The Topologies of Topological Groups

Separation Axioms and Topological Groups

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Transformative Optics

January 25, 2024

Abstract
A topological group is group with a topology for which the group operations are continuous. Using group operations we can construct special open sets that restrict the types of topologies allowed. We shall show that many of the Tychonoff separation axioms are, in fact, equivalent for topological groups. For example, we shall show that if a topology is T_0 then it must be Hausdorff. We also outline Kolmogorov's construction showing how to generate a Hausdorff topological group from a group that is not T_0.

Motivation (1/2)

Given a group G and a set X.
A group action on X is a map $G \times X \to X$ denoted

$$(g, x) \to gx \in X.$$

The action is a left action if

$$g_1 \left(g_2(x)\right) = \left(g_1 g_2\right)(x)$$

A group action can give insight into X.

Groups acting on a manifolds is a rich subject in Riemannian geometry.
For example, $SO(3)$ (rotations) acts on S^2 (sphere).

A group acting on itself describes the structure of the group (see [Hun74]).

Motivation (2/2)

Given a group G with a group action

$$G \times X \to X,$$

what kind of space is X?

Differentiable manifold?
Normed linear space? Banach space? Hilbert Space?

In many of the examples, X is (at least) a topological space.

Add structure to G so group action is compatible with X.

If X is topological space then G should be a topological space.
The group action should be a continuous map.

That's a topological group.

Outline

- Motivation
- Definition of Topological group
- Separation axioms in topological groups
- What do we do if G is not T_0?
- Second countable and normal
- Hilbert's Fifth Problem

We do not proceed in the most direct way,
but instead show how the group operations interact with the open sets.

Manifold Taxonomy

Start with a topological space X, and add successive structures to X.

Manifold Structures

- *Topological Space: Set X equipped with topology.*
- *Topological Manifold: Locally Euclidean Topological space This means there is a collection of homeomorphisms $\psi_\alpha : U_\alpha \subset X \to \mathbb{R}^n$, where U_α is open cover of X.*
  Topological manifolds are locally compact.
- *Differentiable Manifold: Topological manifold with a differential structure.*
  The coordinate neighborhoods (U_α, ψ_α) have $\psi_\beta \circ \psi_\alpha^{-1}$ differentiable (e.g. C^∞ or real analytic).
  Add requirement that they are 2nd countable.

Topological Group Taxonomy

There is a group taxonomy related to this.

Group Structures

- *Topological Group: Group is a topological space and group operations are continuous.*
- *Topological groups that are locally Euclidean: groups that are topological manifolds.*
  These groups are locally compact.
- *Lie Group: Group is a differentiable manifold and group operations are smooth.*
  We add the restriction that G is second countable.

If G is a Lie Group and X a differentiable manifold, the Lie Group action

$$G \times X \to X,$$

is a differentiable map.

Topological Group Motivation

If we assume X is a topological space.

We want the mapping

$$G \times X \to X,$$

to be a continuous map.

This means G should have a topology.

If the topology is consistent with the group operation we have a topological group.

Group operations should be continuous maps.

Topological Group: Constituents

Definition

A group is a set G with an associative binary operation and,

1. There is an identity e with $g \cdot e = g = e \cdot g$ for all $g \in G$.
2. For every $g \in G$ there is a two-sided inverse g^{-1} that satisfies $g \cdot g^{-1} = g^{-1} \cdot g = e$.

Definition

A topological space is a set X with a collection of open sets $\mathscr{T}$ that satisfy

1. Union of elements of $\mathscr{T}$ are in $\mathscr{T}$.
2. Finite intersections of elements of $\mathscr{T}$ are in $\mathscr{T}$.
3. $X \in \mathscr{T}$ and $\emptyset \in \mathscr{T}$.

Topological Group

Definition

A group G is a topological group if it is a topological space and the following group operations are continuous:

1. $G \times G \to G$ given by $(g_1, g_2) \to g_1 g_2$,
2. $G \to G$ given by $g \to g^{-1}$.

It is enough to show the following is continuous:

$$(g_1, g_2) \to g_1 g_2^{-1}.$$

References

See [Pon46] and [Hus66].

Pontryagin is great but a bit dated.
His definition of topology is interesting.

Basic Properties of TG

Given any $g \in G$ form the "left-action"

$$L_g : G \to G \text{ given by } L_g(h) = gh.$$

The mapping L_g is continuous.
L_g has an inverse $L_{g^{-1}}$ which is also continuous.
This means L_g is homeomorphism of G onto G.

R_g (where $R_g(x) = xg$) is also homeomorphism of G onto G.

If $U \subset G$ is open then UF and FU are both open, for any $F \subset G$.
They are both unions of open sets.

If $H < G$ is normal subgroup then G/H is a topological group (see below).

Examples of Topological Groups

Every topological vector space is a topological group.
TVS also have scalar multiplication, which results in richer neighborhoods.

Any Lie Group is a topological group. E.g. $GL(V), SO(N), SU(N)$.

Let X be a metric space. The set of isometries is a topological group.
Group Product is composition. Topology is point-wise convergence.

We have a few exotic examples below.

Topologies on G

The topological space G supports a continuous group action.
This restricts the kinds of topologies allowed on topological groups.

Topological Groups also have special neighborhoods.

Topological Groups have group of transitive homeomorphisms.

Neighborhoods: Symmetric

Proposition

Let G be a topological group and U a neighborhood of e.
There is an open set $U_s \subset U$ with

$$U_s = U_s^{-1}. \quad (1)$$

Proof.

The map $g \to g^{-1}$ is continuous and takes $e \to e$, so there is an open neighborhood U_1 of e with $U_1^{-1} \subset U$.

Let $U_s = U_1 \cap U_1^{-1}$, so $e \in U_s \subset U$.

Claim: $U_s = U_s^{-1}$ as $\left(U_1 \cap U_1^{-1}\right)^{-1} = U_1^{-1} \cap U_1$.

□

Neighborhoods: Powers (1/2)

Proposition

Let G be a topological group and U a neighborhood of e. There is a neighborhood $U_2 \subset U$ of e with

$$U_2^2 \subset U. \tag{2}$$

Proof.

The composition $G \to G \times G \to G$ given by $g \to (g, g) \to g^2$ is continuous.

The map $g \to g^2$ is continuous and takes $e^2 \to e$,

so there is a neighborhood U_2 with $U_2^2 \subset U$. □

Neighborhoods: Powers (2/2)

It's easy to generalize this to the following:

Proposition

Let $r_1, r_2, \cdots, r_m \in \mathbb{Z}$, $r_j \neq 0$ *and* $g_1, g_2, \cdots, g_m \in G$ *with*

$$g_1^{r_1} g_2^{r_2} \cdots g_m^{r_m} = g_0. \tag{3}$$

For any neighborhood U_0 *of* g_0 *there are neighborhoods* U_j *of* g_j *with*

$$U_1^{r_1} \cdots U_m^{r_m} \subset U_0. \tag{4}$$

When $g_0 = g_i = e$ we have

Proposition

There are $U_1, \cdots, U_m$ *neighborhoods of e with,*

$$U_1^{r_1} \cdots U_m^{r_m} \subset U_0. \tag{5}$$

Neighborhoods: Conjugation

Proposition

Let G be a topological group and U a neighborhood of e,
For every $g \in G$ there is an open set U_g with

$$e \in U_g \subset U \tag{6}$$

$$gU_g g^{-1} \subset U \tag{7}$$

Proof.

For every $g \in G$ the map $x \to gxg^{-1}$ is continuous and takes $e \to e$, so there is an open neighborhood U_1 of e that satisfies

$$gU_1 g^{-1} \subset U.$$

Define

$$U_g = U \cap U_1. \tag{8}$$

□

Topological Vector Space (1/2)

Real topological vector spaces have continuous scalar multiplication:

$$(s, v) \to sv \in V, \qquad s \in \mathbb{R}, \quad v \in V. \tag{9}$$

You can generate symmetric neighborhoods that are star-shaped.

Given set $\{v_1, \cdots v_n\}$, there is a continuous map $T : \mathbb{R}^n \to V$ by

$$T(s_1, s_2, \cdots, s_n) = \sum_{i=1}^{n} s_i v_i \tag{10}$$

Let $S = \{(s_1)^2 + \cdots + (s_n)^2 = 1\}$ then $T(S)$ is compact,
so $T(S)^c$ is an open set that contains 0, (Hausdorff required)
form a symmetric subset of $T(S)^c$.

Using these two facts we can create conical neighborhoods,

Topological Vector Space (2/2)

A conical neighborhood has compact closure when intersected with $\mathrm{Span}\{v_1, \cdots, v_n\}$,
as it's contained in the compact set

$$\left\{ \sum_{i=1}^{n} s_i v_i \middle| (s_1)^2 + \cdots + (s_n)^2 \leq 1 \right\} \tag{11}$$

Using these neighborhoods you can show:

a topological vector space is finite dimensional
if and only if it is locally compact.

This is false if not T_0.
A TVS with the indiscrete topology fails this theorem.

Topological Closure

The closure of a set F is the smallest closed set containing F.
We use the following definition:

Definition

A point x is in the closure of F (denoted $cl(F)$) if every open set containing x intersects F.

It is easy to show these two definitions are equivalent.

Proposition

If X is Hausdorff and $F \subset X$ is compact then F is closed.

SketchLet $x \in cl(F) - F$.
For every $y \in F$ find open sets $x \in U_x, y \in U_y$ with $U_x \cap U_y = \emptyset$.
A finite subcover means $\cap U_x$ is an open set that does not intersect F.

Topological Separation Axioms

Let X be a topological space and let $x, y \in X$ with $x \neq y$.

0. A_0: X is T_0 if there is an open set that contains one element but not the other.
1. A_1: X is T_1 if there is an open set U_x with $x \in U_x$ and $y \notin U_x$. Since x, y can be exchanged there is an open set U_y with $y \in U_y$ and $x \notin U_y$.
2. A_2: X is T_2 (Hausdorff) if there are open sets $x \in U_x$ and $y \in U_y$ with $U_x \cap U_y = \emptyset$.
3. A_3: X is regular if for every point x and closed set F with $x \notin F$, there are open sets U_x, U_F with $x \in U_x$ and $F \subset U_F$ and $U_x \cap U_F = \emptyset$. (we do not assume Hausdorff)
4. X is normal if, for every closed sets F_1, F_2 there are there are open sets U_1, U_2 with $F_1 \subset U_1, F_2 \subset U_2$ and $U_1 \cap U_2 = \emptyset$.

Examples: Indiscrete Topology

Let X be any set with at least 2 elements. Define the topology to be the collection of sets

$$\mathscr{T}_i = \{X, \emptyset\}. \tag{12}$$

This is called the indiscrete topology.

$(X, \mathscr{T}_i)$ is not T_0 if X has more than 1 element.

1. $(X, \mathscr{T}_i)$ is compact (any open cover contains the set X).
2. $(X, \mathscr{T}_i)$ is 2nd countable.
3. $(X, \mathscr{T}_i)$ is locally compact.
4. Any map into the indiscrete topology is continuous.

Any group can be given the indiscrete topology to create a topological group.

Examples: T_0 but not T_1

There are spaces which are T_0 but not T_1.
Let X be a set with at least two elements and let $x_0 \in X$. Define a topology

$$\mathscr{T}_i = \{A \subset X | \, x_0 \notin A\} \cup \{X, \emptyset\}. \tag{13}$$

There is no open set containing x_0 except X!

The topological space $(X, \mathscr{T}_i)$ is T_0,
for any $x \in X, x \neq x_0$ the set $\{x\}$ is open.

We can separate any two points since one of them is not x_0.

There is no open set containing x_0 other than X so not T_1.

This example is not homogeneous
and cannot be made into a topological group.

For Topological Groups T_0 is equivalent to T_1

Proposition

If a topological group G is T_0 then G is T_1.

Proof.

Let $g_1, g_2 \in G$ and assume there is an open set U_1 with

$$g_1 \in U_1 \text{ and } g_2 \notin U_1. \tag{14}$$

The set $L_{g^{-1}}(U_1) = g_1^{-1} U_1$ is an open neighborhood of e, so there is a symmetric U_s with

$$e \in U_s \subset g_1^{-1} U_1 \Rightarrow g_1 U_s \subset U_1 \Rightarrow g_2 \notin g_1 U_s. \tag{15}$$

Claim: $g_1 \notin g_2 U_s$ as if $g_1 = g_2 u$ then $g_2 = g_1 u^{-1}$ and

$$g_2 \in g_1 U_s^{-1} = g_1 U_s \subset U_1, \text{ contradiction!} \tag{16}$$

So $g_2 \in g_2 U_s$ but $g_1 \notin g_2 U_s$.

In T_1 Singletons Are Closed

In T_1 singletons are always closed.

In fact it's easy to show,

Theorem

A topological space is T_1 if and only if single element sets are closed.

Co-finite is T_1 but not T_2 (Hausdorff)

Let X be an infinite set. Define a topology collection by

$$\mathscr{T}_{cf} = \{\, A \subset X \,|\, X - A \text{ is finite } \} \cup \{\emptyset\}. \tag{17}$$

This is the co-finite topology.
If X is an infinte set the co-finite topology is T_1 but not T_2.
Singletons are closed but two non-empty open sets always intersect.
The cofinite topology is compact and connected.

Cofinite is similar to Zariski topology (zeros of polynomials are closed).

Cofinite topology on the group S^1 is not a topological group.
Addition is not continuous.

Bug-Eyed Line

Identify points in two real lines $\mathbb{R} \times \{a\}$ and $\mathbb{R} \times \{b\}$.
Identify $(x, a) \sim (x, b)$ when $x \neq 0$.

This example is T_1.

This example is not Hausdorff.

This example is a topological manifold of dimension 1.

For Topological Groups T_1 is Hausdorff (1/2)

Proposition

If a topological group G is T_1 then G is T_2 (Hausdorff).

Proof.

Let $g_1, g_2 \in G$ and let U_1 be a neighborhood of g_1 with $g_2 \notin U_1$.

Claim: There is an open set U_0 with $e \in U_0$ and

$$e \in U_0 U_0^{-1} \subset L_{g_1^{-1}}(U_1) = g_1^{-1} U_1. \tag{18}$$

The map $(h_1, h_2) \to h_1 h_2^{-1}$ is continuous so
there are open sets with $W_1 W_2^{-1} \subset g_1^{-1} U_1$, with $e \in W_1, e \in W_2$.
Set $U_0 = W_1 \cap W_2$, so that $e \in U_0$.

Use U_0 to construct separating neighborhoods.

□

For Topological Groups T_1 is Hausdorff (2/2)

Proof.

We have $g_1 \in U_1$ and $g_2 \notin U_1$ and

$$e \in U_0 U_0^{-1} \subset g_1^{-1} U_1. \tag{19}$$

Claim: $g_1 U_0 \cap g_2 U_0 = \emptyset$.
If $g_1 u_1 = g_2 u_2$ with $u_1, u_2 \in U_0$, then

$$g_2 = g_1 u_1 u_2^{-1} \Rightarrow g_2 \in g_1 U_0 U_0^{-1} \subset U_1, \text{ Contradiction!} \tag{20}$$

So $g_1 U_0$ and $g_2 U_0$ are the required open sets. □

Note this proof really only assumes T_0 so proposition on T_1 is not really required.

But it's more clear to include both.

Separation Axiom Summary

Here's what we have so far.

Theorem

Let G be a topological group.
G is T_0 if and only if it is T_1
and is T_1 if and only if it is T_2 (Hausdorf).

Regular

Proposition

If G is a topological group then G is regular (T_0 not assumed!).

Proof.

First assume F is closed with $e \notin F$ so $e \in F^c$ is open
There is an open set U_0 with $e \in U_0$ and $U_0^{-1}U_0 \subset F^c$.
Claim: $U_0 \cap U_0 F = \emptyset$
If $u_1 = u_2 f$ for $u_1, u_2 \in U_0$ then $f = u_2^{-1}u_1 \in U_0^{-1}U_0$ Contradiction!

Now let F be closed $g \notin F$ so that $e \notin Fg^{-1}$, which is a closed set.
There is U_0 with $e \in U_0$ and $U_0^{-1}U_0 \subset \left(Fg^{-1}\right)^c$.
Claim: $U_0 g \cap U_0 F = \emptyset$
If $u_1 g = u_2 f \Rightarrow u_2^{-1}u_1 = fg^{-1}$ Contradiction!.

□

Are Topological Groups Normal?

If a topological group is T_0 then it is Hausdorff.

A topological group is always regular.

Are topological groups normal?

Are topological groups metrizable?

Moore/Niemytzki plane 1

$$H_m = \left\{ (x, y) \in \mathbb{R}^2 \middle| \, y \geq 0 \right\}. \tag{21}$$

Let $B((x, y), \rho)$ be open disk, centered at (x, y), with radius ρ.
Topology H_m generated by the base,

$$B((x, y), \rho) \text{ where } \rho < y \tag{22}$$

$$B((x, y), \rho) \cup \{x, 0\} \text{ where } \rho = y. \tag{23}$$

There are two types of open disks.
These form a base for the topology on H_m.

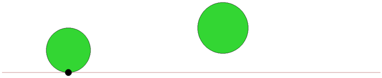

Figure: Open Disks on Moore/Niemytzki Plane. Disk on left includes point.

See [SS70].

Moore/Niemytzki plane 2

We sketch some facts:

1. H_m is NOT a topological group.
2. H_m minus the $x-$axis (open half-plane) is open.
   It is union of disks of equation (22)
3. Every subset of the x-axis is closed.
   If F is subset of $x - axis$.
   F^c is union of open upper half plane and
   Union of disks of equation (23) with point in F^c.
4. H_m is Hausdorff.
   Any two points on (including on x-axis) are separated by disks.
5. H_m is NOT second countable.

Moore/Niemytzki plane 3

Topology is regular (can separate point and closed set).

For example, let $P = (0,0)$ (on the $x-$axis).
and let $F = \{(x,0) | x \neq 0\}$ (subset of $x-$axis).

Let $U_P = B((0,1),1)$, an open disk type (23)), with $(0,0) \in U_P$.

Cover $F = \{(x,0) | x \neq 0\}$ with union of disks of type (23).

If $|x| > 1$ can easily find a tangent disk (say radius $= (\sqrt{2} - 1)/2$)
that does not intersect U_P.
If $|x| \leq 1$ find a tangent disk that fits under $\gamma(t) = 1 - \sqrt{1 - t^2}$.

The details are left as an exercise.

Moore/Niemytzki plane 4

The H_m topology is NOT normal (e.g. you cannot separate two closed sets).

$$F_1 = \{(q,0) | \, q \in Q\}$$
$$F_2 = \{(r,0) | \, r \in Q^c\}$$

You cannot separate these using open sets.

This is obvious but difficult to prove.

There is a more complicated example which is a topological group.

Direct Products

Definition

Let I be an index set and X_α be a topological space for every $\alpha \in I$. The direct product of X_α is defined as

$$\prod_{\alpha \in I} X_\alpha = \left\{ x : I \to \bigcup_{\alpha \in I} X_\alpha \,\middle|\, x(\alpha) \in X_\alpha \right\}$$

The projection maps the product to the constituents.
For any $\beta \in I$ define

$$\pi_\beta : \prod_{\alpha \in I} X_\alpha \to X_\beta \quad \text{by} \quad \pi_\beta(x) = x(\beta) \in X_\beta. \tag{24}$$

Direct Products: Tychonoff Topology

The Tychonoff topology is minimum topology with π_β continuous.

The product or Tychonoff topology has a base given by

$$\pi_{\alpha_1}^{-1}(U_1) \cap \pi_{\alpha_2}^{-1}(U_2) \cap \cdots \cap \pi_{\alpha_a}^{-1}(U_a) \tag{25}$$

One can easily show the following:

1. If X_α are Hausdorff then $\prod_{\alpha\in I} X_\alpha$ is Hausdorff.
2. If X_α are connected then $\prod_{\alpha\in I} X_\alpha$ is connected.
3. If X_α are compact then $\prod_{\alpha\in I} X_\alpha$ is compact (Tychonoff's theorem).

Direct Product of Groups

Let G_α be a collection of topological groups with $\alpha \in I$.
The direct product of groups is

$$\prod_{\alpha \in I} G_\alpha = \left\{ x : I \to \bigcup_{\alpha \in I} G_\alpha \,\middle|\, x(\alpha) \in G_\alpha \right\}$$

If each G_α is a group then define a product by

$$(xy)(\alpha) = x(\alpha)y(\alpha). \tag{26}$$

Inverses and the identity are defined in a natural way:

$$e(\alpha) = e \in G_\alpha$$
$$x^{-1}(\alpha) = (x(\alpha))^{-1} \in G_\alpha$$

One can easily show product and inverse are both continuous maps.

Are Topological Groups Normal?

Let $(Z, +)$ be the topological group with discrete topology.
Let $I = \mathbb{R}$ and $G_\alpha = Z$ for each $\alpha \in I$.

$$G = \prod_{\alpha \in I} G_\alpha = Z^R, \tag{27}$$

is NOT normal (even though $(Z, +)$ is normal).

G is Hausdorff so T_0.

G is not second countable (that's why we chose $\mathbb{R}$).

G is not locally compact even though $(Z, +)$ is.

Every neighborhood of e contains a copy of Z (e.g, there are small subgroups).

$Z^{\mathbb{R}}$ is not normal (1 of 5)

We will show $Z^{\mathbb{R}}$ is NOT normal.

Z has the discrete topology so we have a base of $Z^{\mathbb{R}}$ of the form,

$$U(x_0, B) = \left\{ x \in Z^{\mathbb{R}} \,\middle|\, x(\alpha) = x_0(\alpha) \quad \forall \alpha \in B \right\}$$
$$= \pi_{\alpha_1}^{-1}(\{x_0(\alpha_1)\}) \cap \cdots \cap \pi_{\alpha_k}^{-1}(\{x_0(\alpha_k)\}),$$

where $B = \{\alpha_1, \cdots, \alpha_k\} \subset I$ is a finite set.

$Z^{\mathbb{R}}$ is not normal (2 of 5)

$$P_n = \left\{ x \in Z^R \middle| x \text{ is injective on } x^{-1}(Z - \{n\}) \right\}. \tag{28}$$

- If $x \in P_n$ then $\{\alpha | x(\alpha) \neq n\}$ is countable.
- If $x \in P_n$ then $\{\alpha | x(\alpha) = n\}$ is uncountable.
- so if $n \neq m$ we have $P_m \cap P_n = \emptyset$.

This is why we take $Z^{\mathbb{R}}$ (e.g. $\mathbb{R}$ is uncountable).

Claim: P_n is closed.
$x \notin P_n$ means there is $m \in Z$ and $\alpha, \beta \in I$ with $x(\alpha) = x(\beta) = m \neq n$.
$x \in \pi_\alpha^{-1}(\{m\}) \cap \pi_\beta^{-1}(\{m\})$, so complement of P_n is open.

$Z^{\mathbb{R}}$ is not normal (3 of 5)

P_1 and P_2 are distinct closed sets that do not intersect.

Assume we have open sets U_1, U_2 with $P_1 \subset U_1$ and $P_2 \subset U_2$.
We shall show $U_1 \cap U_2 \neq \emptyset$.

Define $x_1(\alpha) = 1$, so $x_1 \in P_1$.
There is base set with $x_1 \in U(x_1, B_1) \subset U_1$, with $B_1 = \{\alpha_1, \cdots, \alpha_{n_1}\}$.
Now define $x_2(\alpha_i) = i, \alpha_i \in B_1$ and $x_2(\alpha) = 1$ otherwise.
This means $x_2 \in P_1$.

Now iterate $k = 2, 3, \cdots,$
Since $x_k \in P_1$ there is base $U(x_k, B_k) \subset U_1$ and we can choose $B_{k-1} \subset B_k$ (just intersect with $U(x_k, B_{k-1})$), so $B_k = \{\alpha_1, \cdots, \alpha_{n_1}, \cdots \alpha_{n_k}\}$
Define $x_{k+1}(\alpha_i) = i, \alpha_i \in B_k$ and $x_k(\alpha) = 1$ otherwise,
$x_{k+1} \in P_1$.
Exercise: fill in the details.

$Z^{\mathbb{R}}$ is not normal (4 of 5)

Define $y_\infty \in P_2$ by

$$y_\infty(\alpha) = \left\{ \begin{array}{ll} j & \text{for } \alpha = \alpha_j \in \bigcup_{i=1}^{\infty} B_i \\ 2 & \text{else} \end{array} \right\}$$

We have $y_\infty \in P_2 \subset U_2$.
Since $y_\infty \in P_2$ there is a finite $C \subset I$ (base set) with

$$y_\infty \in U(y_\infty, C) \subset U_2.$$

Define $B_a = \bigcup_{i=1}^{\infty} B_i$, (recall $B_k \subset B_{k+1}$)

$$C = (B_a \cap C) \cup (B_a^c \cap C) = (B_k \cap C) \cup (B_a^c \cap C) \text{ for some } k. \tag{29}$$

We have,

$$y_\infty|_{B_k} = x_{k+1}|_{B_k}. \tag{30}$$

$Z^{\mathbb{R}}$ is not normal (5 of 5)

We have

$$x_{k+1} \in U(x_{k+1}, B_{k+1}) \subset U_1 \text{ and } y_\infty \in U(y_\infty, C) \subset U_2. \tag{31}$$

We shall construct an element z with

$$z \in U(x_{k+1}, B_{k+1}) \cap U(y_\infty, C) \subset U_1 \cap U_2.$$

Define z as follows,

$$z(\alpha) = \left\{ \begin{array}{ll} x_{k+1}(\alpha) = j & \text{for } \alpha = \alpha_j \in B_k \\ x_{k+1}(\alpha) = 1 & \text{for } \alpha \in B_{k+1} - B_k \\ y_\infty(\alpha) & \text{else} \end{array} \right\}$$

This means $z \in U(x_{k+1}, B_{k+1}) \subset U_1$.
Since $z(\alpha) = y_\infty(\alpha)$ for all $\alpha \in C$ we have $z \in U(y_\infty, C) \subset U_2$.

This means $z \in U_1 \cap U_2 \neq \emptyset$ and $Z^{\mathbb{R}}$ is NOT normal.

Normal Spaces

We state a theorem from topology.

Theorem

Given a Hausdorff topological space X. If X is regular and second countable then X is normal.

We can use the following formulation for regular.
For ever $x \in X$ and F closed with $x \in F^c$ there is an open U with

$$x \in U \subset cl(U) \subset F^c. \tag{32}$$

For completeness we include a proof of the theorem,
but the theorem and the proof are topology and are a bit of an aside.

Sketch of proof

Let X be a Hausdorff, regular second countable topological space.
Let $\mathscr{B}$ be a countable base and let F_1, F_2 be disjoint closed sets.

For every $a \in F_1$ there is an open U_a with $a \in U_a \subset cl(U_a) \subset F_2^c$.
We can choose $U_a \in \mathscr{B}$, which is a countable set.
For $b \in F_2$ there is an open $V_b \in \mathscr{B}$ with $b \in V_b \subset cl(V_b) \subset F_1^c$.

Since $\mathscr{B}$ is countable there are countable U_a, V_b denoted U_j, V_j.
Iterate:

$$A_k = U_k \cap cl(V_1)^c \cap \cdots \cap cl(V_k)^c \qquad B_k = V_k \cap cl(U_1)^c \cap \cdots \cap cl(U_k)^c \quad (33)$$

Show that $A_k \cap F_1 = U_k \cap F_1$ and $B_k \cap F_1 = V_k \cap F_2$.
Show that $A_k \cap B_m = \emptyset$ for all k, m.

Show that $\bigcup_{j=1}^{\infty} A_j$ and $\bigcup_{j=1}^{\infty} B_j$ are the desired separating open sets.

Metrizable

Urysohn's metrization theorem states that every Hausdorff, regular, second countable topological space is metrizable.

Going from open and closed sets to functions is an exacting enterprise, so we leave it to the professionals.

There are refinements of this theorem which make it more precise.
See Nagata-Smirnov Metrization Theorem and the Bing Theorem.

Topological groups that are T_0 and second countable are metrizable.

Quotient Space: Cosets

Let $H < G$ be a subgroup.

Form the coset space G/H.

Put a topology on G/H!

$$\pi : G \to G/H \tag{34}$$

defined by

$$\pi(g) = [gH] \in G/H. \tag{35}$$

A subset $[V] \subset G/H$ is open iff $\pi^{-1}([V]) = VH$ is open.

Proposition

This definition of open sets defines a topology on G/H.

This is a straightforward exercise.

Quotient Space: H is closed.

Proposition

The coset space G/H is T_1 iff H is closed.

Proof.

Let H be a subgroup of G (not necessarily normal).

$$
\begin{aligned}
& H \text{ is closed }. && (36)\\
& \Leftrightarrow gH \text{ is closed for all } g \in G. && (37)\\
& \Leftrightarrow (gH)^c \text{ is open }, && (38)\\
& \Leftrightarrow \pi^{-1}([gH]^c) = (gH)^c \text{ is open}, && (39)\\
& \Leftrightarrow [gH]^c \text{ is open}, && (40)\\
& \Leftrightarrow [gH] \text{ is closed for all } g \in G. && (41)
\end{aligned}
$$

$$\text{as } x \in \pi^{-1}([gH]^c) \Leftrightarrow \pi(x) \in [gH]^c \Leftrightarrow xH \cap gH = \emptyset \Leftrightarrow x \notin gH. \qquad (42)$$

All singletons are closed if and only if G/H is T_1. □

Quotient Space: T_0 is Hausdorff (1/2)

Proposition

If the coset space G/H is T_0 then the coset space G/H is Hausdorff.

Proof.

Let $g_1, g_2 \in G$ and $U_1 H \subset G$ be an open set with

$$[g_1 H] \in [U_1 H] \text{ and } [g_2 H] \notin [U_1 H]. \tag{43}$$

This means $g_1 \in U_1 H$ and $(g_2 H) \cap U_1 H = \emptyset$.

Let $U_0 = U_1 H g_1^{-1} \Rightarrow U_0 g_1 = U_1 H$.
We also have $e \in U_0$ and $(g_2 H) \cap (U_0 g_1) = (g_2 H) \cap (U_1 H) = \emptyset$.

There is an open U with $U^{-1} U \subset U_0$ and $e \in U$. □

Quotient Space: T_0 is Hausdorff (2/2)

Proof.

We have

$$(g_2 H) \cap (U_1 H) = \emptyset \text{ and } U_0 = U_1 H g_1^{-1} \text{ and } U^{-1} U \subset U_0. \qquad (44)$$

Claim$(Ug_1 H) \cap (Ug_2 H) = \emptyset$
If $u_1 g_1 h_1 = u_2 g_2 h_2$ with $u_1, u_2 \in U$ then

$$\begin{aligned} u_2^{-1} u_1 g_1 &= g_2 h_2 h_1^{-1} \\ &\Rightarrow \left(U^{-1} U g_1\right) \cap (g_2 H) \neq \emptyset, \\ &\Rightarrow \left(U_1 H g_1^{-1} g_1\right) \cap (g_2 H) \neq \emptyset, \\ &\Rightarrow (U_1 H) \cap (g_2 H) \neq \emptyset, \end{aligned}$$

which is a contradiction.

So $[Ug_1 H]$ and $[Ug_1 H]$ are disjoint open sets and G/H is Hausdorff.

□

Quotient Application

Let G act on X, $G \times X \to G$, denoted gx.
The isotropy sub-group of $x_0 \in X$ is

$$G_{x_0} = \{ g \in G | \, gx_0 = x_0 \} .$$

If the action is transitive (for every $x, y \in X$ there is a $gx = y$), we have

$$G/G_{x_0} \cong X. \tag{45}$$

This is true for finite groups, topological spaces and even Lie Groups.
For finite groups see Hungerford [Hun74].

Example, $SO(3)$ acts on S^2 with isotropy group $SO(2)$.
This is a homogeneous space,

$$SO(3)/SO(2) \cong S^2. \tag{46}$$

This is for another talk.

Quotient Group

Theorem

If G is a topological group and $H < G$ a normal subgroup. The quotient group G/H is a topological group.

Sketch: G/H has the quotient topology.
We need to show product and inverse are continuous.

For any $g_1, g_2 \in G$ and any neighborhood VH of $g_1 g_2 H$.
There are open sets $g_1 \in U_1$ and $g_2 \in U_2$ with $U_1 U_2 \subset VH$.
Claim: $(U_1 H)(U_2 H) \subset VH$.

$$u_1 h_1 u_2 h_2 = u_1 u_2 (u_y^{-1} h_1 u_y) h_2 = u_1 u_2 h_3 \in VH.$$

If VH is arbitrary neighborhood of $g_1^{-1} H$
there is an U_1 with $U_1^{-1} \subset VH \Rightarrow (U_1 H)^{-1} = H U_1^{-1} = U_1^{-1} H \subset VH$.

What if not T_0? (1 of 2)

What if a topological group G is not T_0?

We will form a quotient group that is T_0 (and so Hausdorff).

We shall look at $cl(\{e\})$.

Recall the definition of closure:

Definition

Let X be a topological space and $S \subset X$. The closure of S, denoted $cl(S)$, is defined as

$$cl(S) = \{x \in X | \text{ every neighborhood of } x \text{ intersects } S\}$$

In Pontryagin [Pon46] the definition of topology on X is a closure function, that maps $S \subset X$ to $cl(S) \subset X$.

What if not T_0? (2 of 2)

If $cl(\{e\}) = \{e\}$ then the singleton set $\{e\}$ is closed.

If the singleton $\{e\}$ is closed then $L_g(\{e\}) = \{g\}$ is also closed.

If all singletons are closed then G is T_1 and so T_0 and Hausdorff.

So if G is NOT T_0 then $cl(\{e\}) \neq \{e\}$ and the singleton is NOT closed.

$cl(\{e\})$ is a group (1 of 2)

Proposition

Let G be a topological group. $cl(\{e\})$ is a subgroup of G and so is a topological group.

Proof.

Let $K = cl(\{e\})$. If $g_1, g_2 \in K$ and V is any neighborhood of $g_1 g_2$, there are neighborhoods $g_1 \in U_1, g_2 \in U_2$ with

$$g_1 g_2 \in U_1 U_2 \subset V. \tag{47}$$

But $e \in U_1, e \in U_2$ so $e \in U_1 U_2 \subset V$,
and $g_1 g_2 \in K$.

Now we show that K is closed under inverse.

□

$cl(\{e\})$ is a group (2 of 2)

Proof.

If $g_1 \in K = cl(\{e\})$ and V is any neighborhood of g_1^{-1},
there is a neighborhood U of g_1 with

$$g_1^{-1} \in U^{-1} \subset V. \tag{48}$$

But $g_1 \in K$ so $e \in U \Rightarrow e \in V$
so $g_1^{-1} \in K$ and K is closed under inverses.

This means $K = cl(\{e\})$ is a group.

□

$cl(\{e\})$ is Normal Subgroup

Proposition

Let G be a topological group. The subgroup $K = cl(\{e\})$ is a normal subgroup.

Proof.

Let $h \in cl(\{e\}), g \in G$ and V any neighborhood of ghg^{-1}.

There is U_g with $e \in U_g$ and $gU_gg^{-1} \subset V$ (conjugation neighborhood).

Since $e \in U_g \Rightarrow e \in gU_gg^{-1} \subset V$.

This means $ghg^{-1} \in cl(\{e\})$, and $cl(\{e\})$ is normal. □

What if G is not T_0?

Let G be a topological group that is not T_0.

The subgroup $cl(\{e\})$ is normal.

The subgroup $cl(\{e\})$ is closed.

The group $G/cl(\{e\})$ is a topological group.

The group $G/cl(\{e\})$ is T_1 and so also Hausdorff.

This is Kolmogorov's procedure for creating Hausdorff spaces out of integrable functions.

Topological Groups that are not T_0

Example from L^p spaces.

$$\tilde{L}^2(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{C} |\, f \text{ is measurable and } \int_R \|f(t)\|^2 dt < \infty \right\}$$

Form topology from semi-norm $\|f\|_2$.

If f and g differ on a set of measure zero
then they are in the same open sets.
You cannot separate f and g so $\tilde{L}^p(\mathbb{R})$ is NOT T_0.

This construction was first discussed by Kolmogorov (I think!?).

Topological Groups That Are manifolds

- Let G be a topological group.
- If G is not T_0 then form $G/cl(e)$ instead.
- Assume that G is T_0, this implies G is Hausdorff.
- G is regular.
- Now assume that G is second countable.
- This means G Is normal and metrizable.

What do we need to insure that G is a topological manifold (e.g. locally Euclidean).

Hilbert's Fifth Problem (1/3)

If G is a topological manifold then it must be locally compact.

To guarantee that G is a Lie Group we must assume

- G is a topological Group.
- G is T_0 (so Hausdorff).
- G is second countable (so normal and metrizable).
- G is locally compact.

But this is not sufficient.

Hilbert's Fifth Problem (2/3)

Definition

A topological group has no small groups if there is a neighborhood U of e with the property that if H is a non-trivial subgroup of G then H is not a subset of U.

Our example $Z^{\mathbb{R}}$ has small sub-groups.
Topological Groups that are topological manifolds (e.g. locally Euclidean) have no small subgroups.

This is another important necessary condition.

If there are no small subgroups then $cl(\{e\}) = \{e\}$ and the group is T_1.

Hilbert's Fifth Problem (3/3)

Theorem

Let G be a locally compact, second countable topological group. If G has no small subgroups then G is locally Euclidean.

The proof is result of work done in the 1950's by,
A. Gleason and by D. Montgomery and L. Zippin (see [MZ74]).
One can also show the following,

Theorem

If G is a topological group that is locally Euclidean then G admits a unique Lie Group structure.

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