Routh–Hurwitz stability criterion

Are the roots of a polynomial Stable?

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Abstract The Routh-Hurwitz stability criterion is a test applied to a polynomial to determine the number of roots located in the left half complex plane. This problem comes up often in stability questions of dynamical systems. Although the Routh-Hurwitz recipe is well known, the proof of its veracity is not. In this presentation we will sketch a proof of the Routh-Hurwitz criteria. The outline includes a construction of a Sturm sequence from the polynomial and uses methods from complex analysis, calculus, and algebra.

Problem Given a polynomial $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$.
When are the roots of $p(z)$ in the left half complex plane?

Rough outline:

- Direct manipulation of polynomials
- Complex analysis
- Calculus and Sturm sequences
- Some algebra (after all it is an algebra problem)

Put this all in a blender to get Routh–Hurwitz stability criterion.
One of the most used theorems in Control theory - but rarely proved.

See the excellent treatment in [Bar08].

> *Thus, the Sturm method solves problems of algebra by means of analysis. As a result, it is disliked by both algebraists and analysts. – Barkovsky*

Motivation: Stability

We start with a linear ODE system in $\mathbb{R}^n$,

$$\frac{dX}{dt} = MX, \tag{1}$$

where $X : \mathbb{R} \to \mathbb{R}^n$ and M is an $n \times n$ matrix.
The solution is,

$$X(t) = \exp(Mt)\, X_0,$$

If X_0 is an eigenvector of M with eigenvalue $\lambda \in \mathbb{C}$ then,

$$X(t) = \exp(\lambda t)\, X_0 = \exp(\Re(\lambda) t) \exp(i \Im(\lambda) t)\, X_0. \tag{2}$$

If $\Re(\lambda) < 0$ then the eigenvector is stable.

The system is stable if all the roots of the characteristic polynomial lie in the left complex plane.

Problem Statement

Given a polynomial

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0, \tag{3}$$

with $a_k \in \mathbb{R}$.

Do the roots of $p(z)$ lie in the left half complex plane?

Definition

A polynomial is called stable if all the roots have negative real parts.

What happens when the roots lie on the imaginary axis?
Understanding this detail is fruitful and interesting, but we don't have time and we will ignore this case.

What About Complex Polynomials?

Given a polynomial

$$p(z) = w_n z^n + w_{n-1} z^{n-1} + \cdots + w_1 z + w_0, \tag{4}$$

where $w_k \in \mathbb{C}$. Form the polynomial,

$$\bar{p}(z) = \bar{w}_n z^n + \bar{w}_{n-1} z^{n-1} + \cdots + \bar{w}_1 z + \bar{w}_0. \tag{5}$$

z_r is a root of $p(z)$ if and only if $\bar{z}_r$ is a root of $\bar{p}(z)$.

$$p(z)\bar{p}(z) \tag{6}$$

- Is a $2n$ degree polynomial with real coefficients.
- Has $2\times$ the number of 'stable' roots and $2\times$ the number of unstable roots.

Stability of Quadratics

$$z^2 + bz + c = 0 \quad \Rightarrow \quad z = \frac{-b \pm \sqrt{b^2 - 4c}}{2},$$

where $b, c \in \mathbb{R}$.

If the roots are stable then,

- If $c \leq 0$ then $\sqrt{b^2 - 4c} \geq |b|$. Not all roots are stable.
- If $c > 0$ then $\sqrt{b^2 - 4c} < |b|$ or imaginary. Either way we must have $b > 0$.

A second degree polynomial is stable iff all coefficients are positive (for simplicity we ignore the cases $c = 0$ or $b = 0$ which give roots on the imaginary axis).

Is stability equivalent to positive coefficients?

Theorem

If $p(z)$ is degree n real stable polynomial and has lead coefficient $a_n > 0$, then all the other coefficients a_k are positive.

Proof.

Let $-r, -\alpha + i\beta, -\alpha - i\beta$ be 3 stable roots of $p(z)$, where $r, \alpha, \beta \in \mathbb{R}$. and r, α are positive. All stable roots have one of these forms. The polynomial has a factor,

$$(z + r)(z + \alpha - i\beta)(z + \alpha + i\beta) = (z + r)(z^2 + 2\alpha z + \alpha^2 + \beta^2).$$

This cubic has positive coefficients. $p(z)$ is a product of terms similar to these (multiplied by a_n). If $\alpha, r > 0$ then none of the a_k are zero. □

If a_k is zero for some k then there are roots on the imaginary axis. If $p(z)$ is stable then coefficients are positive. Is the converse true?

Unstable polynomial with positive coefficients

Assume three roots, $-r, \alpha + i\beta, \alpha - i\beta$ where $\alpha, \beta, r \in \mathbb{R}$, where $r > 0$. The cubic with these roots has the form,

$$\begin{aligned}&(z+r)(z-\alpha-i\beta)(z-\alpha+i\beta)\\ &\qquad = z^3 + (r-2\alpha)z^2 + (\alpha^2+\beta^2-2\alpha r)z + r(\alpha^2+\beta^2)\end{aligned}$$

This has non-negative coefficients when

$$\begin{aligned} r \geq 2\alpha \qquad\qquad & \qquad \alpha^2+\beta^2 \geq 2\alpha r \geq 4\alpha^2 \\ \Rightarrow \left(\tfrac{1}{2}\right)^2 \geq \left(\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}\right)^2 & \quad \Rightarrow |arg(\alpha+i\beta)| > 60(degrees).\end{aligned}$$

Pick $0 < \alpha < \frac{1}{2}$ with $r = 1, \alpha^2+\beta^2 = 1$. The coefficients are positive:

$$r - 2\alpha > 0 \quad \alpha^2+\beta^2 - 2\alpha r > 1 - r = 0.$$

More generally, one can show $p(z)$ with positive coefficients has no roots in $|arg(z)| \leq \frac{\pi}{n}$.

We need a better way

Descartes, Hermite, Stodola and Routh had more to say on this subject. In this talk we are less concerned with real roots - just stable roots.

Historically: Sturm (1803 - 1855)
then Edward Routh (1831 – 1907)
then Adolf Hurwitz (1859 – 1919).

On to Complex Analysis ...
We move on to complex logarithms.

See [Gam01].

Some Complex Analysis

Complex logarithm is the inverse of e^z and should satisfy,

$$\ln(Re^{i\theta}) = \ln(R) + \ln(e^{i\theta}) = \ln(R) + i\theta.$$

We define,

$$\ln(z) = \ln(|z|) + i\arg(z). \tag{7}$$

The imaginary part of $d\ln(z)$ forms a closed form that is not exact so is sensitive to 'holes' in the domain.

Theorem

Let $D \subset \mathbb{C}$ be a bounded domain with a piece-wise smooth boundary ∂D. Let f be analytic on D except for a finite number of poles (Meromorphic) and f extends to analytic function on ∂D, such that $f(z) \neq 0$ for $z \in \partial D$. We have,

$$\frac{1}{2\pi i} \int_{\partial D} \frac{f'(z)}{f(z)} dz = N_0 - N_\infty, \tag{8}$$

where N_0 is the number of zeros and N_∞ is the number of poles in D, counting multiplicities.

This theorem explains why hodographs are so important.

Pseudo-proof

Let f have a single pole or zero at z_0, so there is analytic g with,

$$f(z) = (z - z_0)^\alpha g(z).$$

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\partial D} \frac{f'(z)}{f(z)}dz &= \frac{1}{2\pi i}\int_{\partial D} \frac{\alpha(z-z_0)^{\alpha-1}g(z) + (z-z_0)^\alpha g'(z)}{(z-z_0)^\alpha g(z)}dz \\
&= \frac{1}{2\pi i}\int_{\partial D} \left(\alpha\frac{1}{(z-z_0)} + \frac{g'(z)}{g(z)}\right)dz = \frac{1}{2\pi i}\int_{\partial D} \alpha\frac{1}{(z-z_0)}dz \\
&= \alpha.
\end{aligned}$$

The value α can be positive (zero) or negative (pole). The value tracks multiplicities.

It is easy to turn into an actual proof.

We can use this theorem to count poles and zeros in D.

$$\frac{1}{2\pi i}\int_{\partial D}\frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\int_{\partial D} d\log(f(z))dz$$
$$= \frac{1}{2\pi i}\int_{\partial D} d\log(|f(z)|)dz + \frac{1}{2\pi i}\int_{\partial D} di\arg(f(z))dz$$

The first integrates to $\log(|f(z)|)$ (e.g. is exact) and so vanishes on closed loop.
We get,

$$\int_{\partial D} d\arg(f(z))dz = 2\pi\left(N_0 - N_\infty\right) \tag{9}$$

Application: Polynomial roots

Start with a polynomial $p(z)$. Polynomials have no poles.

Form piece-wise smooth ∂D as follows:

- γ_1 goes along imaginary axis from $-iR$ to iR (hodograph).
- γ_2 arcs counter-clockwise along circle radius R centered at 0 connecting iR to $-iR$.

For large R, $N_0 = N_-$ is all the 'stable' zeros.

For large R, $a_n z^n$ dominates the integral over γ_2.

$$\int_{\gamma_2} d\arg(p(z))dz = \int_{\gamma_2} d\arg(a_n z^n)dz = n\pi. \tag{10}$$

hodograph

(Hermite) Let $p(z)$ be a real polynomial of degree n.

$$\int_{\gamma_1} d\arg(p(z))dz = 2\pi N_- - \pi(N_- + N_+) = \pi\left(N_- - N_+\right) \qquad (11)$$

- N_+ = number of roots in the right half plane.
- N_- = number of roots in the left half plane.

Definition

The Nyquist-Mikhailov hodograph is the curve,

$$\Gamma_p(t) = i^{-n} p(it). \qquad (12)$$

Δ_p = total phase change of $\Gamma_p(t)$ as $t = -\infty \to \infty$.

$$\Delta_p = \pi\left(N_- - N_+\right). \qquad (13)$$

hodograph examples

$$p_1(z) = z - z_1 \quad \Rightarrow \tag{14}$$

$$\Gamma_{p_1}(t) = \frac{1}{i}\,(it - \Re(z_1) - i\Im(z_1)) = (t - \Im(z_1)) + i\Re(z_1) \tag{15}$$

Phase change from $-\infty \to \infty$ is $-\pi\,\mathrm{sgn}(\Re(z_1))$.

$$p_T(z) = (z - z_1)(z - z_2)\cdot(z - z_n)$$
$$\Rightarrow \Gamma_{p_T}(t) = (t - \Im(z_1) + i\Re(z_1))\cdots(t - \Im(z_n) + i\Re(z_n))$$

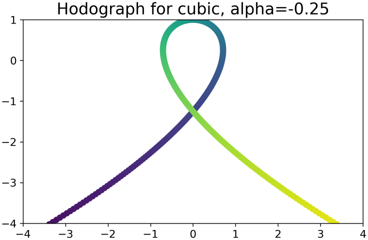
$$\Delta_p = \Delta_{p_1} + \cdots + \Delta_{p_n} = \pi\,(N_- - N_+) \tag{16}$$

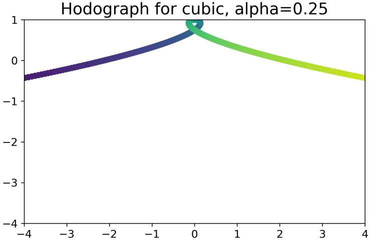
hodograph examples

We use the cubic we analyzed previously,

$$p(z) = z^3 + (r - 2\alpha)z^2 + (\alpha^2 + \beta^2 - 2\alpha r)z + r(\alpha^2 + \beta^2)$$

has roots $-r, \alpha \pm i\beta$.





Phase change for $\alpha = -0.25$ is 3π $(N_- = 3, N_+ = 0)$.
Phase change for $\alpha = 0.25$ is $-\pi$ $(N_- = 1, N_+ = 2)$.

$$\Delta_p = \pi\,(N_- - N_+) \tag{17}$$

Compute the Phase Change I

$$\Delta_p = \int_{-\infty}^{\infty} d \arg\left(\Gamma_p(t)\right) = \pi \left(N_- - N_+\right) \tag{18}$$

From the definition we have,

$$\Gamma_p(t) = i^{-n} p(it) = f_0(t) - i f_1(t) \tag{19}$$

where,

$$f_0(t) = a_n t^n - a_{n-2} t^{n-2} + a_{n-4} t^{n-4} + \cdots + 0$$
$$f_1(t) = a_{n-1} t^{n-1} - a_{n-3} t^{n-3} + a_{n-5} t^{n-5} + \cdots + 0$$

We will, WLOG, assume that $a_n > 0$.

Compute the Phase Change II

To compute the change in phase of $\Gamma_p(t)$ we note $(a_n > 0)$,

$$\lim_{t \to \infty} \arg\left(\Gamma_p(t)\right) = \lim_{t \to \infty} \arg\left(f_0(t)\right) = 0$$

$$\lim_{t \to -\infty} \arg\left(\Gamma_p(t)\right) = \lim_{t \to -\infty} \arg\left(f_0(t)\right) = \begin{Bmatrix} 0 & n = 2, 4, 6, \cdots \\ \pi & n = 1, 3, 5, \cdots \end{Bmatrix}$$

$\arg(\Gamma_p(t))$ starts $(t = -\infty)$ at π when n is odd and 0 when even.
In either case, $\arg(\Gamma_p(t)) \to 0$ as $t \to \infty$, so ends at 0.

To compute the phase change we will track how many times $\Gamma(t)$ crosses the imaginary axis.

$\Gamma(t)$ crosses the imaginary axis when $f_0(t) = 0$.

Count Imaginary Axis Crossings

(a) Phase change $+\pi$ when $f_0(t_r) = 0$ $f_1(t_r) < 0$. $-\pi$ if direction reversed.

(b) Phase change $+\pi$ when $f_0(t_r) = 0$ $f_1(t_r) > 0$. $-\pi$ if direction reversed.

Figure: Phase change is $\pm\pi$ every time imaginary axis is crossed

Remember $\Im(\Gamma_p(t)) = -f_1(t)$.

Track the number of times the hodograph crosses the imaginary axis. Remember the hodograph asymptotically approaches the real axis.

Figure 1a has phase change π, reverse results in $-\pi$.

Track the angles

The signs of $\{f_0(t), f_1(t)\}$ determine phase shift, where

$$\Gamma(t) = f_0(t) - if_1(t). \tag{20}$$

Figure 1a has transition $\{+, -\} \rightarrow \{-, -\}$, so lose sign change.

Figure 1b has transition $\{-, +\} \rightarrow \{+, +\}$, so lose sign change.

Reverse the arrows (change by $-\pi$) and you gain a sign.

We need to count all the sign changes.
Can we keep sign changes even when f_1 changes sign.

Track the angles

Let $\omega_0 < \omega_1 < \cdots < \omega_m$ be zeros of $f_0(t)$ of odd multiplicity.

Define i_k at imaginary axis crossing:
Is $+1$ if curve passes through counter-clockwise,
Is -1 if curve passes through clockwise.

i_k is the number of sign losses at ω_k.

The total phase change of $\Gamma_p(t)$ is,

$$\Delta_p = \pi\,(i_0 + i_1 + \cdots + i_m) = \pi\,(N_- - N_+)$$

Can we track the phase changes, without computing the roots of $f_0(t)$?

We use Sturm Sequences.

What have we got so far?

How many roots of a polynomial are in the left half complex plane? Let

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0, \tag{21}$$

with $a_k \in \mathbb{R}$. Form the hodograph,

$$\begin{aligned} \Gamma_p(t) &= i^{-n} p(it) = f_0(t) - i f_1(t) \\ f_0(t) &= a_n t^n - a_{n-2} t^{n-2} + \cdots, \\ f_1(t) &= a_{n-1} t^{n-1} - a_{n-3} t^{n-3} + \cdots \end{aligned} \tag{22}$$

$$\Delta_p = \int_{\gamma_1} d \arg(p(z)) dz = \pi (N_- - N_+) \tag{23}$$

- Track the number of times $\Gamma_p(t)$ crosses the imaginary axis.
- Crossing correspond to zeros of f_0 of odd multiplicity.
- Crossings corresponds to sign changes of $\{f_0, f_1\}$.

Can we keep track of the sign changes? Yes, using Sturm sequences.

Sturm Sequences: Definition

Definition

A finite sequence of continuous functions $\{f_0, f_1, \cdots, f_n\}$ is called a Sturm sequence on (a, b) if each f_k has a finite number of zeros (e.g. polynomials) and if for every $c \in (a, b)$ we have:

1. If $f_0(c) = 0$ then $f_1(c) \neq 0$.
2. $f_n(c) \neq 0$ for all $a < c < b$.
3. If $f_k(c) = 0$ for some k, then $f_{k-1}(c)f_{k+1}(c) < 0$.

For every $x \in (a, b)$ form the sequence $\{f_0(x), f_1(x), \cdots, f_n(x)\}$.
$V(x)$ is the number of sign changes of this sequence.
Exclude c when $f_k(c) = 0$ for some k. There are a finite number.

For $x \in (a, b)$ there are no adjacent zeros, $\{f_k(x), f_{k+1}(x)\}$ in the sequence (by 3rd property).

What Sturm Sequence?

$$\Gamma_p(t) = i^{-n} p(it) = f_0(t) - i f_1(t), \tag{24}$$

so $f_0(x)$ and $f_1(x)$ cannot both be zero,
as the hodograph doesn't go through origin.

So f_0 and f_1 are relatively prime in polynomial ring (only units divide both).

Repeatedly divide, as in f_0/f_1 to get $f_0 = d_1 f_1 - f_2$

Sturm Sequences: Sign Changes I

Proposition

If $c \in (a,b)$ is not a zero of $f_0(x)$ then $V(c-) = V(c+)$.

Proof.

If, for some $0 < k < n$, $f_k(c) = 0$ then $f_{k-1}(c)f_{k+1}(c) < 0$. Assume $f_{k-1}(c) < 0$ and $f_{k+1}(c) > 0$. There is a neighborhood of c where the signs of f_{k-1}, f_{k+1} do not change.

As x goes through c we get the sign sequences,

$$\{f_{k-1}(c-), f_k(c-), f_{k+1}(c-)\} = \{-, f_k(c-), +\}$$
$$\to \{f_{k-1}(c+), f_k(c+), f_{k+1}(c+)\} = \{-, f_k(c+), +\}.$$

There is a single sign change whatever the sign of f_k. □

Sturm Sequences: Sign Changes II

Proposition

If $c \in (a, b)$ is a zero of $f_0(x)$ of even multiplicity then $V(c-) = V(c+)$.

Proof.

For x in a neighborhood of c, the sign of $f_0(x)$ does not change ($x \neq c$). The sequence $\{f_0(x), f_1(x)\}$ does not change. □

The zeros of $f_0(x)$ with odd multiplicity are exactly the hodograph points that cross the imaginary axis!

The Sturm sequence 'saves' these results.

Construct a Sturm Sequence I

The hodograph has the form,

$$\Gamma_p(t) = f_0(t) - if_1(t). \tag{25}$$

Since $\deg(f_0) > \deg(f_1)$ so we divide,

$$f_0 = d_1 f_1 - f_2 \quad \Rightarrow \quad f_2(x) = d_1(x) f_1(x) - f_0(x). \tag{26}$$

$$f_{k-1} = d_k f_k - f_{k+1} \quad \Rightarrow \quad f_{k+1}(x) = d_k(x) f_k(x) - f_{k-1}(x). \tag{27}$$

$$\deg(f_0) > \deg(f_1) > \cdots > \deg(f_m)$$

We will use quotients $d_k(x) = xd_k$ (zero constant term).

We get a Sturm sequence $\{f_0, f_1, \cdots, f_m\}$.

Prove a Sturm Sequence

Proposition

At some point the sequence stops and we get $\{f_0, f_1, \cdots, f_m\}$ *which is a Sturm sequence.*

Recall that $f_{k-1} = d_k f_k - f_{k+1}$.
If f_m is a constant (a unit) then $f_{m-1} = \left(\frac{f_{m-1}}{f_m}\right) f_m$, so $f_{m+1} = 0$.

Property 1 Let $f_0(c) = f_1(c) = 0$. But $\Gamma_p(c) \neq 0$ as Γ does not go through the origin.

Property 3 Assume $f_k(c) = 0$ then $f_{k-1}(c) = -f_{k+1}(c)$.
If $f_{k-1}(c) = 0$ then $f_{k-2}(c) = d_{k-1}(c) f_{k-1}(c) - f_k(c) = 0$.
We get $f_0(c) = f_1(c) = \cdots = f_{k+1}(c) = 0$.
This is impossible so $f_{k-1}(c) f_{k+1}(c) < 0$.

Property 2 is the Chinese remainder theorem.

Sturm Property 2

At some point process ends. There is m such that,

$$f_{m+1} = d_m f_m - f_{m-1} = 0 \;\; \Rightarrow \;\; d_m f_m = f_{m-1}$$

Proposition

The polynomial f_m is the greatest common divisor of f_0 and f_1.

Using this,

The hodograph $\Gamma_p(t) = f_0(t) - if_1(t)$ does not go through origin so f_0 and f_1 have no common zeros.

Therefore $f_m(x) = d_m$ is just a constant.

This means $f_m(c) \neq 0$, which is Property 2 of the Sturm sequence.

Proof of Proposition

Proof.

f_m divides both f_0 and f_1.

$$f_{m-1} = d_m f_m$$
$$f_{m-2} = d_{m-1} f_{m-1} - f_m = (d_{m-1} d_m - 1) f_m$$

Proceed up the chain: $m-3, \cdots 1, 0$ to $f_1 = r_1 r_m$ and $f_0 = r_0 f_m$.

To show divisor is 'greatest' write,

$$f_m = d_{m-1} f_{m-1} - f_{m-2} = \cdots = q_0 f_0 + q_1 f_1$$

A common factor h has, $f_0 = e_0 h$ and $f_1 = e_1 h$.

$$f_m = q_0 e_0 h + q_1 e_1 h = (q_0 e_0 + q_1 e_1) h \Rightarrow h | f_m.$$

□

Regular Sturm Sequence

A polynomial Sturm sequence, $\{f_0, f_1, \cdots, f_n\}$ is regular if

$$\deg(f_k) = n - k, \quad \text{for} \quad k = 0, 1, \cdots n.$$

This is the 'typical' case. In this case, $\deg(d_k) = 1$.

We will assume a 'regular' polynomial sequence,

$$f_{k+1} = d_k f_k - f_{k-1} = (x d_k) f_k - f_{k-1} \tag{28}$$

Some of the non-regular cases are handled as Routh-Hurwitz exceptions. There are a number of these exceptions.

Routh Table: First Row

Routh table has $n+1$ rows. One row for $f_0, f_1, \cdots, f_n$. The first three rows are shown. We remove all the polynomial signs.

t^n	a_n	a_{n-2}	a_{n-4}	$\cdots$	0
t^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$	0
t^{n-2}	b_1	b_2	b_3	$\cdots$	0

The coefficients are defined by,

$$b_1 = \frac{(a_{n-1}a_{n-2} - a_n a_{n-3})}{a_{n-1}} \quad b_k = \frac{\left(a_{n-1}a_{n-2k} - a_n a_{n-(2k+1)}\right)}{a_{n-1}} \tag{29}$$

The $k+1$th row determines f_k (when regular) by adding the signs back in.

$$f_2(t) = b_1 t^{n-2} - b_2 t^{n-4} + b_3 t^{n-6} + \cdots$$

Routh Table: Fine Print

$$f_{k+1} = d_k f_k - f_{k-1}. \tag{30}$$

$$
\begin{aligned}
f_0 &= a_n t^n - a_{n-2} t^{n-2} + a_{n-4} t^{n-4} - \cdots \\
f_1 &= a_{n-1} t^{n-1} - a_{n-3} t^{n-3} + a_{n-5} t^{n-5} - \cdots \\
f_2 &= \left(\frac{a_n}{a_{n-1}} t \right) \left(a_{n-1} t^{n-1} - \right) - \left(a_n t^n - \cdots \right) \\
&= \left(\frac{a_{a-1} a_{n-2} - a_n a_{n-3}}{a_{n-1}} \right) t^{n-2} - \left(\frac{a_{a-1} a_{n-4} - a_n a_{n-5}}{a_{n-1}} \right) t^{n-4} + \cdots
\end{aligned}
$$

The Routh table is a short hand for computing the Sturm sequence.

Routh Table: Repeat for each line

Row 4 corresponds to $f_3 = d_2 f_2 - f_1$,

t^n	a_n	a_{n-2}	a_{n-4}	$\cdots$	0
t^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$	0
t^{n-2}	b_1	b_2	b_3	$\cdots$	0
t^{n-3}	c_1	c_2	c_3	$\cdots$	0

The coefficients are defined by 2 previous rows,

$$c_1 = \frac{(b_1 a_{n-3} - b_2 a_{n-1})}{b_1} \quad c_k = \frac{\left(b_1 a_{n-(2k+1)} - b_{k+1} a_{n-1}\right)}{b_1} \tag{31}$$

Alternate signs to get the next element in Sturm sequence,

$$f_3(t) = c_1 t^{n-3} - c_2 t^{n-5} + c_3 t^{n-7} + \cdots \tag{32}$$

Using The Sturm Sequence

Given hodograph $\Gamma_p(t) = f_0(t) - if_1(t)$, we get regular Sturm sequence,

$$\{f_0, f_1, \cdots, f_n\}$$

The number of sign changes $V(x)$ does not change unless x is a zero of $f_0(x)$ of odd degree. Call these points $\omega_0 < \omega_1 < \cdots < \omega_m$.

These are the sign changes when $\Gamma_p(t)$ crosses the imaginary axis. The sign change at ω_k is $-i_k$,

$$V(\omega_k+) - V(\omega_k-) = -i_k. \tag{33}$$

The Sturm sequence 'adds' these results.

$$\begin{aligned} V(\infty) - V(-\infty) = &(V(\omega_m+) - V(\omega_m-)) + (V(\omega_{m-1}+) - V(\omega_{m-1}-)) \\ &+ \cdots + (V(\omega_0+) - V(\omega_0-)) \end{aligned}$$

This means that the sum of the sign changes is,

$$\pi\left(V(\infty) - V(-\infty)\right) = -\pi\left(i_0 + i_1 + \cdots + i_m\right) = -\pi\left(N_- - N_+\right). \quad (34)$$

The sum of the sign changes is,

$$V(-\infty) - V(\infty) = N_- - N_+ \quad (35)$$

We need to compute $V(-\infty)$ and $V(\infty)$.

For that we only need highest order term of polynomial.

Leading Coefficients I

Let h_k be the leading coefficient for the polynomial $f_k(x)$.

$$
\begin{aligned}
f_0(x) &= a_n t^n - a_{n-2} t^{n-2} + a_{n-4} t^{n-4} + \cdots + 0 \\
f_1(x) &= a_{n-1} t^{n-1} - a_{n-3} t^{n-3} + a_{n-5} t^{n-5} + \cdots + 0 \\
f_2(x) &= b_1 t^{n-2} - b_2 t^{n-4} + \cdots \\
f_3(x) &= c_1 t^{n-3} - c_2 t^{n-5} + \cdots
\end{aligned}
$$

The leading terms are the first column of the Routh table,

$$
\begin{aligned}
h_0 &= a_n \\
h_1 &= a_{n-1} \\
h_2 &= b_1 \\
h_3 &= c_1
\end{aligned}
$$

Leading Coefficients II

Let $v\left(h_0, h_1, \cdots h_n\right)$ be the number of sign changes of the sequence so,

$$V(x) = v\left(f_0(x), f_1(x), \cdots f_n(x)\right).$$

The change in $\arg\left(\Gamma(t)\right)$ on (a, b) is π times the number of sign changes,

$$\begin{aligned}\Delta \arg\left(\Gamma(t)\right) &= -\pi\left(v\left(f_0(b-), \cdots, f_n(b-)\right) - v\left(f_0(a+), \cdots, f_n(a+)\right)\right)\\ &= \pi\left(v\left(f_0(a+), \cdots, f_n(a+)\right) - v\left(f_0(b-), \cdots, f_n(b-)\right)\right)\end{aligned}$$

Leading Coefficients III

As $t \to \infty$ the number of sign changes depends only on h_k,

$$V(\infty) = v\left(h_0, h_1, \cdots, h_n\right)$$
$$V(-\infty) = v\left(h_0(-1)^n, h_1(-1)^{n-1}, \cdots h_n\right)$$

where $v\left(h_0, h_1, \cdots h_n\right)$ is the number of sign changes of the sequence. Use

$$v\left(h_0, h_1, \cdots h_n\right) + v\left(h_0(-1)^n, h_1(-1)^{n-1}, \cdots h_n\right) = n. \tag{36}$$

Put this all together,

$$\begin{aligned} V(-\infty) - V(\infty) &= n - 2v\left(h_0, h_1, \cdots, h_n\right) \\ &= \left(N_- + N_+\right) - 2v\left(h_0, h_1, \cdots, h_n\right) = N_- - N_+ \end{aligned} \tag{37}$$

$$N_+ = v\left(h_0, h_1, \cdots, h_n\right). \tag{38}$$

And the answer is ...

Fill out the Routh table.

When the $n + 1$ rows are complete. Look at the sign changes in the first column,

$$(a_n, a_{n-1}, b_1, c_1, \cdots) \tag{39}$$

The number of sign changes is the number of roots that are in the right half complex plane.

If there are no sign changes then the polynomial is stable.

Where we've been

Given a polynomial $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$, with $a_k \in \mathbb{R}$.
When are the roots of $p(z)$ in the left half complex plane?

- Form hodograph $\Gamma(t) = f_0(t) - i f_1(t)$.
- Imaginary axis crossing correspond to sign changes in $\{f_0, f_1\}$.
- Create Sturm sequence $\{f_0, f_1, \cdots, f_n\}$.
- Sturm sequence accumulates the sign changes, so we only need signs at $\pm\infty$.
- Use Routh table to compute Sturm sequence.

Example: General Cubic I

Let $p(z)$ be a cubic polynomial with real coefficients,

$$p(z) = z^3 + a_2 z^2 + a_1 z + a_0.$$

We know that a necessary condition for stability is $a_k \geq 0$.

t^3	1	a_1	0
t^{3-1}	a_2	a_0	0
t^{3-2}	b_1	0	0
t^{3-3}	c_1	0	0

$$b_1 = \frac{1}{a_2}(a_2 a_1 - a_0) > 0,$$
$$c_1 = \frac{1}{b_1}(b_1 a_0) = a_0 > 0.$$

Stability means first column are positive,

$$a_2 a_1 > a_0. \tag{40}$$

Example: General Cubic II

Compare to our previous cubic,

$$z^3 + (r - 2\alpha)z^2 + (\alpha^2 + \beta^2 - 2\alpha r)z + r(\alpha^2 + \beta^2)$$

The condition is,

$$(r - 2\alpha)(\alpha^2 + \beta^2 - 2\alpha r) > r(\alpha^2 + \beta^2). \tag{41}$$

We get,

$$\begin{aligned} 2\alpha^2 r &> \alpha(\alpha^2 + \beta^2) + \alpha r^2 \\ &\Rightarrow 0 > \alpha\left((\alpha - r)^2 + \beta^2\right) \end{aligned}$$

This means that p is stable iff $\alpha < 0$, as expected.
A degenerate condition is $\alpha = r, \beta = 0 \Rightarrow r = 0$.
Notice, if $\alpha = 0$ then $b_1 = 0$ and the Routh-Hurwitz test fails.

Issues not covered

- What if a row in Routh table is zero?
- What if there roots on the imaginary axis?
- Hurwitz matrix methods

References I

- Yury S. Barkovsky, *Letures on the routh-hurwitz problem*, arXiv:0802.1805v1 (2008).
- Theodore W. Gamelin, *Complex analysis*, Springer-Verlag, 2001.