Szemerédi's Theorem via Ergodic Theory

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Arithmetic Progressions

Definition (Arithmetic Progressions)

An *arithmetic progression* is a finite sequence of integers

$$\{a + jb\}_{j=0}^{k-1}$$

where $b \neq 0$. The *length* of an arithmetic progression is the number of terms in the sequence.

- For example, $\{2, 4, 6\}$ and $\{-100, 5, 110, 215\}$ are arithmetic progressions of length three and four.

van der Waerden's theorem

Theorem (van der Waerden, 1927)

Suppose that the integers are partitioned into finitely many sets. Then some member of the partition contains an arithmetic progression arbitrary length.

$$\ldots, -3, \; -2, \; -1, \; 0, \; 1, \; 2, \; 3, \; \ldots$$

Density

Definition (Upper Density)

Let $A \subset \mathbb{Z}$. The *upper density* of A is the real number

$$\overline{d}(A) = \limsup_{N \to \infty} \frac{1}{2N+1} |A \cap \{-N, ..., N\}|.$$

- $\overline{d}(2\,\mathbb{Z}) = \frac{1}{2}$.
- $\overline{d}(\text{primes}) = 0$.

Density

- Why lim sup?
- The natural density $\lim_N \frac{1}{2N+1}|A \cap \{-N, ..., N\}|$ does not exist in general.
- Example: Set $I_n = \{1 - 2^{n+1}, ..., -2^n\} \cup \{2^n, ..., 2^{n+1} - 1\}$. Then $\bigcup_{n\in\mathbb{N}} I_{2n}$ does not have a natural density.

Hungarian School

- Ask hard questions and see what theory develops.

- Conjecture (Erdös, Turán (1936))
  Let $A \subset \mathbb{Z}$ have $\overline{d}(A) > 0$. Then A contains an arithmetic progression of length k for all $k \in \mathbb{N}$.
  - Roth (1953): Case $k = 3$ using harmonic analysis.
  - Szemerédi (1969): Case $k = 4$ using combinatorics.
  - Szemerédi (1975): Any k, using combinatorics.

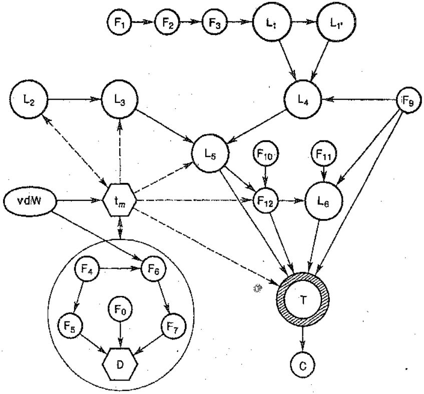
Szemerédi

Theorem (Szemerédi (1975))

Let $A \subset \mathbb{Z}$ have $\overline{d}(A) > 0$. Then A contains an arithmetic progression of length k for all $k \in \mathbb{N}$.

- Only restriction on A is that it have positive size or content.
- Szemerédi's proof is notable for only using "elementary methods", though is very intricate.

Szemerédi



Known Methods of Proof

- Combinatorics (Szemerédi, 1975)
- Ergodic theory (Furstenberg, 1977 / 1979 with Katznelson)
- Harmonic analysis (Gowers, 2001)
- Graph and hypergraph theory (Szemerédi, Rusza for k =3 1978 / general case Gowers et al 2006)

Furstenberg's Proof

- Furstenberg (1977): General case using ergodic theory.
- Furstenberg's proof is simpler than Szemerédi's in many ways.
- Generalized to produce many new interesting results.

Sequence Spaces

- Let $\Omega = 2^{\mathbb{Z}} = \{\{\omega_n\}_{n=-\infty}^{\infty} : \omega_n \in \{0,1\}\}$ be the space of bi-infinite sequences with coordinates in $\{0,1\}$.
- The map $A \mapsto \chi_A$ sends a subset $A \subset \mathbb{Z}$ into Ω.
- The shift map $S : \Omega \to \Omega$ is defined by

$$S(\ldots, \omega_{-1},\ \hat{\omega_0},\ \omega_1, \ldots) = (\ldots, \omega_{-1},\ \omega_0,\ \hat{\omega_1}, \ldots).$$

Sequence Spaces

- When given the product topology, the space $\Omega = 2^{\mathbb{Z}}$ is compact and metrizable.
- One such metric is

$$d(\omega, \omega') = \inf\{\frac{1}{k+1} : \omega_j = \omega'_j \ \text{for} \ |j| < k\}$$

when $\omega_0 = \omega'_0$, and $d(\omega, \omega') = 1$ if $\omega_0 \neq \omega'_0$

- The shift map S is a homeomorphism with respect to this topology

Arithmetic Progressions in Sequence Space

- $\{m + jb\}_{j=0}^{k} \subset A$ if and only if $\chi_A(m) = \chi_A(m + b) = ... = \chi_A(m + kb) = 1$
- That is, if and only if $(S^m \chi_A)(0) = (S^{m+b}\chi_A)(0) = ... = (S^{m+kb}\chi_A)(0) = 1.$
- Let $\Omega_0 = \{\omega : \omega_0 = 1\}$. Then

$$\{m + jb\}_{j=0}^{k} \subset A \iff S^m \chi_A \in \bigcap_{j=0}^{k} S^{-jb}(\Omega_0)$$

A Focused Sequence Space

- Let $X = \text{cl}\,\{S^n \chi_A\}_{n=-\infty}^{\infty}$.
- X is a compact metric space, and is S-invariant.
- The set $X_0 = \Omega_0 \cap X = \{x \in X : x_0 = 1\}$ is open in X.
- So $\bigcap_{j=0}^{k} S^{-jb}(X_0)$ is open in X.

A Focused Sequence Space

- Suppose $\bigcap_{j=0}^{k} S^{-jb}(X_0) \neq \emptyset$.
- As $\{S^n \chi_A\}_{n=-\infty}^{\infty}$ is dense in X, then there is an $m \in \mathbb{Z}$ so that $S^m(\chi_A) \in \bigcap_{j=0}^{k} S^{-jb}(X_0)$.
- Therefore, to show that A contains an arithmetic progression of length $k+1$, it suffices to show that $\bigcap_{j=0}^{k} S^{-jb}(X_0) \neq \emptyset$ for some b depending on k.

Measure Preserving Systems

Definition

Let $(Y, \mathcal{B}, \mu)$ be a probability space. Let $T : Y \to Y$ be a measurable map so that $\mu(T^{-1}(E)) = \mu(E)$ for all $E \in \mathcal{B}$. Then the quadruple $(Y, \mathcal{B}, \mu, T)$ is called a *measure preserving system*.

$(Y, \mathcal{B}, \mu, T)$ is *ergodic* if $T^{-1}(B) = B$, where $B \in \mathcal{B}$, implies $\mu(B) = 0$ or 1.

- Furstenberg noticed that $\bigcap_{j=0}^{k} S^{-jb}(X_0) \neq \emptyset$ will follow from a generalization of the Poincaré recurrence theorem applied to $(X, \mathcal{B}, \mu, S)$ for some suitable measure μ, and where $\mathcal{B}$ are the Borel subsets of X.

Recurrence

Theorem (Poincaré-Carathéodory, 1919)

Let $(Y, \mathcal{B}, \mu, T)$ *be a measure preserving system. Let* $B \in \mathcal{B}$ *with* $\mu(B) > 0$. *Then* $\mu(B \cap T^{-n}B) > 0$ *for some* $n \geq 1$.

Theorem (Multiple Recurrence, Furstenberg, 1977)

Let $(Y, \mathcal{B}, \mu, T)$ *be a measure preserving system. Let* $B \subset \mathcal{B}$ *with* $\mu(B) > 0$. *Then for all* k, *there is an* $n \geq 1$ *so that*

$$\mu(\bigcap_{j=0}^{k} T^{-jn}(B)) > 0$$

The Correspondence

Theorem

Furstenberg's multiple recurrence theorem implies Szemerédi's theorem.

Outline of proof:

- $(X, \mathcal{B}, *, S)$ is close to being a measure preserving system.
- Use functional analysis to summon a measure μ which is S-invariant, and so that $\mu(X_0) > 0$.
- Multiple recurrence theorem will then imply $\mu(\bigcap_{j=0}^{k} S^{-jb}(X_0)) > 0$.

Proof of the Correspondence

Proof.

- Suppose $A \subset \mathbb{Z}$ has $\overline{d}(A) > 0$.
- Let δ_n be the probability measures on X given by $\delta_n(B) = 1$ if $S^n(\chi_A) \in B$ and 0 otherwise.
- Define probability measures on $(X, \mathcal{B})$

$$\mu_N(B) = \frac{1}{2N+1} \sum_{n=-N}^{n=N} \delta_n.$$

Proof of Correspondence

Proof continued.

- Notice that $\delta_n(X_0) = 1$ if and only if $S^n(\chi_A) \in X_0$.
- That is, if and only if $\chi_A(n) = 1$
- Therefore

$$\mu_N(X_0) = \frac{1}{2N+1} \sum_{n=-N}^{n=N} \delta_n(X_0) = \frac{1}{2N+1} |A \cap \{-N, ..., N\}|.$$

Proof of Correspondence

Proof continued.

- Choose $\{\mu_{N_j}\}$ so that $\mu_{N_j}(X_0) \to \overline{d}(A) = \limsup_N \frac{1}{2N+1}|A \cap \{-N, ..., N\}|$.
- The sequence $\{\mu_{N_j}\}$ lives in the closed unit ball of $M(X)$.
- As X is a compact metric space, $C(X)$ is separable.
- By Alaoglu's theorem, the closed unit ball of $M(X)$ is sequentially compact in the weak* topology.

Proof of Correspondence

Proof continued.

- Let μ be a weak* limit point of $\{\mu_{N_j}\}$.
- Then $\mu(X_0) = \overline{d}(A) > 0$.
- The only thing left to show is that $\mu \circ S^{-1} = \mu$.

Proof of Correspondence

Proof continued.

- Notice that $\mu_N \circ S^{-1} = \frac{1}{2N+1} \sum_{n=-N}^{N} \delta_{n+1}$.
- So $\mu_N \circ S^{-1} - \mu_N = \frac{1}{2N+1}(\delta_{N+1} - \delta_{-N})$.
- $||\delta_{N+1} - \delta_{-N}|| \leq 2$ so $\mu_N \circ S^{-1} - \mu_N \to 0$
- So $\mu \circ S^{-1} - \mu = 0$.

□

Bernoulli Schemes

- Bernoulli schemes are sequence spaces which model infinite trials.
- Let $F = \{1, ..., r\}$ be a finite set and let $p = (p_1, ..., p_r)$ be a probability vector on F; so $p_j \geq 0$ and $\sum_j p_j = 1$.
- A *cylinder set* $C \subset F^{\mathbb{Z}}$ is defined on finitely many coordinates $C = \{(..., x_{-1}, x_0, x_1, ...) : x_{i_1} = j_1, ..., x_{i_n} = j_n\}$, $j_k \in F$.
- E.g. $C = \{(..., x_{-1}, x_0, x_1, ...) : x_{-1} = 3,\ x_{10} = 2\}$ is a cylinder set.

Bernoulli Schemes

Definition

Let $F = \{1, ..., r\}$ be a finite set, let $p = (p_1, ..., p_r)$ be a probability vector on F and let $\mathcal{B}$ be the σ-algebra generated by cylinder sets. Define μ to be

$$\mu(\{x : x_{i_1} = j_1, ..., x_{i_n} = j_n\}) = p_{j_1} \cdots p_{j_n}$$

on cylinder sets. Extend μ to $\mathcal{B}$, and let S be the shift map $S(..., x_{-1}, \hat{x_0}, x_1, ...) = (x_{-1}, x_0, \hat{x_1}, ...)$. Then $(F^{\mathbb{Z}}, \mathcal{B}, \mu, S)$ is a measure preserving system, and is called a *Bernoulli scheme*.

Bernoulli Schemes

Example: Infinite Coin Flips

- Let $F = \{H, T\}$, $p = (1/2, 1/2)$.
- $(F^{\mathbb{Z}}, \mathcal{B}, \mu)$ models flipping a coin infinitely many times.
- $\mu(\{x : x_{-10} = T, x_2 = H, x_3 = T\}) = \frac{1}{2}\frac{1}{2}\frac{1}{2} = \frac{1}{8}$

Bernoulli Schemes

- Bernoulli schemes satisfy the multiple recurrence theorem.
- If C is a cylinder set, then C is defined on finitely many coordinates.
- Therefore there are infinitely many n so that $S^{-n}(C)$ and C are defined on different coordinates.
- For such n, then $\mu(S^{-n}(C) \cap C) = \mu(C)\mu(C) = \mu(C)^2$.
- For any k, we can find n (which will be large) so that $\mu(C \cap S^{-n}(C) \cap ... \cap S^{-kn}(C)) = \mu(C)^{k+1} > 0$.

Bernoulli Schemes

- If A is measurable and $\mu(A) > 0$, we can approximate A by cylinder sets.
- As n gets very large, the cylinder sets approximating A will be all defined on different coordinates
- So $\lim_{n\to\infty} \mu(A \cap S^{-n}(A) \cdots \cap S^{-kn}(A)) = \mu(A)^{k+1} > 0$.
- This implies that multiple recurrence holds for Bernoulli schemes.

Circle Rotations

- Let $(S^1, \mathcal{B}, \mu)$ be the probability space on the circle $S^1 \cong \mathbb{R} / \mathbb{Z}$ with Borel sets $\mathcal{B}$ and normalized Lebesgue measure μ.
- For $\alpha \in [0, 1)$ define $T_\alpha : S^1 \to S^1$ by $T_\alpha(x) = x + \alpha \pmod 1$
- The quadruple $(S^1, \mathcal{B}, \mu, T_\alpha)$ is a measure preserving system.
- If α is rational then T_α is periodic.
- Taking n to be the period of T_α shows $\mu(A \cap T^{-n}(A) \cap \ldots \cap T^{-kn}(A)) = \mu(A)$ for any k.
- Link

Circle Rotations

- If $\alpha \notin \mathbb{Q}$, then $\{m\alpha\}_{m=0}^{\infty}$ is dense in S^1.
- For any $\delta > 0$ we can choose $n \geq 1$ so that $0, n\alpha, 2n\alpha, ..., kn\alpha \in (-\delta, \delta)$ mod 1.
- In fact, the set of n so that $0, n\alpha, ..., kn\alpha \in (-\delta, \delta)$ mod 1 has positive lower density.

Circle Rotations

- For these n, $T^{-n}, ..., T^{-kn}$ are close to the identity.
- So $A \cap T^{-n}(A) \cap ... \cap T^{-kn}(A)$ has measure close to that of A.
- In fact,

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T^{-n}(A) \cap \cdots \cap T^{-kn}(A)) > C\mu(A) > 0$$

for some positive constant C, and where $\mu(A) > 0$.

SZ System

Definition (SZ)

A measure preserving system $(X, \mathcal{B}, \mu, T)$ is *SZ* or is a *SZ system* if for $A \in \mathcal{B}$ with $\mu(A) > 0$,

$$\liminf_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T^{-n}(A) \cap \ldots \cap T^{-kn}(A)) > 0$$

for any $k \in \mathbb{N}$.

- Any SZ system satisfies the multiple recurrence theorem.
- This definition allows us to not worry about n.

SZ Systems

- Bernoulli scheme: $\mu(A \cap T^{-n}(A) \cap ... \cap T^{-kn}(A)) \sim \mu(A)^{k+1}$ for large n.
- Circle rotation: $\mu(A \cap T^{-n}(A) \cap ... \cap T^{-kn}(A)) \sim \mu(A)$ for infinitely many n.

Weak Mixing Systems

The Bernoulli scheme is an example of a *weak mixing system*.

Definition

A measure preserving system $(X, \mathcal{B}, \mu, T)$ is *weak mixing* if the product system $(X \times X, \mathcal{B} \otimes \mathcal{B}, \mu \times \mu, T \times T)$, where $T \times T(x, x') = (Tx, Tx)$, is ergodic.

Weak Mixing Systems

- The circle rotation is not weak mixing.
- The function $f : S^1 \times S^1 \to \mathbb{C}$ given by $f(x, y) = e^{2\pi i(x-y)}$ is non-constant.
- Yet $(f \circ T_\alpha \times T_\alpha)(x, y) = e^{2\pi i(x+\alpha-y-\alpha)} = f(x, y)$.
- So the product system $(S^1 \times S^1, \mathcal{B} \times \mathcal{B}, \mu \times \mu, T_\alpha \times T_\alpha)$ is not ergodic.

Compact Systems

The circle rotation is an example of a *compact* system.

Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system. Given $f \in L^2(X, \mathcal{B}, \mu)$, let $U_T : L^2(X, \mathcal{B}, \mu) \to L^2(X, \mathcal{B}, \mu)$ be the map $U_T(f) = f \circ T$.

Definition

$(X, \mathcal{B}, \mu, T)$ is *compact* if $\mathrm{cl}\ \{U_T^n(f)\}_{n=0}^{\infty}$ is compact in $L^2(X, \mathcal{B}, \mu)$ for all f in $L^2(X, \mathcal{B}, \mu)$.

Compact and Weak Mixing systems are SZ

- Both compact and weak mixing systems are SZ.
- Unfortunately, not every system is weak mixing or compact.
- Example: Take the product of a Bernoulli scheme with a circle rotation.

Dichotomy

Theorem

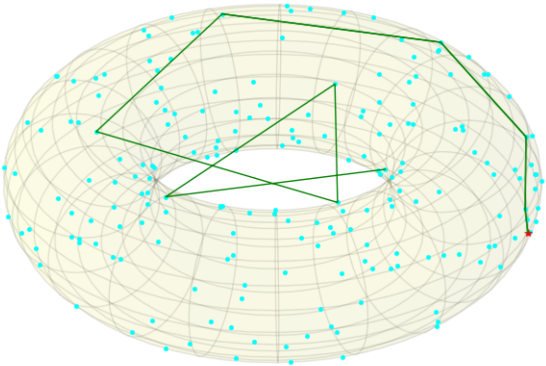
Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system. X is weak mixing if and only if there is no non-trivial T-invariant sub-σ-algebra $\mathcal{B}' \subset \mathcal{B}$ so that $(X, \mathcal{B}', \mu, T)$ is compact.

T-invariant sub-σ-algebras are known as *factors* of a measure preserving system.

Skew Torus

- The measure preserving system $(S^1 \times S^1, \mathcal{B} \otimes \mathcal{B}, \mu \times \mu, T)$, where $T(x, y) = (x + \alpha, x + y)$, is called the *skew torus*.
- The factor $\mathcal{B} \times \{\emptyset, S^1\}$ defines a system which is isomorphic to the circle system.
- So the skew torus is not weak mixing.

Skew Torus



Skew Torus

- The skew torus is not compact.
- $T^n(x,y) = (x + n\alpha, y + nx + \frac{n(n-1)}{2}\alpha) \pmod 1$
- If $f(x,y) = e^{2\pi iy}$, then $(U_T^n e^{2\pi iy})(x,y) = e^{2\pi iy + \frac{n(n-1)}{2}\alpha} e^{2\pi inx}$.
- $\langle U_T^n(f), U_T^m(f) \rangle = C \int e^{2\pi iy} e^{-2\pi iy} e^{2\pi i(nx-mx)}\, d(\mu \times \mu) = 0$ for $m \neq n$.
- $\implies \{U_T^n(f)\}_n$ not totally bounded.

Conditional Expectation

Definition

- $(X, \mathcal{B}, \mu, T)$ a measure preserving system and $\mathcal{A} \subset \mathcal{B}$ be a factor; so $\mathcal{A}$ is T-invariant and is a sub-σ-algebra.
- $L^2(X, \mathcal{A}, \mu)$ is a closed subspace of $L^2(X, \mathcal{B}, \mu)$
- Let $E(*|\,\mathcal{A}) : L^2(X, \mathcal{B}, \mu) \to L^2(X, \mathcal{A}, \mu)$ be the orthogonal projection.
- The *conditional expectation* of $f \in L^2(X, \mathcal{B}, \mu)$ is $E(f|\,\mathcal{A})$.

Example: $\mathcal{A} = \{\emptyset, X\}$, then $E(f|\,\mathcal{A}) = \int f \; d\mu$.

Skew Torus Revisited

- $\mathcal{A} = \mathcal{B} \times \{\emptyset, S^1\}$ defines factor isomorphic to the circle system.
- f is $\mathcal{A}$-measurable iff $f(x_0, y)$ is constant for fixed x_0.
- $E(f|\,\mathcal{A})(x) = \int f(x,y)\; d(\delta_x \times \mu)(x,y) \quad$ a.e.
- Note that $\int E(f|\,\mathcal{A})(x)\; d\mu(x) = \int f\; d(\mu \times \mu)$
- Also notice $\delta_x \times \mu \circ T^{-1} = \delta_{x+\alpha} \times \mu$
- The family $\{\delta_x \times \mu\}_{x \in S^1}$ is called a *disintegration* of $(S^1, \mathcal{B}, \mu)$ with respect to $\mathcal{A}$.

Disintegration Theorem

Under some assumptions:

- $\mathcal{A} \subset \mathcal{B}$ of a measure preserving system $(X, \mathcal{B}, \mu, T)$, there is a system $(Y, \mathcal{D}, \nu, S)$ and a measure preserving map $\pi : X \to Y$ so that: $\pi^{-1}(\mathcal{D}) = \mathcal{A}$, and $\pi \circ T = S \circ \pi$.
- π is called a *factor map*, and Y is called a factor of X or X is called an extension of Y.
- We can identify $L^2(X, \mathcal{A}, \mu)$ with $L^2(Y, \mathcal{D}, \nu)$.

Theorem

Let $(X, \mathcal{B}, \mu, T) \xrightarrow{\pi} (Y, \mathcal{D}, \nu, S)$ be an extension. Then there is a measurable map $y \mapsto \mu_y$ from Y into the Borel probability measures on X so that

$$E(f | \mathcal{A})(y) = \int f \, d\mu_y \quad a.e.$$

Moreover, $\mu_y \circ T^{-1}(B) = \mu_{Sy}(B)$ for all $B \in \mathcal{B}$, and μ_y is supported on $\pi^{-1}(y)$.

Furstenberg's Big Idea

- Using the disintegration theorem, we can define notions of compactness and weak mixing *relative* to a factor.
- We know one factor is SZ (the maximal compact factor)
- So if being SZ lifts through these relative extensions, then the multiple recurrence theorem is proved.

Details on Relative WM and Compactness

- An extension $(X, \mathcal{B}, \mu, T) \xrightarrow{\pi} (Y, \mathcal{D}, \nu, S)$ is *relatively weak mixing* if $(X \times X, \mathcal{B} \otimes \mathcal{B}, \mu \times_Y \mu, T \times T)$ is ergodic, where $\mu \times_Y \mu(B) = \int_Y \mu_y \times \mu_y(B)\, d\nu(y)$
- Given extension $(X, \mathcal{B}, \mu, T) \xrightarrow{\pi} (Y, \mathcal{D}, \nu, S)$, $f \in L^2(X, \mathcal{B}, \mu)$ is *almost periodic* with respect to Y if for each $\epsilon > 0$, there are finitely many $g_j \in L^2(X, \mathcal{B}, \mu)$ so that for almost all y,

$$\min_j ||U_T^n f - g_j||_y < \epsilon,$$

  where $||h||_y^2 = \int |h|^2 \, d\mu_y$.
- X is a *compact extension* of Y if the collection of almost periodic functions in $L^2(X, \mathcal{B}, \mu)$ are dense.

SZ in the relative cases

The main theorems from the absolute case generalize:

- If $(X, \mathcal{B}, \mu, T) \to (Y, \mathcal{D}, \nu, S)$ is a weak mixing extension, and Y is SZ, then so is X.
- If $(X, \mathcal{B}, \mu, T) \to (Y, \mathcal{D}, \nu, S)$ is a compact extension, and Y is SZ, then so is X.
- Dichotomy: if $X \to Y$ is not a weak mixing extension, then there is an intermediate factor $X \to Z \to Y$ which is a non-trivial compact extension of Y.

Limits of SZ factors

- Given $(X, \mathcal{B}, \mu, T)$, let $\{\mathcal{B}_\alpha\}_\alpha$ be a family of factors totally ordered by inclusion. Define $\sup_\alpha \mathcal{B}_\alpha = \sigma(\bigcup_\alpha \mathcal{B}_\alpha)$.
- If each $\mathcal{B}_\alpha$ is SZ, then so is $\sup_\alpha \mathcal{B}_\alpha$.
- Zorn's lemma $\implies$ there is a maximal SZ factor in X

Conclusion of Multiple Recurrence theorem

- Call the maximal SZ factor Y and suppose Y is a proper factor of X
- If $X \to Y$ is weak mixing, then X is SZ, contradicting maximality of Y.
- By dichotomy, there is a nontrivial compact extension $Z \to Y$.
- But then Z is SZ and contains Y properly, a contradiction.
- $\implies X$ must be the maximal SZ factor.
- This shows the multiple recurrence theorem, and therefore Szemerédi's theorem.

Furstenberg-Zimmer Structure Theorem

- An interesting and useful structure has come from this proof.
- This characterizes the factors of a measure preserving system into orderly and random factors.

$$(X, \mathcal{B}, \mu, T)$$

$\downarrow$ *Rel WM Ext.*

Limits of Limits of ... Compact Ext.

$\vdots$

Limit of Compact Ext

$\vdots$

Compact Ext

$\downarrow$

Max Compact Factor

$\downarrow$

Trivial Factor

Further Results

- Furstenberg and Katsnelson (1979): Commuting groups of measure preserving transformations and a multi-dimensional Szemerédi's theorem
- Bergelson and Leibman (1996): Multi-dimensional and polynomial version of Szemerédi's theorem
- Green and Tao (2004): Arithmetic progressions in the primes
- Host and Kra (2005): Showed $\lim \frac{1}{N} \sum_{n=1}^{\infty} U_T^n(f_1) \cdots U_T^{kn}(f_k)$ exists in L^2 for bounded functions.

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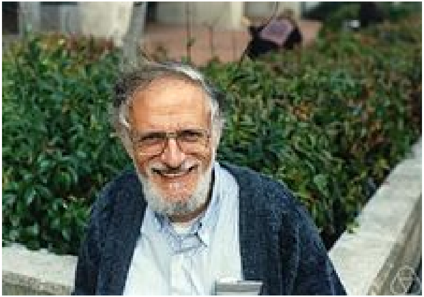
Thank You!



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