Hilbert-Schmidt Operators Are Compact

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Modulus and Partial Isometry

Let H be a complex Hilbert space and $A : H \to H$ is a bounded linear operator.

A is called positive and we write $A \geq 0$ if $\langle Ax, x \rangle \geq 0$ for all $x \in H$.
Fact: Every positive operator A on a complex-Hilbert space H is self-adjoint.

The modulus of A, denoted by $|A|$, is the unique positive operator $S : H \to H$ such that $S^2 = A^*A$, that is, $|A| = (A^*A)^{1/2}$ (The positive square root of A^*A.)

An operator $W \in B(H)$ is called a partial isometry if $\|Wh\| = \|h\|$ for all $h \in (\ker W)^\perp$. The space $(\ker W)^\perp$ is called the initial space of W and the space ran W is called the final space of W.

Theorem: (Polar decomposition)

If $A \in B(H)$, then there is a partial isometry W with $(\ker A)^{\perp}$ as its initial space and $\overline{\operatorname{ran} A}$ as its final space such that $A = W|A|$. Moreover, if $A = UP$ where $P \geq 0$ and U is a partial isometry with $\ker U = \ker P$, then $P = |A|$ and $U = W$.

The representation $A = UP$ as the product of the unique operators U and P satisfying the conditions of the theorem is called the polar decomposition of A.

Theorem: Spectral Theorem

Every positive operator A on a complex Hilbert space H has a (unique) positive square root.

Theorem: Basis-Independence I

If $T \in B(H)$, then the (possibly infinite) value of the positive-term sum $\sum_n \|Te_n\|^2$ does not depend on the choice (e_n) of basis.

Proof. First note that for any $x \in H$, we have $\|Tx\| = \||T|x\|$. Indeed:

$$\|Tx\|^2 = \langle Tx, Tx\rangle = \langle T^*Tx, x\rangle = \langle |T|^2 x, x\rangle = \langle |T|x, |T|x\rangle = \| \, |T|x \, \|^2.$$

Thus in proving our theorem, we may assume that T is a positive operator. Now suppose that (e_n) and (x_k) are two bases. By Parseval identity, $\|Te_n\|^2 = \sum_k |\langle Te_n, x_k\rangle|^2$ for each n. Hence,

$$\sum_n \|Te_n\|^2 = \sum_n \sum_k |\langle Te_n, x_k\rangle|^2 = \sum_k \sum_n |\langle Te_n, x_k\rangle|^2$$

$$= \sum_k \sum_n |\langle e_n, Tx_k\rangle|^2 = \sum_k \|Tx_k\|^2,$$

where in the third equality we use the fact that for T, positivity guarantees self-adjointness.

Corollary: Basis-Independence II

If $T \in B(H)$ is a positive operator, then the (possibly infinite) value of the positive-term sum $\sum_n \langle Te_n, e_n\rangle$ is independent of the basis (e_n).

Proof. Proof. By the Spectral Theorem, we know that T, being a positive operator, has a (unique) positive square root S.

Thus for each $x \in H$, we have

$$\langle Tx, x\rangle = \langle S^2 x, x\rangle = \langle Sx, Sx\rangle = \|Sx\|^2.$$

Hence for every basis (e_n), $\sum_n \langle Te_n, e_n\rangle = \sum_n \|Se_n\|^2$ and by the theorem, we know that the sum $\sum_n \|Se_n\|^2$ is independent of the basis.

- We say that $T \in B(H)$ belongs to the Hilbert-Schmidt class whenever $\sum_n \|Te_n\|^2 < \infty$ for some basis (e_n).

We use the notation HS or $B_2(H)$ to denote the Hilbert-Schmidt class.

- For $T \in HS$, we call $\|T\|_2 := \left(\sum_n \|Te_n\|^2\right)^{1/2}$ the Hilbert-Schmidt Norm of T, noting that it is basis independent.

Hilbert-Schmidt Operators

Note that HS is linear subspace of $B(H)$, and the above defined "norm" really is a norm on that subspace. Indeed, for $T \in HS$ and (e_n) a basis for H,

$$\|T\|_2^2 := \sum_j \|Te_j\|^2 = \sum_{j,k} |\langle Te_j, e_k \rangle|^2$$

which establishes an isometric isomorphism taking HS onto the "infinite-matrix space" $l^2(\mathbb{N} \times \mathbb{N})$.

Proposition: HS is an ideal in B(H)

The Hilbert-Schmidt class is a $\star$-ideal in $B(H)$, more precisely: if $T \in HS$ then $T^* \in HS$, with $\|T\|_2 = \|T^*\|_2$, and if $A \in B(H)$, then $AT \in HS$ with $\|AT\|_2 \leq \|A\| \|T\|_2$.

Proof. For (e_n) a basis in H,

$$\|AT\|_2^2 = \sum_j \|ATe_j\|^2 \leq \|A\|^2 \sum_j \|Te_j\|^2 = \|A\|^2 \|T\|_2^2.$$

It follows that $\|AT\|_2 \leq \|A\| \|T\|_2$, which establishes the "ideal-ness" of HS. To prove it's a $\star$-ideal, recall that for $T \in B(H)$ we have

$$\|T\|_2^2 = \sum_{j,k} |\langle Te_j, e_k \rangle|^2 = \sum_{j,k} |\langle e_j, T^* e_k \rangle|^2 = \|T^*\|_2^2.$$

So $T \in HS$ if and only if $T^* \in HS$, with equality of norms.

HS Implies Compact

Proposition

Every Hilbert-Schmidt operator is compact.

Proof. Fix a basis (e_n) for H, and a Hilbert-Schmidt operator T on H. So, $\sum_n \|Te_n\|^2 < \infty$. Each $f \in H$ has Fourier expansion $f = \sum_n \langle f, e_n \rangle e_n$ with the series convergent in the norm metric of H. Thus the continuity of T guarantees that $Tf = \sum_n \langle f, e_n \rangle Te_n$ with the series once again convergent in the norm metric of H.

For $N \in \mathbb{N}$ and $f \in H$, let $T_N f := \sum_{n=1}^{N} \langle f, e_n \rangle Te_n$. Then

$$\|Tf - T_N f\| = \| \sum_{n=N+1}^{\infty} \langle f, e_n \rangle Te_n \|.$$

$$
\begin{aligned}
\|Tf - T_N f\| &= \|\sum_{n=N+1}^{\infty} \langle f, e_n \rangle Te_n\| \\
&\leq \sum_{n=N+1}^{\infty} |\langle f, e_n \rangle| \|Te_n\| \\
&\leq \left(\sum_{n=N+1}^{\infty} |\langle f, e_n \rangle|^2\right)^{1/2} \left(\sum_{n=N+1}^{\infty} \|Te_n\|^2\right)^{1/2} \\
&\leq \|f\| \left(\sum_{n=N+1}^{\infty} \|Te_n\|^2\right)^{1/2}.
\end{aligned}
$$

Thus, $\|T - T_N\| \leq \left(\sum_{n=N+1}^{\infty} \|Te_n\|^2\right)^{1/2}$ which (since T is Hilbert-Shmidt) $\to 0$ as $N \to \infty$. This exhibits T as operator-norm limit of finite-rank operators, hence T is compact.

Hilbert-Schmidt Integral Operators

Let (X, μ) be a (separable) σ-finite measure space and let $k \in L^2(X \times X, \mu \times \mu)$ be a square integrable function of two variables on X. We want to define an integral operator T on $L^2(X, \mu)$ by

$$T\xi(x) = \int_X k(x, y)\xi(y)d\mu(y), \qquad \xi \in L^2(X, \mu).$$

Since

$$\int_{X \times X} |k(x, y)|^2 d\mu(x)d\mu(y) < \infty,$$

the Fubini theorem implies that for almost every $x \in X$, the section $y \mapsto k(x, y)$ belongs to $L^2(X, \mu)$, and for such x the function $y \mapsto k(x, y)\xi(y)$ belongs to $L^1(X, \mu)$ (by Holder inequality.)

Next, we estimate the norm.

$$
\begin{aligned}
\|T\xi\|^2 &= \int_X |T\xi(x)|^2 d\mu(x) \\
&= \int_X |\int_X k(x,y)\xi(y)d\mu(y)|^2 d\mu(x) \\
&\leq \int_X \left(\int_X |k(x,y)||\xi(y)|d\mu(y)\right)^2 d\mu(x) \\
&\leq \int_X \left(\int_X |k(x,y)|^2 d\mu(y) \cdot \int_X |\xi(y)|^2 d\mu(y)\right) d\mu(x) \\
&= \int_X \left(\int_X |k(x,y)|^2 d\mu(y) \cdot \|\xi\|^2\right) d\mu(x) \\
&= \|\xi\|^2 \int_{X\times X} |k(x,y)|^2 d\mu(x)d\mu(y) \\
&= \|\xi\|^2 \|k\|^2
\end{aligned}
$$

Thus $\|T\| \leq \|k\|$. This inequality shows that T is a bounded linear operator on $L^2(X.\mu)$.

Next, choose an orthonormal basis $e_1, e_2, \ldots$ for $L^2(X, \mu)$. For every $m, n = 1, 2, \ldots,$ we have

$$\begin{aligned} \langle Te_m, e_n \rangle &= \int_X Ae_m(x)\bar{e}_n(y)d\mu(y)d\mu(x). \\ &= \int_{X\times X} k(x,y)\bar{e}_n(x)e_m(y)d\mu(y)d\mu(x). \end{aligned}$$

Writing $u_{mn}(x,y) = e_n(x)\bar{e}_m(y)$, we find that $\{u_{mn} : m, n = 1, 2, \ldots\}$ is an orthonormal basis for $L^2(X \times X, \mu \times \mu)$, and the preceding formula becomes

$$\langle Te_m, e_n \rangle = \langle k, u_{mn} \rangle,$$

the inner product of the right being that of $L^2(X \times X, \mu \times \mu)$. It follows that

$$\sum_m \|Te_m\|^2 = \sum_m \sum_n |\langle Te_m, e_n \rangle|^2 = \sum_m \sum_n |\langle k, u_{mn} \rangle|^2 = \|k\|^2.$$

Therefore $\sum_m \|Te_m\|^2 < \infty$ and thus T is a Hilbert-Schmidt operator.

Thank you