosov's Lemma (see page 5), that $\mathscr{A}$ is transitive iff $\mathscr{A}x$ is dense in X for every non-zero $x \in X$.

Our work on the Invariant Subspace Problem can be rephrased in terms of transitive algebras. Associated with an operator T on a Banach space X there are two natural subalgebras of $\mathscr{L}(X)$:

- $\mathrm{alg}(T)$, the algebra *generated by* T, the linear span of the operators $\{T^n : n \in \mathbb{N}\}$, i.e. the collection of operators $p(T)$ where p runs through all complex polynomials in one variable. This is the smallest subalgebra of $\mathscr{L}(X)$ that contains both T and the identity operator.

- $\mathrm{com(T)}$, the *commutator of* T, i.e. the collection of all operators on X that commute with T.

To say that T has no invariant subspace is to say that $\mathrm{alg}(T)$ is transitive, and to say T has no hyperinvariant subspace is to say that $\mathrm{com(T)}$ is transitive. Thus our invariant subspace questions are subsumed under the more general one:

Which subalgebras of $\mathscr{L}(X)$ are transitive?

Some things to know about transitivity.

(a) *The algebra $\mathscr{F}(X)$ of all finite rank operators on X (and therefore also the algebras of compact operators and of all operators on X) is transitive.*

Proof. Suppose M is a nontrivial subspace of $\mathscr{L}(X)$. We want to find a finite rank operator on X that takes M outside of itself. Since $M \neq X$ there's a vector $y \in X$ that lies outside M, and since M is closed in X the Hahn-Banach Theorem supplies a continuous linear functional φ on X that vanishes on M but not at y. Thus the rank-one operator $y \otimes \varphi$, defined by

$$(y \otimes \varphi)(x) = \varphi(x)y \qquad (x \in X)$$

takes M into the one dimensional subspace spanned by y, which intersects M only in the zero-subspace. □

(b) *If $T \in \mathscr{L}(X)$ is not a scalar multiple of the identity operator, then $\mathrm{com(T)} \neq \mathscr{L}(\mathrm{X})$.*

Proof. Since T is not a scalar multiple of the identity operator there is a vector $y \in X$ for which Ty is not a scalar multiple of y (this is actually a curious little exercise, which I leave to you). By the Hahn-Banach Theorem there is a continuous linear functional φ on X that takes the value zero at y, but not at Ty. Let $S = y \otimes \varphi$. Then $TS = Ty \otimes \varphi$ and $ST = y \otimes T^*\varphi$, so $TSy = \varphi(y)Ty = 0$, but $STy = \varphi(Ty)y \neq 0$. Thus $S \notin \mathrm{com(T)}$.

(c) Burnside's Theorem [6, 1905], *If $\dim(X) < \infty$ then the only transitive subalgebra of $\mathscr{L}(X)$ is $\mathscr{L}(X)$ itself.*

Burnside's theorem is usually stated for algebras of square matrices over the an algebraically closed field. In our case it's the field of complex numbers, and the matrices are operators on $\mathbb{C}^N$, which is linearly homeomorphic to any complex Banach space of dimension N. See Appendix B below for a short proof of Burnside's Theorem.

An infinite dimensional Burnside Theorem? In its original form Burnside's Theorem cannot be generalized to infinite dimensional Banach spaces. From (a) above we see that the algebra $\mathscr{F}(X)$ of finite rank operators is transitive, yet when $\dim(X) = \infty$ this algebra is not equal to $\mathscr{L}(X)$. One might object that $\mathscr{F}(X)$ is not closed in the operator-norm topology of $\mathscr{L}(X)$. So close it up: the resulting algebra is still not transitive, but is contained in the algebra $\mathscr{K}(X)$ of compact operators on X, which is not equal to $\mathscr{L}(X)$.

On the other hand, if we trade in the operator-norm topology of $\mathscr{L}(X)$ for the *strong operator topology,* i.e. the topology of pointwise convergence on X (a.k.a the restriction to $\mathscr{L}(X)$ of the product topology on X^X), then $\mathscr{F}(X)$ *is* dense in $\mathscr{L}(X)$.

This is most easily seen if X is a separable Hilbert space. For this case let $(e_n)_1^\infty$ be an orthonormal basis, and for N a positive integer let P_N be the orthogonal projection of X onto the linear span of $\{e_1, e_2, \ldots, e_N\}$. That is:

$$P_N x = \sum_{n=1}^{N} \langle x, e_n \rangle e_n \qquad (x \in X),$$

where $\langle \cdot, \cdot \rangle$ is the inner product in the Hilbert space X. In the norm topology of X we have for each $x \in X$:

$$\lim_N P_N x = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n = x\,,$$

i.e. the sequence (P_N) of finite rank operators converges pointwise to the identity operator on X. From this it follows that for each $T \in \mathscr{L}(X)$ the sequence (TP_N) of finite rank operators converges pointwise, i.e. in the strong operator topology, to T. Thus $\mathscr{F}(X)$ is dense in X.

For general Banach spaces the proof needs more work. I leave it as an interesting exercise (Hint: replace the orthonormal basis above by a—necessarily uncountable—Hamel basis, and use the Hahn-Banach theorem to patch up the mostly discontinuous coordinate functionals of this basis to get a net of finite rank operators that converges pointwise to the identity operator.)

The above discussion suggests that it might be reasonable to try for a version of Burnside's theorem where the algebras in question are *unital* (i.e. contain the identity operator) and closed in the strong operator topology. The *commutant* of any Banach space operator is a prime example of such an algebra.

The question of whether or not the conclusion of Burnside's theorem holds for such algebras on a Banach space X is what's called the *Transitive Algebra Problem* for X. Here's a complete statement:

The Transitive Algebra Problem for a Banach Space X. *Suppose X is a complex Banach space and $\mathscr{A}$ is a unital subalgebra of $\mathscr{L}(X)$ that is closed in the strong operator topology. If $\mathscr{A} \neq \mathscr{L}(X)$, does $\mathscr{A}$ have a nontrivial invariant subspace?*

Let's call a Banach space for which the answer to the above question is "Yes" a "Burnside Space."

If it's known that some operator on X has no nontrivial invariant subspace (e.g. Read's operator R on the sequence space $X = \ell^1$), then the same is true of the strong closure of the algebra generated by that operator (also a unital algebra). Thus X is not a Burnside space.

On the other hand, if X *is* a Burnside space, then since the commutant of a non-scalar operator can't be the whole space, that algebra must have an invariant subspace, i.e. *every nonscalar operator on a Burnside space has a hyperinvariant subspace.*

The "Transitive Algebra Problem" for Hilbert spaces—*Is every Hilbert space a Burnside space?*—is still open, as is the corresponding problem for reflexive Banach spaces.

Is there *any* infinite dimensional Burnside space? There's a very strange Banach space X constructed just recently by Argyros and Haydon [1] that might be a candidate. These authors prove that every operator on their space has the form "compact plus scalar multiple of the identity," and so by Lomonosov's Theorem must have a hyperinvariant subspace. This gives lots of nontransitive algebras, but it doesn't prove that *every* proper subalgebra of $\mathscr{L}(X)$ is transitive.

5. Notes

The standard reference, circa 1970, on the Invariant Subspace Problem for Hilbert Space is [16]. In this 2003 Dover edition there is a supplementary chapter surveying some of the many advances made up to that date. More recent research on the problem is set out in the 2011 book [7] of Chalendar and Partington.

The material on Lomonosov's Theorem presented here was taken from [15], where you can find further results based on those techniques.

The Transitive Algebra Problem was introduced in Arveson's 1967 paper [3]. Although the Invariant Subspace Problem has meaning only for separable Banach spaces, the same cannot be said for the Transitive Algebra Problem. As we pointed out earlier, even the Hyperinvariant Subspace Problem seems to be be "nonseparably nontrivial."

A. Partitions of unity

Theorem. *Suppose C is a compact metric space and $\mathscr{U} = \{U_1, U_2, \ldots, U_N\}$ is a finite open cover of C. Then there exists a* partition of unity on C subordinate to $\mathscr{U}$*, i.e. a family of continuous real-valued functions $\{p_1, p_2, \ldots, p_N\}$ each of which takes C into the closed unit interval $[0, 1]$, and for which*

- $p_j \equiv 0$ *on the complement of U_j $(1 \leq j \leq N)$, and*
- $\sum_{j=1}^{N} p_j \equiv 1$ *on C.*

Proof. Let d denote the metric on C. Define q_j on C by

$$q_j(x) = d(x, C \backslash U_j) := inf\{d(x, y) : y \notin U_j\} \qquad (x \in C),$$

where because of compactness, the "inf" is actually a "min".

For each j the function q_j is, by its definition positive on U_j and zero off U_j; one checks easily that q_j, for each j, is continuous on C. (*Picture:* If C as a subset of the real line and the U_j's are open intervals, then the graph of q_j looks like a tent over U_j, staked down at the endpoints). The required functions p_j are then defined by

$$p_j(x) = \frac{q_j(x)}{\sum_{k=1}^{N} q_k(x)} \qquad (x \in C),$$

where the denominator is non-vanishing thanks to the fact that the U_j's cover C and q_j vanishes nowhere on U_j. □

B. A proof of Burnside's Theorem

Burnside's Theorem says that if V is a finite dimensional vector space over an algebraically closed field, and $\mathscr{A}$ is a subalgebra of linear transformations of V for which the only invariant subspaces are the zero-subspace and the whole space, then $\mathscr{A}$ is the algebra of *all* linear transformations on V.

Let's use $\mathscr{L}(V)$ to denote the space of all linear transformations on the finite dimensional vector space V, and call a subalgebra of $\mathscr{L}(V)$ *irreducible* when it has no invariant subspaces other than $\{0\}$ and V. Thus Burnside's theorem asserts that the only irreducible subalgebra of $\mathscr{L}(V)$ is $\mathscr{L}(V)$ itself.

For Banach spaces the idea of irreducibility has already surfaced, without being named, in the work on Lomonosov's theorems. There we dealt with algebras of continuous linear transformations that were assumed to have no invariant *closed* subspaces other than the zero-subspace and the whole space. As we've pointed out earlier, in the finite dimensional setting all linear transformations are continuous and all linear subspaces are closed.

For Banach spaces we showed that if an algebra of operators $\mathscr{A}$ has no nontrivial invariant subspace, then for each nonzero vector x in the space the linear subspace $\mathscr{A}x$ is dense in X (the CLAIM near the bottom of page 5). The same argument shows that:

> *If V is a finite dimensional complex vector space and $\mathscr{A}$ an irreducible subalgebra of $\mathscr{L}(X)$, then $\mathscr{A}v = V$ for each nonzero vector $v \in V$.*

To matters as familiar as possible, let's prove Burnside's Theorem for the special case where the scalar field is $\mathbb{C}$. Thus our vector space V is linearly homeomorphic with $\mathbb{C}^N$, where $N = \dim V$, and we'll be able to avail ourselves of the inner product $\langle \cdot, \cdot \rangle$ that $\mathbb{C}^N$ induces on V.

So suppose $\mathscr{A}$ is an irreducible algebra of linear transformations of V. We wish to show that $\mathscr{A} = \mathscr{L}(V)$. The key is to show that $\mathscr{A}$ contains a rank-one operator.

Suppose we've established this. Thus there exist nonzero vectors v and w in V such that the rank-one operator S defined by

$$Sx = \langle x, w \rangle v \qquad (x \in V)$$

lies in $\mathscr{A}$. It's usual to write $S = v \otimes w$, whereupon it's easy to check that for any $T \in \mathscr{L}(V)$:

$$T(v \otimes w) = (Tv) \otimes w \qquad \text{and} \qquad (v \otimes w)T = v \otimes T^* w \tag{B.1}$$

From the first of these equations we see that $Tv \otimes w \in \mathscr{A}$ for every $T \in \mathscr{A}$, and since $\{Tv : T \in \mathscr{L}(V)\} = V$ (irreducibility of $\mathscr{A}$) we conclude that $x \otimes w \in \mathscr{A}$ for every $x \in V$.

Now the fact that $\mathscr{A}$ has no nontrivial invariant subspace implies that the same is true of the algebra $\mathscr{A}^*$, the collection of (Hilbert space) adjoints of the operators in $\mathscr{A}$. Indeed, if M *were* a nontrivial invariant subspace for $\mathscr{A}^*$, then its orthogonal complement $M^\perp = \{x \in V : \langle x, m \rangle = 0 \; \forall\, m \in M\}$ would be a nontrivial invariant subspace for $\mathscr{A}$, contradicting our assumption that no such subspace exists.

Thus an argument similar to the one above, but this time employing the second equation of (B.1), shows that $x \otimes y \in \mathscr{A}$ for every pair x, y of vectors in V, so that $\mathscr{A}$ contains all the rank-one operators on V, hence all the finite rank operators, i.e. all the operators on V.

It remains to prove that $\mathscr{A}$ contains a rank-one operator. We'll do this by induction on the dimension of V.

For $\dim V = 1$ the statement is trivially true, so suppose it holds for all complex vector spaces of dimension between 1 and $N-1$, where $N > 1$. Let $\dim V = N$ and suppose $\mathscr{A}$ is an irreducible subalgebra of $\mathscr{L}(V)$.

Claim: $\mathscr{A}\backslash\{0\}$ contains a non-invertible transformation.

Proof of Claim. Fix $T \in \mathscr{A}$, not a scalar multiple of the identity. We know there is such a T since $\dim V > 1$ and $\mathscr{A}$ is irreducible. If T is not invertible, set $S = T$. If T *is* invertible then it has an eigenvalue $\lambda \neq 0$ (here we use, for the first and only time, the fact that our scalar field $\mathbb{C}$, is algebraically closed). If $\mathscr{A}$ contains the identity, then $S = T - \lambda I$ is a noninvertible operator in $\mathscr{A}$, and it's not the zero-operator by our choice of T. If $\mathscr{A}$ does *not* contain the identity transformation, then $S = T^2 - \lambda T$ belongs to $\mathscr{A}$, is non-invertible and (because T is invertible) is not the zero-operator.

Now that we know $\mathscr{A}\backslash\{0\}$ contains a noninvertible operator $S \neq 0$, let's focus on the subspace $S\mathscr{A} = \{SA : A \in \mathscr{A}\}$ of $\mathscr{L}(V)$. It's an algebra, and it takes the subspace $S\mathbb{C}^N = \operatorname{ran} S$ into itself. Thus the restriction of $S\mathscr{A}$ to $\operatorname{ran} S$ is an algebra of operators on a complex vector space which, because S is noninvertible, has dimension $< N$. By our induction hypothesis this algebra contains a rank-one operator, i.e. there exists $T \in \mathscr{L}(V)$ such that the restriction to $\operatorname{ran} S$ of ST has rank one. Stated differently: STS is an operator of rank one on V. Since it clearly belongs to $\mathscr{A}$, we're done. □

The proof given above is taken from [14]; it can be easily modified to work for vector spaces over any algebraically closed field. Lomonosov has proved a result for Banach spaces [13] which, when applied to finite dimensional ones, recovers Burnside's Theorem. "Lomonosov's Burnside Theorem" improves some of his earlier results, but doesn't settle any of the open versions of the Transitive Algebra Problem or the Invariant Subspace Problem.

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