$SU(2)$ and $SO(3)$

A Taste of Lie Groups

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Abstract
A Lie Group is a differentiable manifold that is also a group with smooth group operations. We start with a quick taste of Lie Groups and their Lie Algebras. Then we introduce our main examples: $SU(2)$ and $SO(3)$. $SU(2)$ and $SO(3)$ have isomorphic Lie Algebras and so are locally isomorphic. We can make this isomorphism concrete by constructing a homomorphism from $SU(2)$ to $SO(3)$ using the Adjoint Representation. This mapping forms the basis of the Hopf Fibration. Along the way we introduce a variety of concepts that are applicable to any Lie Group.

In this brief presentation we will not include complex proofs but instead try to get a taste of the modern language of Lie Groups. The talk should be understandable to an advanced undergraduate.

Outline

Manifolds

Review of Coordinates, Tangent Vectors and differentials

Manifolds

An $n-$dimensional manifold M is a topological space that is locally homeomorphic to $\mathbb{R}^n$.
For this talk we take $M \subset \mathbb{R}^m$ (example: $S^2 \subset \mathbb{R}^3$ has $n = 2, m = 3$).

A chart on M is an open set $U \subset M$ and a homeomorphism

$$\varphi : U \to \mathbb{R}^n. \tag{1}$$

This coordinate neighborhood has coordinates,

$$x^k : U \to \mathbb{R}, \text{ given by } x^k(x) = r^k(\varphi(x)) = \varphi^k(x). \tag{2}$$

The formal definition is

Definition

A C^∞ differentiable manifold is a Hausdorff, second-countable Topological space with a (maximal) C^∞ atlas.

Tangents to Curves

We will associate tangent vectors in M with curves.
Let (U, φ) be a coordinate neighborhood with coordinates $\{x^k\}$.

If $\gamma : (-\delta, \delta) \to M$ is a curve with $\gamma(0) = x$,
the derivative is a vector with coordinates,

$$X^k(\gamma(t)) = \left. \frac{d(x^k \circ \gamma)}{dt} \right|_t . \tag{3}$$

where $x^k : U \subset M \to \mathbb{R}$.

A smooth vector field $X : \mathbb{R}^n \to \mathbb{R}^n$ determines a unique curve γ in $(-\delta, \delta)$,

$$\frac{d\gamma^k}{dt} = X^k(\gamma(t)). \tag{4}$$

The curve γ is called the integral curve of X.
We're being loose on the details.

Tangent Vectors (2/2)

For any definition we have the following important theorem.

Theorem

Let M be a differentiable manifold of dimension n and let $x \in M$. The tangent space at x, denoted $T_x(M)$, is an n dimensional real vector space.

Proof.

See [War70] for a proof when tangents are derivations.
See [Spi99] for proofs of the other definitions. □

Differential of Function

Given manifolds M, N and a smooth map $F : M \to N$.

Define a mapping $dF : T_x(M) \to T_{F(x)}(N)$ by

$$dF\left(\frac{d\gamma}{dt}\right) = \frac{d(F \circ \gamma)}{dt}. \qquad (5)$$

In coordinates this is just the Jacobian of F.

The Bracket (Lie Derivative) (1 of 2)

Given two vector fields X, Y, defined on a neighborhood of $x_0 \in M$.
We construct a Lie Derivative, denoted by $[X, Y]$.

Denote the integral curves of X by α and Y by β.
The system of curves α and β form a weave.
If the weave is consistent then we can construct a smooth surface.

The Bracket (Lie Derivative) (1 of 2)

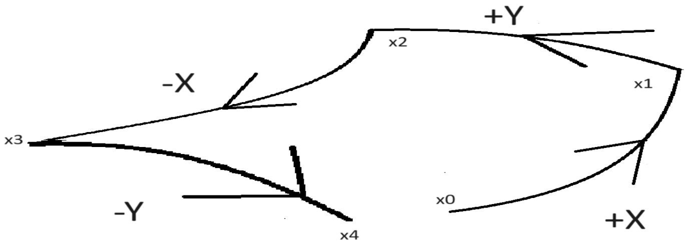

Figure: Lie Derivative at x_0

1. Move from x_0 along α (along X) to $\alpha(s) = x_1 \in M$,
2. Move from x_1 along β (along Y) to $\beta(t) = x_2 \in M$,
3. Move from x_2 along reversed α (along $-X$) to $\alpha(-s) = x_3 \in M$,
4. Move from x_3 along reversed β (along $-Y$) to $\beta(-t) = x_4 \in M$,
5. The difference $x_4 - x_0$ as $s, t \to 0$ is $[X, Y]$.

Frobenius Theorem

The Lie Derivative is a condition for integrability.

Let X and Y be vector fields on M and $x_0 \in M$.
If $[X, Y] = 0$ then there is a $2-$dimensional manifold $S \subset M$ with $T_y(S) = \text{Span}\{X(y), Y(y)\}$, for all $y \in S$.

S is only guaranteed to exist on a neighborhood of x_0.

Frobenius Theorem is a very nice generalization of this.

Lie Groups

Basic facts for Lie Groups, especially $GL(n, V)$.

Lie Groups

Recall our definition of Topological Groups.

Definition

A topological group is a group where the group action is continuous. This means both $(x, y) \to xy$ and $x \to x^{-1}$ are continuous.

For a Lie Group the group operation is smooth.

Definition

A Lie Group is a second countable, Hausdorff topological group, that is a differentiable manifold and the group operations are smooth.

Example: $GL(m, \mathbb{R})$

Let F be either $\mathbb{R}$ or $\mathbb{C}$.
The set of invertible $m \times m$ matrices on F^m form a group.
Denoted this group by $GL(m, F)$.

$GL(m, \mathbb{R})$ is an open subset of $\mathbb{R}^{m^2}$ and is a real manifold.
$GL(m, \mathbb{C})$ is an open subset of $\mathbb{R}^{2m^2}$ and is also a real manifold.

Matrix multiplication is algebraic and so smooth.
This makes $GL(m, F)$ into a Lie Group.

For simplicity, in this talk, we will only talk about Lie Groups that are subgroups of $GL(m, F)$.

For the general case see [Kna96] or [Bum04].

General Properties

Let G be a Lie Group.

The mapping $L_g : G \to G$ given by $L_g(x) = gx$ is a diffeomorphism.

For any $g \in G$ there is a diffeomorphism $L_g : G \to G$ with $L_g(e) = g$. This means that there is an isomorphism

$$dL_g : T_e(G) \to T_g(G). \tag{6}$$

Left Invariant Vector Fields

Let $X_e \in T_e(G)$ and define a global vector field $X(g)$ by

$$X(g) = dL_g\left(X_e\right). \tag{7}$$

For sub-groups of $GL(n, F)$ we let $X_e = \left.\frac{d\gamma}{dt}\right|_{t=0}$,
where γ is a curve with $\gamma(0) = e$.

$$X(g) = dL_g\left(\frac{d\gamma}{dt}\right) = \frac{d(L_g \circ \gamma)}{dt} = \frac{d(g\gamma)}{dt} \tag{8}$$

The derivative of the linear map g is itself, so by the chain rule,

$$X(g) = g\frac{d\gamma}{dt} = gX_e \in T_g(G). \tag{9}$$

This is a linear in the components of g, and so smooth.

One can show X_g is smooth for any G (see [Spi99]).

Lie Algebra (1/2)

Definition

A Lie Algebra is a vector space with a bilinear, anti-symmetric bracket,

$$X, Y \in \mathfrak{g} \to [X, Y] \in \mathfrak{g}.$$

that satisfies the Jacobi identity

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0. \tag{10}$$

Lie Algebra (2/2)

For every $X_e \in T_e(G)$ we have a left-invariant vector field $X_g = dL_g(X_e)$.

We must show the bracket is consistent,

$$[dL_g X_e, dL_g Y_e] = dL_g\left([X_e, Y_e]\right). \quad (11)$$

This says the bracket at $e \in G$ determines the bracket on all of G.

We prove this for subgroups of $GL(n, F)$ at the end of the slides.

Definition

Let G be a Lie Group. The Lie Algebra of G is the tangent space at e,

$$\mathfrak{g} = T_e(G).$$

For $X_e, Y_e \in T_e(G)$ we have the bracket $[dL_g(X_e), dL_g(Y_e)]\big|_e$.

Example: $GL(n, \mathbb{R})$ and $\mathfrak{gl}(n, \mathbb{R})$

We know $GL(n, \mathbb{R})$ is a Lie Group.

What is the Lie Algebra?

Define $\gamma : (-\delta, \delta) \to GL(n, \mathbb{R})$ by

$$\gamma(t) = I + tX_{ij} \in GL(n, \mathbb{R}). \quad (12)$$

Compute the derivative at $\gamma(0) = I$.

$$\left.\frac{d\gamma}{dt}\right|_{t=0} = X_{ij} \in \mathfrak{gl}(n, \mathbb{R}). \quad (13)$$

We define $\mathfrak{gl}(n, F)$ to be $n \times n$ matrices over field F.
One can show that the bracket is just the commutator,

$$[X, Y] = XY - YX, \text{ for } X, Y \in \mathfrak{gl}(n, R). \quad (14)$$

Integral curves of Left-invariant vector fields

The solution to the ODE

$$\frac{d\gamma}{dt} = X_g(\gamma(t)) = \gamma(t)X_e, \tag{15}$$

Is

$$\gamma(t) = \exp\left(tX_e\right). \tag{16}$$

These are the integral curves for the left invariant vector fields.

The exponential maps the Lie Algebra to the Lie Group,

$$\exp : T_e(G) = \mathfrak{g} \to G. \tag{17}$$

Sub-Algebras of $\mathfrak{g}$

Let G be a Lie Group with Lie Algebra $\mathfrak{g}$.

If $X, Y \in \mathfrak{g}$ have $[X, Y] = 0$ then
there is a $2-$dimensional abelian subgroup $H < G$.

We can integrate $dL_g(X)$ and $dL_g(Y)$ to get a manifold.
Show that multiplication is smooth.

In general we have Lie's 3rd Theorem

Theorem

Let G be a Lie Group with Lie algebra $\mathfrak{g}$. If $\mathfrak{h} < \mathfrak{g}$ is a sub-algebra, then there is a unique connected subgroup H that has a Lie Algebra $\mathfrak{h}$.

$SU(2), SO(3)$ and the Hopf fibration

Definitions

The sphere

$$S^n = \left\{ x \in \mathbb{R}^{n+1} \middle| \, \|x\| = 1 \right\}. \tag{18}$$

The set S^n is a differentiable manifold of dimension n.

The closed ball

$$B^n(R) = \{ x \in \mathbb{R}^n | \, \|x\| \leq R \}. \tag{19}$$

The interior of $B^n(R)$ is an open submanifold of $\mathbb{R}^n$.

The boundary of $B^n(1)$ is S^{n-1}.

Useful Facts

Proposition

For T an $n \times n$ matrix we have

$$\exp\left(\mathrm{Tr}(T)\right) = \det\left(\exp(T)\right) \tag{20}$$

Proof.

Find an invertible S where STS^{-1} is upper triangular. $\square$

Since $\exp : \mathfrak{gl}(n, F) \to GL(n, F)$,
if $g = \det\left(\exp(X)\right) = 1$ then

$$\exp\left(\mathrm{Tr}(X)\right) = 1 \Rightarrow \mathrm{Tr}(X) = 0. \tag{21}$$

If the group is "Special" (e.g. determinant $= 1$)
then trace of a Lie Algebra element is 0.

$SO(n)$

Definition

The Orthogonal group $O(n)$ is the set of $n \times n$ real matrices that preserve the inner product on $\mathbb{R}^n$,

$$(Rx, Ry) = (x, y) \tag{22}$$

$SO(n) < O(n)$ is the subgroup that satisfies $\det(R) = 1$.

This is equivalent to the condition

$$R^T R = I \Rightarrow R^T = R^{-1}. \tag{23}$$

From this we have $\det(R) = \pm 1$.
There is an identity component with $\det(R) = +1$
and a second diffeomorphic component with $\det(R) = -1$ (not a subgroup!).

Lie Algebra of $SO(n)$

Proposition

The Lie Algebra of $SO(n)$ is the set of $n \times n$ real matrices X with

$$X + X^T = 0 \text{ so } \mathrm{Tr}(X) = 0. \tag{24}$$

Denote the Lie Algebra by $\mathfrak{so}(n)$. Note, $\mathrm{Tr}(X) = 0$ *holds on* $\mathfrak{o}(n) = \mathfrak{so}(n)$.

Let $R(t)$ be a curve $R : (-\delta, \delta) \to SO(n)$ with $R(0) = I$.

Take the derivative of the condition $R(t)^T R(t) = I$, and set $t = 0$,

$$0 = \frac{dR^T}{dt} R(t)\Big|_{t=0} + R(t)^T \frac{dR}{dt}\Big|_{t=0} \tag{25}$$

$$= \left(\frac{dR}{dt}\Big|_{t=0} \right)^T R(0) + R(0)^T X \tag{26}$$

$$\Rightarrow X^T + X = 0. \tag{27}$$

Here $X = \frac{dR}{dt}\Big|_{t=0} \in \mathfrak{so}(n)$.

A Basis for the Lie Algebra $\mathfrak{so}(3)$

The Lie Algebra $\mathfrak{so}(3)$ is 3×3 anti-symmetric traceless matrices.
We write down a basis for the Lie Algebra,

$$B_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, B_y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, B_z = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (28)$$

These satisfy

$$[B_x, B_y] = B_z \quad [B_z, B_x] = B_y \quad [B_y, B_z] = B_x. \quad (29)$$

and $\exp : \mathfrak{so}(3) \to SO(3)$ maps X to rotation around x-axis.

$$\exp(tB_x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(t) & -\sin(t) \\ 0 & \sin(t) & \cos(t) \end{bmatrix}, \quad (30)$$

and similarly for B_y and B_z.

$SU(2)$

Let's look at a different Lie Group.

Definition

The unitary group $U(n)$ is the set of $n \times n$ complex matrices that preserve the Hermitian inner product,

$$\langle Ux, Uy \rangle = \langle x, y \rangle \tag{31}$$

$SU(n) < U(n)$ is the subgroup that satisfies $\det(U) = 1$.

This is equivalent to the condition

$$U^\dagger U = I \Rightarrow U^\dagger = U^{-1}. \tag{32}$$

Proposition

The Lie Algebra of $SU(n)$ is the set of $n \times n$ complex matrices with

$$X + X^\dagger = 0 \text{ and } \mathrm{Tr}(X) = 0. \tag{33}$$

Denote this Lie Algebra by $\mathfrak{su}(n)$.

Let $U(t)$ be a curve $U : (-\delta, \delta) \to SU(n)$ with $U(0) = I_n$.

Take the derivative of the condition $U(t)^\dagger U(t) = I_n$ at $t = 0$,

$$0 = \frac{dU^\dagger}{dt}(0)U(0) + U(0)^\dagger \frac{dU}{dt}(0) = X^\dagger + X, \tag{34}$$

where $X = \frac{dU}{dt}\big|_{t=0} \in T_e(SU(n)) = \mathfrak{su}(n)$.

A Basis for the Lie Algebra $\mathfrak{su}(2)$

Write a basis for the vector space of all 2×2 complex matrices X, with $X + X^T = 0$ and $\mathrm{Tr}(X) = 0$

$$D_x = \frac{1}{2}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad D_y = \frac{1}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad D_z = \frac{1}{2}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \tag{35}$$

One can verify they satisfy the following Lie Algebra Bracket conditions,

$$[D_x, D_y] = D_z, \quad [D_y, D_z] = D_x, \quad [D_z, D_x] = D_y. \tag{36}$$

This is the isomorphic to the Lie Algebra $\mathfrak{so}(3)$. Just map

$$aB_x + bB_y + cB_z \to aD_x + bD_y + cD_z, \tag{37}$$

and show this map preserves the bracket $[,]$.
The factor $\frac{1}{2}$ is required to make commutation relations work.

$SO(3)$ and $SU(2)$ the same (at least locally)

The Lie Algebras $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$ are isomorphic!

The groups $SO(3)$ and $SU(2)$ are the isomorphic
in a neighborhood of the identity,
but they are NOT the same group!

The Adjoint Representation

Conjugation by $g \in G$ is defined by

$$C_g : G \to G \quad \text{defined by} \quad C_g(x) = gxg^{-1}, \tag{38}$$

Since $C_g(e) = e$, we have $dC_g : T_e(G) \to T_e(G)$. Define

$$\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g} \text{ by } \mathrm{Ad}_g(X) = dC_g(X) \tag{39}$$

What is Ad_g action when $G < GL(n, F)$?
Let $X = \frac{d\gamma}{dt}$ where $\gamma(0) = I$.

$$\mathrm{Ad}_g(X) = dC_g\left(\frac{d\gamma}{dt}\right) = \frac{d(C_g \circ \gamma)}{dt} = \frac{d(g\gamma g^{-1})}{dt} \tag{40}$$

$$= g\frac{d\gamma}{dt}g^{-1} = gXg^{-1} \tag{41}$$

The operation gXg^{-1} is just matrix multiplication.

Inner Product on $\mathfrak{su}(2)$

Using the "Killing form" we get an inner product

$$(V, W) = -2\,\mathrm{Tr}(VW) \text{ for } V, W \in \mathfrak{su}(2). \tag{42}$$

We can find an orthonormal basis, where, for example,

$$(D_x, D_x) = 1 \quad (D_x, D_y) = (D_x, D_z) = 0. \tag{43}$$

where our basis is

$$D_x = \frac{1}{2}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad D_y = \frac{1}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad D_z = \frac{1}{2}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \tag{44}$$

Adjoint Representation of $SU(2)$ on $\mathfrak{su}(2)$ (1 of 2)

Recall that the adjoint map is $\mathrm{Ad}_U : \mathfrak{su}(2) \to \mathfrak{su}(2)$, where

$$\mathrm{Ad}_U(X) = UXU^{-1}, \text{ for } U \in SU(2), X \in \mathfrak{su}(2). \tag{45}$$

Proposition

Ad_U *is an orthogonal linear mapping on the vector space* $\mathfrak{su}(2)$.

$$\begin{aligned} (\mathrm{Ad}_U(X), \mathrm{Ad}_U(Y)) &= -2\,\mathrm{Tr}(UXU^{-1}UYU^{-1}) && (46) \\ &= -2\,\mathrm{Tr}(XY) = (X, Y)\,. && (47) \end{aligned}$$

So $\mathrm{Ad}_U \in SO(3)$.
This is a representation $\rho : SU(2) \to SO(3)$.

Adjoint Representation of $SU(2)$ on $\mathfrak{su}(2)$ (1 of 2)

Notice that

$$\mathrm{Ad}_{-U}(X) = (-U)X(-U)^T = UXU^T = \mathrm{Ad}_U(X), \tag{48}$$

Using this you can show $\rho : SU(2) \to SO(3)$ is $2:1$.

You can show the mapping is onto.

Generic form for $SU(2)$

Proposition

If $U \in SU(2)$ then U has the form

$$U = \begin{bmatrix} z & w \\ -w^* & z^* \end{bmatrix} \tag{49}$$

where $z, w \in \mathbb{C}$ satisfy $|z|^2 + |w|^2 = 1$.

Proof.

The with the generic form

$$U = \begin{bmatrix} z_1 & w_1 \\ w_2 & z_2 \end{bmatrix} \tag{50}$$

Write out the conditions $UU^T = U^T U = I_2$.
From the diagonals $|z_1| = |z_2|$, $|w_1| = |w_2|$ and $|z_1|^2 + |w_1|^2 = 1$.
To finish use conditions of the off diagonal elements and $\det(U) = 1$. □

Topology of $SU(2)$

We saw that a general element $U \in SU(2)$ has

$$U = \begin{bmatrix} z & w \\ -w^* & z^* \end{bmatrix} \tag{51}$$

where $z, w \in \mathbb{C}$ satisfy $|z|^2 + |w|^2 = 1$.

Think of this as a mapping

$$(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \to \frac{1}{2} \begin{bmatrix} x_1 + ix_2 & -x_3 + ix_4 \\ x_3 + ix_4 & x_1 - ix_2 \end{bmatrix} \tag{52}$$

with $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$.

This is a diffeomorphism.
$SU(2)$ is homeomorphic to S^3 (it's actually diffeomorphic).

Topology of $SO(3)$

A rotation is determined by an axis $\hat{n}$ and an angle $\theta \in [-\pi, \pi]$.
The axis is a unit vector in $\mathbb{R}^3$, so is a point on S^2

Use the right hand rule.
Note that the rotation $(\hat{n}, \theta)$ is the same as $(-\hat{n}, -\theta)$.
To capture all rotations take $\hat{n} \in S^2$ and $\theta \in [0, \pi]$.

But $(\hat{n}, \pi)$ is the same rotation as $(-\hat{n}, -\pi) = (-\hat{n}, \pi)$.

Map the rotation $(\hat{n}, \theta)$ to the point $\frac{\theta}{\pi}(\hat{n}) \in B^3(1)$.
This is a homeomorphism if we identify antipodal points in $B^3(1)$.

The topology of $SO(3)$ is same as
the unit ball with antipodal points identified.

Topology of $\mathrm{Ad}_U : SU(2) \to SO(3)$

Antipodal points on S^3 map to the same element of $SO(3)$.

A cap on the north pole of S^3 is homeomorphic to a ball $B^3(r)$.

If you pull the cap down to the equator of S^3,
you have to identify boundary points (antipodal on S^3).
This is homeomorphic to $B^3(1)$ with antipodal points identified.

The Hopf Fibration

$$D_z = \frac{1}{2}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \tag{53}$$

Map the element $U \in SU(2) \cong S^3$ to the point

$$\mathrm{Ad}_U\left(D_z\right) = U D_z U^{-1} \tag{54}$$

We have

$$\mathrm{Ad}_U\left(D_z\right) = aD_x + bD_y + cD_z, \text{ with } a^2 + b^2 + c^2 = 1. \tag{55}$$

So $\mathrm{Ad}_U\left(D_z\right) \in S^2$.

We have a map $H : S^3 \to S^2$ given by

$$H(U) = \mathrm{Ad}_U\left(D_z\right). \tag{56}$$

This is the Hopf Fibration.

Computational Details

Some extra computations and details.

Flow Lines for Left Invariant Vector Fields

Let G be a subgroup of $GL(n, F)$ and $\mathfrak{g}$ its Lie Algebra.
Let $X_e \in T_e(G)$ and form the left invariant vector field

$$X_g = dL_g(X_e) = gX_e. \tag{57}$$

The "flow lines" of X_g are functions $\Psi_t(x)$ the satisfy

$$\frac{\partial \Psi}{\partial t} = X_g(\Psi_t(x)) = \Psi_t(x)X_e, \text{ where } t \in (-\delta, \delta), x \in G. \tag{58}$$

The solution is similar to integral curves through I,

$$\Psi_t(x) = x \exp(tX_e). \tag{59}$$

Bracket of Left Invariant Vector Fields (1/4)

Proposition

If $X_e, Y_e \in T_e(G)$ *then*

$$[dL_g X_e, dL_g Y_e] = dL_g\left([X_e, Y_e]\right). \tag{60}$$

Proof.

We sketch a proof for G a subgroup of $GL(n, F)$.
The flow for $dL_g(X_e)$ is $\Psi_t(x) = x\exp(tX_e)$ and
the flow for $dL_g(Y_e)$ is $\Psi_s(x) = x\exp(sY_e)$.

$$dL_g X_e\left(dL_g Y_e(f(x))\right) = \lim_{s\to 0} dL_g X_e\left(\frac{1}{s}(f(\Psi_s x) - f(x))\right) \tag{61}$$

$$= \lim_{s,t\to 0}\frac{1}{ts}\left(f(\Psi_s\Phi_t x) - f(\Psi_s x) - f(\Psi_t x) + f(x)\right) \tag{62}$$

□

This means

$$[dL_g X_e, dL_g Y_e](f)(x) = \lim_{s,t\to 0} \frac{1}{ts}\left(f(\Psi_s(\Phi_t x)) - f(\Phi_t(\Psi_s x))\right) \tag{63}$$

But we know the flows.

$$= \lim_{s,t\to 0} \tfrac{1}{ts}\left(f(\Phi_t(x)\exp(sY_e)) - f(\Psi_s(x)\exp(tX_e))\right)) \tag{64}$$

$$= \lim_{s,t\to 0} \tfrac{1}{ts}\left(f(x\exp(tX_e)\exp(sY_e)) - f(x\exp(sY_e)\exp(tX_e))\right) \tag{65}$$

$$= \lim_{s,t\to 0} \tfrac{1}{ts}\left((f\circ L_x)(\exp(tX_e)\exp(sY_e)) - (f\circ L_x)(x\exp(sY_e)\exp(tX_e))\right) \tag{66}$$

At $x = I$ we have,

$$[X_e, Y_e](f) = \lim_{s,t\to 0} \frac{1}{ts}\left(f(\exp(tX_e)\exp(sY_e)) - f(\exp(sY_e)\exp(tX_e))\right). \tag{67}$$

Bracket of Left Invariant Vector Fields (3/4)

Let $X_e = \frac{d\gamma}{dt} \in T_e(G)$ and $dL_x : T_e(G) \to T_x(G)$.

For any $f \in G(x)$ we have

$$dL_x(X_e)(f) = dL_x\left(\frac{d\gamma}{dt}\right)(f) = \frac{d(L_x \circ \gamma)}{dt}(f) \quad (68)$$

$$= \frac{d(f \circ L_x \circ \gamma)}{dt} = \frac{d\gamma}{dt}(f \circ L_x) \quad (69)$$

$$[dL_g X_e, dL_g Y_e](f) = [X_e, Y_e](L_g \circ f) = dL_g([X_e, Y_e])(f) \quad (70)$$

Since this is true for all f we have,

$$[dL_g X_e, dL_g Y_e] = dL_g([X_e, Y_e]). \quad (71)$$

See [Bum04].

Bracket of Left Invariant Vector Fields (4/4)

So, if you think of $\mathfrak{g} = T_e(G)$ and you have $X_e, Y_e \in T_e(G)$ then

$$[dL_g X_e, dL_g Y_e] = dL_g\left([X_e, Y_e]\right). \qquad (72)$$

So that associated left invariant vector field of $[X_e, Y_e]$ is

$$[dL_g X_e, dL_g Y_e].$$

The Lie Algebra of Left invariant vector fields
is the same as $\mathfrak{g} = T_e(G)$.

References I

Daniel Bump, *Lie groups*, Springer-Verlag, 2004.

Anthony Knapp, *Lie groups: Beyond an introduction*, Birkhauser, 1996.

Michael David Spivak, *A comprehensive introduction to differential geometry, volume 1, third edition*, Publish or Perish, Inc., 1999.

Frank Warner, *Foundations of differentiable manifolds and lie groups*, Scott Foresman and Company, 1970.