Every composition operator is (mean) asymptotically Toeplitz*

Joel H. Shapiro

Department of Mathematics and Statistics
Portland State University, Portland, OR 97207

Abstract

It was shown in [8] that, while composition operators on the Hardy space H^2 can only trivially be Toeplitz, or even "Toeplitz plus compact," it is an interesting problem to determine which of them can be "asymptotically Toeplitz." I show here that if "asymptotically" is interpreted in, for example, the Cesàro (C, α) sense ($\alpha > 0$), then *every* composition operator on H^2 becomes asymptotically Toeplitz.

Key words: Composition operator, Toeplitz operator, regular summation method
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1 Introduction

Every holomorphic function φ mapping the open unit disc $\mathbb{U}$ into itself induces, by means of composition on the right, a linear *composition operator* C_φ on the space $\mathrm{Hol}(\mathbb{U})$ consisting of all functions holomorphic on $\mathbb{U}$. Explicitly:

$$C_\varphi f = f \circ \varphi \qquad (f \in \mathrm{Hol}(\mathbb{U})).$$

Littlewood's Subordination Principle [7] yields the remarkable fact that every composition operator restricts to a bounded operator on the Hardy space H^2 (see also [5, Theorem 1.7, page 10] or [11, pp. 13–15]), a phenomenon that has inspired intense interest in relating operator-theoretic properties of C_φ on H^2, and other spaces of analytic functions, with function-theoretic properties of φ (see [3,6,11] for more on this).

* For William F. Ames, on the occasion of his 80th birthday
Email address: `shapiroj@pdx.edu` (Joel H. Shapiro).
URL: `www.mth.pdx.edu/~shapiroj` (Joel H. Shapiro).

The work I will describe here originates in an intriguing paper [1] of Barría and Halmos, which introduced the notion of "asymptotic Toeplitz operator." A *Toeplitz operator* on H^2 is a bounded linear operator whose matrix, relative to the orthonormal basis of monomials $\{z^n : n \geq 0\}$, has constant diagonals. Such an operators T can be characterized by the equation $S^*TS = T$, where S is the forward shift on H^2 (i.e. "multiplication by the independent variable z"), and S^* is its Hilbert space adjoint, easily seen to be the *backward shift* defined by: $S^*z^n = z^{n-1}$ if $n > 0$, and $= 0$ if $n = 0$.

Barría and Halmos call an operator T on H^2 *asymptotically Toeplitz* if the sequence of operators $(S^{*n}TS^n)$ converges strongly (i.e. pointwise) on H^2. Clearly every Toeplitz operator is asymptotically Toeplitz, and so is every compact perturbation of a Toeplitz operator. Recently Fedor Nazarov and I [8] characterized, among the holomorphic self-maps of $\mathbb{U}$ that *fix the origin*, those that induce asymptotically Toeplitz composition operators on H^2. The answer is simple, as is the proof [8, Propositions 3.1 and 3.2]:

> *If $\varphi(0) = 0$ then C_φ is asymptotically Toeplitz on H^2 if and only if the set of points of $\partial\mathbb{U}$ at which $|\varphi| = 1$ has measure zero .*

By contrast, the case $\varphi(0) \neq 0$ yields up the surprising fact that, while the "measure zero" condition remains sufficient for asymptotic toeplitzness, it is no longer necessary [8, Theorem 3.4]:

> *C_φ can be asymptotically Toeplitz on H^2 even in some cases where $|\varphi| = 1$ on a subset of $\partial\mathbb{U}$ having positive measure.*

In what follows I show that if, in the definition of "asymptotically Toeplitz," the mode of convergence is weakened to "(C, α) for all $\alpha > 0$," then all subtlety disappears: *every* composition operator on H^2 becomes asymptotically Toeplitz. This result implies an intriguing fact about the matrices of composition operators with respect to the standard orthonormal basis $\{z^n\}_0^\infty$ of H^2: *all their diagonals converge (C, α) for each $\alpha > 0$*. For the special case $\alpha = 1$, this corollary was proved in [8] by an argument different from the one used here. It is not known if the result is true for $\alpha = 0$. In other words:

> *Do the diagonals of every "composition operator matrix" converge?*

The (C_α)-results in this paper emerge from a theorem that deals with quite general convergence methods. These methods are explained in the next section, where further necessary prerequisites are recorded. The proof of the main result takes place in Section 3, and the paper closes with a short discussion of complementary results and open problems.

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2 Mean Asymptotic Toeplitzness

Let us call an infinite matrix $A = [a_{i,j}]_{i,j=0}^{\infty}$ *regular* if it transforms each convergent sequence, viewed as a column vector, into another convergent one with the same limit. According to a classical theorem associated with Hausdorff, Toeplitz, Silverman, Steinhaus, and perhaps others (see, e.g., [10, Chapter 5, Exercise 15]), regular matrices are characterized by the conditions:

$$\lim_{i\to\infty} a_{i,j} = 0 \qquad \text{(columns } \to 0\text{)}, \tag{1}$$

$$\sup_i \sum_{j=0}^{\infty} |a_{i,j}| < \infty \qquad \text{(rows bounded in } \ell^1\text{)}, \tag{2}$$

and

$$\lim_{i\to\infty} \sum_{j=0}^{\infty} a_{i,j} = 1 \qquad \text{(row-sums} \to 1\text{)}. \tag{3}$$

Let A be such a matrix. Let's say a bounded linear operator T on H^2 is A-*asymptotically Toeplitz* if A transforms the operator sequence $(S^{*n}TS^n)$ into one that converges strongly in H^2. The special case $A = I$ (the identity matrix) recovers the original Halmos-Barría notion of asymptotic Toeplitzness, while the regularity of A insures that every asymptotically Toeplitz operator is A-asymptotically Toeplitz.

A matrix is called *strongly regular* if, in addition to the regularity conditions (1)–(3) its *row variations* converge to zero, i.e.,

$$\lim_{i\to\infty} \sum_{j=0}^{\infty} |a_{i,j} - a_{i,j+1}| = 0. \tag{4}$$

The strongly regular matrices are precisely the ones that are *translative* in the sense that whenever one of them transforms a bounded sequence into one that has a limit, it transforms the backward shift of that sequence to into one that has the same limit [9, Theorem 1.5.4, page 21].

Here is the main result of this paper:

Theorem 1 *If A is a strongly regular matrix then every composition operator on H^2 is A-asymptotically Toeplitz.*

The proof of this result will occupy the next section, with the rest of this one devoted to its consequences for Cesàro, and more generally Nörlund, matrices.

Nörlund matrices. Let $p = (p_n)_0^\infty$ be a sequence of non-negative real numbers. For each non-negative integer n set $P_n = \sum_{k=0}^n p_k$. The sequence p generates a *Nörlund matrix* $N(p) = (\nu_{i,j}(p))_{i,j=0}^\infty$, where

$$\nu_{i,j}(p) = \begin{cases} \frac{p_{i-j}}{P_i} & \text{if } 0 \le j \le i \\ 0 & \text{if } j > i. \end{cases}$$

Thus $N(p)$ is a lower-triangular matrix with non-negative entries, and it is not difficult to show that

$N(p)$ is regular if and only if $p_n/P_n \to 0$

(see [2, Theorem 3.3.3, page 127], for example).

Cesàro matrices. For $\alpha > 0$ the *Cesàro (C,α)-matrix* is the Nörlund matrix generated by the sequence $p^{(\alpha)} = (p_n^{(\alpha)})_0^\infty$, where $p_n^{(\alpha)}$ is the n-th coefficient in the MacLaurin expansion of $(1-z)^{-\alpha}$, that is, $p_0^\alpha = 1$ and

$$p_n^{(\alpha)} = \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!} \qquad (n = 1, 2, \ldots)\,. \tag{5}$$

Note that $p^{(1)}$ is the sequence that is identically 1, so the $(C,1)$-matrix is the classical one that takes a sequence to its sequence of arithmetic means. If α is an integer > 1, then the (C,α)-matrix is *not* the α-th power of the $(C,1)$-matrix, but it is known to be equivalent to this matrix in the sense that the two matrices produce the same convergent sequences with the same limits (this is the Knopp-Schnee Theorem; see, e.g., [2, Theorem 3.1.16, page 111] for more details). Our application of Cesàro convergence to the mean toeplitzness of composition operators is a special case of the following corollary of Theorem 1.

Corollary 2 *If*

$$\sum_n p_n = \infty \quad \textit{and} \quad \lim_n \frac{1}{P_n} \sum_{j=0}^n |p_j - p_{j+1}| = 0 \tag{6}$$

then every composition operator on H^2 is $N(p)$-asymptotically Toeplitz.

Proof By Theorem 1 it is enough to show that $N(p)$ is strongly regular. Let us recall the notation $P_n = p_0 + p_1 + \cdots + p_n$, and set

$$V_n = \sum_{j=0}^{n} |p_j - p_{j+1}| \qquad (n = 0, 1, 2, \ldots).$$

Then the first hypothesis of (6) states that $P_n \to \infty$. This, along with the second hypothesis and the fact that $p_n \leq V_n + p_0$, implies that $p_n/P_n \to 0$, hence $N(p)$ is regular.

The total variation of the i-th row of $N(p)$ is

$$\frac{V_{i-1} + p_0}{P_i} \leq \frac{V_i - p_0}{P_i} \to 0 \qquad (i \to \infty)$$

by (6), hence the row-variation condition (4) required for Theorem 1 is satisfied. □

Corollary 3 *Every composition operator on H^2 is (C, α)-asymptotically Toeplitz for each $\alpha > 0$.*

Proof Recall that $p_n^{(\alpha)}$ is the n-th Maclaurin coefficient of $(1-z)^{-\alpha}$. Thus $P_n^{(\alpha)}$, which is $\sum_{k=0}^{n} p_k^{(\alpha)}$, is the n-th coefficient of $(1-z)^{-1}$ times this function, i.e., the n-th coefficient of $(1-z)^{-(\alpha+1)}$. Thus $P_n^{(\alpha)} = p_n^{(\alpha+1)}$. Now by (5),

$$p_{n+1}^{(\alpha)} = \frac{n+\alpha}{n+1}\, p_n^{(\alpha)} \qquad \text{and} \qquad p_n^{\alpha+1} = \frac{\alpha+n}{\alpha} p_n^{(\alpha)}\,. \tag{7}$$

The first of these equations shows that $p_n^{(\alpha)} \nearrow \infty$ whenever $\alpha > 1$. In particular, $P_n^{(\alpha)} = p_n^{(\alpha+1)} \nearrow \infty$ for any $\alpha > 0$, so the first condition of (6) is established.

As for the second condition of (6), note from the first equality of (7) that the sequence $p(\alpha)$ is monotone increasing if $\alpha > 1$, decreasing if $\alpha > 1$, and (as noted earlier) constant if $\alpha = 1$. Thus, borrowing notation from the proof of Corollary 2),

$$V_n = |p_0^{(\alpha)} - p_{n+1}^{(\alpha)}| = |1 - p_{n+1}^{(\alpha)}|,$$

so

$$\frac{V_n}{P_n^{(\alpha)}} = \pm \frac{1 - p_n^{(\alpha)}}{p_{n+1}^{(\alpha)}}\,,$$

which, by the second equality of (7), converges to zero as $n \to \infty$. Thus the hypotheses of Corollary 2 are satisfied, and the result is proved. □

3 Proof of Main Theorem

Recall from the Introduction that a bounded linear operator T on H^2 is:

- *Asymptotically Toeplitz* if, for every $f \in H^2$, the sequence of vectors $(S^{*n}TS^n f)_0^\infty$ converges, and
- A-*asymptotically Toeplitz* for a regular matrix $A = [a_{i,j}]_{i,j=0}^\infty$ if, for every $f \in H^2$, the sequence of vectors

$$\left(\sum_{j=0}^{\infty} a_{i,j} S^{*j} T S^j f \right)_{i=0}^{\infty}$$

converges.

In both cases, the convergence is in the norm topology of H^2.

Our goal in this section is to prove the following more precise version of Theorem 1:

Theorem 4 *If A is a strongly regular matrix and φ is a holomorphic self map of $\mathbb{U}$ that is not the identity map, then for every $f \in H^2$:*

$$\lim_{i \to \infty} \| \sum_{j=0}^{\infty} a_{i,j} S^{*j} C_\varphi S^j f \| = 0.$$

In the excluded case, that of the identity map on $\mathbb{U}$, the induced composition operator is the identity operator on H^2, which is Toeplitz, hence trivially A-asymptotically Toeplitz. Thus Theorem 4 implies Theorem 1.

For the proof of Theorem 4, recall that each Toeplitz operator on H^2 has the form

$$T_g f = P(gf) \qquad (f \in H^2),$$

where $g \in L^\infty(\partial\mathbb{U})$ and P is the orthogonal projection of $L^2(\partial\mathbb{U})$ onto H^2, where now we identify H^2 with the collection of $L^2(\partial\mathbb{U})$-functions whose Fourier coefficients of negative index vanish [4, Chapter 7, page 177]. Thus the shift S is the Toeplitz operator T_z, the Toeplitz operator induced by the identity map on $\partial\mathbb{U}$, and $S^* = T_{\overline{z}}$. Crucial to our enterprise is the observation that

$$S^{*j} C_\varphi S^j = T_{\psi^j} C_\varphi \qquad (j = 0, 1, 2, \ldots)$$

where

$$\psi(\zeta) = \overline{\zeta}\varphi(\zeta) \qquad (\zeta \in \partial\mathbb{U}).$$

Thus, for each non-negative integer i,

$$\sum_{j=0}^{\infty} a_{i,j}\, S^{*j} C_\varphi S^j = T_{\Psi_i} C_\varphi$$

where $\Psi_i = \sum_{j=0}^{\infty} a_{i,j}\psi^j$, and so our goal becomes that of showing, for each $f \in H^2$, that

$$0 = \lim_{i\to\infty} \|T_{\Psi_i} C_\varphi f\| = \lim_{i\to\infty} \|P[\Psi_i \cdot (f \circ \varphi)]\|.$$

For this it will suffice, in view of the boundedness of both the projection P and the composition operator C_φ, to show that

$$0 = \lim_{i\to\infty} \|\Psi_i f\| \qquad (f \in H^2),$$

where now the norm is that of $L^2(\partial\mathbb{U})$.

Our argument will focus on the set $E = \{\zeta \in \partial\mathbb{U} : \psi(\zeta) \neq 1\}$ which, in view of the definition $\psi(\zeta) = \overline{\zeta}\psi(\zeta)$, is also the set of points $\zeta \in \partial\mathbb{U}$ at which $\varphi(\zeta) \neq \zeta$. Two bounded analytic functions coincide once their radial limit functions coincide on a subset of $\partial\mathbb{U}$ having positive measure ([5, Theorem 2.2, page 17], [10, Theorem 17.18, page 340]) so, because we are assuming that φ is not the identity map on $\mathbb{U}$, the set E has full measure in $\partial\mathbb{U}$.

By a straightforward calculation, aided by the fact that $|\psi| \leq 1$ a.e on $\partial\mathbb{U}$ and the convergence to zero of the rows of the matrix A (they are summable, recall), we see that for each index i:

$$(1-\psi)\Psi_i = (1-\psi)\sum_{j=0}^{\infty} a_{i,j}\psi^j = a_{i,0} + \sum_{j=1}^{\infty}(a_{i,j} - a_{i,j-1})\psi^j$$

at each point of $\partial\mathbb{U}$ where the radial limit of φ exists. In particular, at each point of E we have

$$|\Psi_i| \leq \frac{1}{|1-\psi|}\left(|a_{i,0}| + \sum_{j=1}^{\infty}|a_{i,j} - a_{i,j-1}|\right),$$

and since the sum on the right can be rewritten as $\sum_{j=0}^{\infty}|a_{i,j} - a_{i,j+1}|$, conditions (1) and (4) guarantee that $\Psi_i \to 0$ pointwise on E, hence pointwise a.e. on $\partial\mathbb{U}$.

Now for each index i observe that a.e. on $\partial\mathbb{U}$:

$$|\Psi_i| \leq \sum_{j=0}^{\infty} |a_{i,j}||\psi|^j \leq \sum_{j=0}^{\infty} |a_{i,j}| \leq M,$$

where $M = \sup_i \sum_{j=0}^{\infty} |a_{i,j}|$ is finite by the regularity condition (2) on A. Thus for each $f \in H^2$ we have a.e. on $\partial\mathbb{U}$:

$$\sup_i |\Psi_i f| \leq M|f| \quad \text{and} \quad \lim_{i\to\infty} \Psi_i f = 0.$$

From this and the Lebesgue Dominated Convergence Theorem:

$$\lim_{i\to\infty} \|\Psi_i f\|^2 = \lim_{i\to\infty} \int_{\partial\mathbb{U}} |\Psi_i f|^2 \, dm = 0$$

where m denotes normalized Lebesgue measure on $\partial\mathbb{U}$. This establishes the desired is A-asymptotic toeplitzness for C_φ. □

4 Closing Remarks

Here are some issues raised by the preceding work.

Weak Asymptotic Toeplitzness. Theorem 4 asserts that if φ is not the identity map then each strongly regular matrix transforms the operator sequence $(S^{*n}C_\varphi S^n)$ into one that converges strongly (i.e., pointwise on H^2) to zero. We know from [8], however, that the question of strong convergence of the *un-transformed* operator sequence is an interesting one with some surprising twists. Furthermore, it is still not known—this time for φ not a rotation—if the un-transformed sequence always converges *weakly.* Note that by our Theorem 4, if φ is not the identity then the only possible limit—weak, strong, or uniform—for that sequence is the zero-operator. Here is a succinct statement of the question:

> *If φ is neither the identity nor a rotation about the origin, then must the operator sequence $(S^{*n}C_\varphi S^n)$ converge weakly to zero?*

More precisely, the question asks whether or not, for the indicated maps φ,

$$\lim_{n\to\infty} \langle S^{*n}C_\varphi S^n f,\, g\rangle = 0$$

for every pair of functions $f, g \in H^2$. In the case where φ is a rotation it is easy to see that the sequence in question does not converge weakly, while for φ the identity map, C_φ is the identity operator which is obviously Toeplitz.

G-matrices. Let us call a regular matrix a "G-matrix" if, for all but perhaps finitely many $\omega \in \partial\mathbb{U}$, it transforms the geometric sequence (ω^n) into a sequence that converges to zero. Although this was not stated explicitly in the proof of Theorem 4, a crucial step involved showing that *every strongly regular matrix is a G-matrix.* The rest of the proof of Theorem 4 actually showed that the result is true for the class of G-matrices. To see that this class of matrices is strictly larger than the strongly regular ones, just take any strongly regular matrix and insert, between each of its original columns, a column of zeros. The result is still a G-matrix, but its row variations no longer tend to zero.

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