Nuclear Operators on Hilbert Space are Trace Class

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Nuclear Vs Trace-Class

X is a Banach space.
An operator T on X is said to be nuclear if there exists a sequence (h_n) in X and a sequence (γ_n) in X^* with $\sum_n \|h_n\|\|\gamma_n\| < \infty$ such that

$$T = \sum_n h_n \otimes \gamma_n.$$

For $h \in X$ and $\gamma \in X^*$, the rank-one operator $h \otimes \gamma$ on X is defined by

$$(h \otimes \gamma)(x) = \gamma(x)h \quad (x \in X).$$

H is a Hilbert space.
An operator T on H is said to be of trace class if it has the form

$$T(x) = \sum_n s_n \langle x, f_n \rangle g_n \quad (x \in H) \tag{1}$$

where (f_n) and (g_n) are orthonormal sequences in H, and (s_n) is a summable sequence of positive real numbers.

Nuclear implies trace class

For $f, g \in H$, we form the rank-one operator $g \otimes f$ on H by

$$(g \otimes f)(x) = \langle x, f \rangle g \quad (x \in H).$$

An operator T on H is said to be nuclear if, in (1) above, we require that $\|f_n\| = 1$, $\|g_n\| = 1$ for all n.

Theorem

If T is nuclear on the Hilbert space H, then T is trace class.

Proof. We will show that the singular sequence of T is summable. Since T is nuclear, it is the operator-norm limit of finite rank operators, hence compact. It therefore has a Singular-Value Decomposition $T = \sum_j s_j(f_j \otimes e_j)$, with each $s_j \geq 0$, and with (f_j) and (e_j) orthonormal lists of vectors in H. Fix an index n and observe that

$$Te_n = \sum_j s_j \langle e_n, e_j \rangle f_j = s_n f_n.$$

So,

$$s_n = \langle Te_n, f_n \rangle. \tag{2}$$

Now the nuclear representation of T is $T = \sum_j h_j \otimes \gamma_j$, where $\sum_j \|h_j\| \|\gamma_j\| < \infty$. (Now γ_j is just a vector in H.)
Thus

$$Te_n = \sum_j (h_j \otimes \gamma_j) e_n = \sum_j \langle e_n, \gamma_j \rangle h_j.$$

It follows from (2) above that

$$s_n = \langle Te_n, f_n \rangle = \sum_j \langle e_n, \gamma_j \rangle \langle h_j, f_n \rangle \leq \sum_j |\langle e_n, \gamma_j \rangle| |\langle h_j, f_n \rangle|.$$

Therefore,

$$\begin{aligned} \sum_n s_n &\leq \sum_n \sum_j |\langle e_n, \gamma_j \rangle| |\langle h_j, f_n \rangle| = \sum_j \sum_n |\langle e_n, \gamma_j \rangle| |\langle h_j, f_n \rangle| \\ &\leq \sum_j \Big(\sum_n |\langle e_n, \gamma_j \rangle|^2 \Big)^{1/2} \Big(\sum_n |\langle h_j, f_n \rangle|^2 \Big)^{1/2} \\ &= \sum_j \|\gamma_j\| \|h_j\| < \infty. \end{aligned}$$

Trace-Class Operators

Recall: For each $T \in B(H)$, the Spectral Theorem provides for the positive operator T^*T a unique positive square root $|T| := \sqrt{T^*T}$.

- We say that $T \in B(H)$ belongs to the Trace-Class ($B_1(H)$) whenever $\sum_n \langle |T| e_n, e_n \rangle < \infty$ for some basis (e_n). Here $|T|$ is the positive square root of the positive operator T^*T.

- If $T \in B_1(H)$, we call the convergent and basis-independent series $\sum_n \langle |T| e_n, e_n \rangle$ the trace norm of T, and denote it by $\|T\|_1 := \sum_n \langle |T| e_n, e_n \rangle$.

Indeed, since T is of trace-class, it is compact (see below!). Therefore T has a Singular-Value Decomposition $T = \sum_j s_j (f_j \otimes e_j)$, with each $s_j \geq 0$, and with (f_j) and (e_j) orthonormal sequences in H.

Then

$$T^*T = \sum_j s_j^2 (e_j \otimes e_j)$$

with $|T| := \sqrt{T^*T} = \sum_j s_j(e_j \otimes e_j)$,
and so $|T|e_k = s_k e_k$.
Now

$$\infty > \|T\|_1 = \sum_k \langle |T|e_k, e_k\rangle = \sum_k s_k.$$

- If (e_n) is a basis for H, define the trace of T, $\mathrm{tr} : B_1(H) \to \mathbb{C}$ by

$$\mathrm{tr}(T) := \sum_n \langle Te_n, e_n\rangle.$$

If $\dim H < \infty$, then $\mathrm{tr}(T)$ is precisely the sum of the diagonal terms of any matrix representation of T.

Theorem (Polar Decomposition).

If $T \in B(H)$ then:

(a) There exists a partial isometry V such that $T = V|T|$, and

(b) V^* is a partial isometry with $V^*T = |T|$.

Theorem

Every Trace-class operator is compact.

Proof. Fix a trace-class operator T on H, and a basis (e_n) for H. Let S denote the unique positive square root of $|T|$. Then

$$\infty > \sum_n \langle |T|e_n, e_n\rangle = \sum_n \langle S^2 e_n, e_n\rangle = \sum_n \langle Se_n, Se_n\rangle = \sum_n \|Se_n\|^2.$$

Conclusion: S is Hilbert-Schmidt, hence compact, so $|T| = S^2$ is compact. By the Polar Decomposition, $T = W|T|$, where W is a partial isometry. Therefore, T is the product of a bounded operator and a compact one, so is compact.

Corollary ($B_1(H) \subset B_2(H) \times B_2(H)$).

Every trace-class operator is the product of two Hilbert-Schmidt operators.

Proof. From the proof above, we have, for $T \in B_1(H)$:

$$T = W|T| = (WS)S$$

where S is the Hilbert-Schmidt and W a bounded operator on H. Since the Hilbert-Schmidt operators form an ideal in $B(H)$, the operator WS is also Hilbert-Schmidt, and we are done.

Theorem

The Trace Theorem. If $T \in B_1(h)$, and (e_n) is a basis for H, then the series $tr(T) := \sum_n \langle Te_n, e_n \rangle$ is absolutely convergent, with its sum independent of the choice of the basis (e_n).

Proof. Write $T = QS$, a product of two Hilbert-Schmidt operators. Then

$$\begin{aligned}
\sum_n |\langle Te_n, e_n\rangle| &= \sum_n |\langle Se_n, Q^* e_n\rangle| \leq \sum_n \|Se_n\| \|Q^* e_n\| \\
&\leq \left(\sum_n \|Se_n\|^2\right)^{1/2} \left(\sum_n \|Qe_n\|^2\right)^{1/2} \\
&= \|S\|_2 \|Q\|_2 < \infty.
\end{aligned}$$

Theorem ($B_2(H) \times B_2(H) \subset B_1(H)$).

If S and T are Hilbert-Schmidt operators, then ST is trace-class, and $\|ST\|_1 \leq \|S\|_2 \|T\|_2$.

Proof. We need to show that if (x_n) is a basis for H then $\sum_n \langle |ST| x_n, x_n \rangle$ is finite. Recall from The Polar Decomposition that we can write $|ST| = V^*ST$, where V is a partial isometry. Thus,

$$\begin{aligned}
\|ST\|_1 &= \sum_n \langle |ST| x_n, x_n \rangle = \sum_n \langle V^*STx_n, x_n \rangle = \sum_n \langle Tx_n, (V^*S)^* x_n \rangle \\
&\leq \left(\sum_n \|Tx_n\|^2 \right)^{1/2} \left(\sum_n \|(V^*S)^* x_n\|^2 \right)^{1/2} \\
&= \|T\|_2 \|(V^*S)^*\|_2 = \|T\|_2 \|(V^*S)\|_2 \\
&\leq \|T\|_2 \|V^*\| \|S\|_2 = \|T\|_2 \|S\|_2.
\end{aligned}$$

Theorem ($B_2(H) \times B_2(H) = B_1(H)$).

An operator is trace class if and only if it is the product of two Hilbert-Schmidt operators.

Lemma

If $A \in B(H)$, then the following are equivalent:

(a) $A \in B_1(H)$

(b) $|A| = (A^*A)^{1/2} \in B_1(H)$

(c) $|A|^{1/2} \in B_2(H)$

(d) $\mathrm{tr}(|A|) < \infty$.

Proof. $(d) \Rightarrow (c)$. We have

$$\|\,|A|^{1/2}\|_2^2 = \sum_n \|\,|A|^{1/2}e_n\|^2 = \sum_n \langle |A|^{1/2}e_n, |A|^{1/2}e_n\rangle = \sum_n \langle |A|e_n, e_n\rangle = \mathrm{tr}(|A|).$$

Corollary

If $A \in B_1(H)$ and $T \in B(H)$, then

$$\Big|\,\mathrm{tr}(T|A|)\,\Big| \leq \|T\|\|A\|_1.$$

Proof. $|A|^{1/2} \in B_2(H)$ by Lemma and, hence, $T|A|^{1/2}$ and $|A|^{1/2}\ T$ belong to $B_2(H)$. Also

$$\mathrm{tr}(T|A|) = \sum_n \langle T|A|^{1/2}|A|^{1/2}e_n, e_n\rangle = \sum_n \langle |A|^{1/2}e_n, |A|^{1/2}\ T^* e_n\rangle.$$

Using the Cauchy-Schwartz inequality gives

$$\Big|\,\mathrm{tr}(T|A|)\,\Big| \le \sum_n \Big|\langle |A|^{1/2}e_n, |A|^{1/2}\ T^* e_n\rangle\Big| \le \sum_n \|\ |A|^{1/2}e_n\|\ \|\ |A|^{1/2}\ T^* e_n\| \le$$

$$\left(\sum_n \|\ |A|^{1/2}e_n\|^2\right)^{1/2}\left(\sum_n \|\ |A|^{1/2}\ T^* e_n\|^2\right)^{1/2} = \|\ |A|^{1/2}\|_2\ \|\ |A|^{1/2}\ T^*\|_2 \le$$

$$\|\ |A|^{1/2}\|_2\ \|\ |A|^{1/2}\|_2\ \|T\| = \|\ |A|^{1/2}\|_2^2\ \|T\|.$$

But

$$\|\ |A|^{1/2}\|_2^2 := \sum_n \|\ |A|^{1/2}e_n\|^2 = \sum_n \langle |A|^{1/2}e_n, |A|^{1/2}e_n\rangle := tr(|A|) = \|A\|_1.$$

This completes the proof.

Thank you