Nuclear Operators on Banach Spaces

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Rank-One Operators

Lemma

Let X and Y be Banach spaces, $f \in X^*$ and $y \in Y$. Let $f \oplus y : X \to Y$ be defined by the formula $(f \oplus y)(x) = f(x)y$. Then

(1) $f \oplus y \in B(X, Y)$ with $\|f \oplus y\|_* = \|f\|_* \|y\|$ and $\dim(f \oplus y)[X] \leq 1$; indeed $\dim(f \oplus y)[X] = 1$ if $f \neq 0$ and $y \neq 0$.

(2) for any sequence (f_n) in X^* and (y_n) in Y such that $\sum_{n=1}^{\infty} \|f_n\|_* \|y_n\| < \infty$, the sequence $(\sum_{k=1}^{n} f_k \oplus y_k)_n$ is Cauchy for the operator-norm $\| \; \|_*$. (Thus the series $\sum_{n=1}^{\infty} f_n \oplus y_n$ converges for $\| \; \|_*$ to a bounded linear operator on X.)

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(2) for any sequence (f_n) in X^* and (y_n) in Y such that $\sum_{n=1}^{\infty} \|f_n\|_* \|y_n\| < \infty$, the sequence $(\sum_{k=1}^{n} f_k \oplus y_k)_n$ is Cauchy for the operator-norm $\| \ \|_*$. (Thus the series $\sum_{n=1}^{\infty} f_n \oplus y_n$ converges for $\| \ \|_*$ to a bounded linear operator on X.)

Proof. (1) Linearity of $f \oplus y$ is easily verified. If $x \in X$ with $\|x\| \leq 1$, then

$$\|(f \oplus y)(x)\| = \|f(x)y\| = |f(x)| \|y\| \leq \|f\| \|y\|.$$

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Furthermore,

$$\begin{aligned}
\|f \oplus y\|_* &= \sup_{||x||\leq 1} \{\|(f \oplus y)(x)\| : x \in X\} \\
&= \sup_{||x||\leq 1} \{||f(x)y|| : x \in X\} \\
&= \sup_{||x||\leq 1} \{|f(x)|\|y\| : x \in X\} \\
&= \|f\|_*\|y\|.
\end{aligned}$$

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Let $E = \{\alpha y : \alpha \in \mathbb{F}\}$ be the subspace of Y generated by y.
So dim $E \leq 1$.
Clearly $(f \oplus y)[X] = \{f(x)y : x \in X\} \subset E$
$\Rightarrow$ dim $(f \oplus y)[X] \leq \dim E \leq 1$.

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If $f \neq 0$, then f is surjective. Hence $(f \oplus y)[X] = E$; but $y \neq 0 \Rightarrow \dim E = 1$ so that $\dim(f \oplus y)[X] = \dim E = 1$.

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(2) Let $T_n = \sum_{k=1}^{n} f_k \oplus y_k \ \forall n \in \mathbb{N}$. By (1), $T_n \in B(X, Y) \ \forall n \in \mathbb{N}$.
Let $c_n = \sum_{k=1}^{n} \|f_k\|_* \|y_k\| \ \forall n \in \mathbb{N}$.

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Let $c_n = \sum_{k=1}^{n} \|f_k\|_* \|y_k\| \ \forall n \in \mathbb{N}$.By the hypothesis, $(c_n)_{n \geq 1}$ is a Cauchy sequence. To see that $(T_n)_{n \geq 1}$ is a Cauchy sequence for $\| \ \|_*$, let $\varepsilon > 0$. Then

$$\exists K \in \mathbb{N} \text{ st } n, m \geq K \Rightarrow |c_n - c_m| < \varepsilon.$$

For $n \geq m \geq K$ we have

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$$\begin{aligned}
\|T_n - T_m\|_* &= \|\sum_{t=1}^{n} f_t \oplus y_t - \sum_{t=1}^{m} f_t \oplus y_t\|_* \\
&= \|\sum_{t=m+1}^{n} f_t \oplus y_t\|_* \\
&\leq \sum_{t=m+1}^{n} \|f_t \oplus y_t\|_* \\
&= \sum_{t=m+1}^{n} \|f_t\|_* \|y_t\| \\
&= c_n - c_m = |c_n - c_m| < \varepsilon.
\end{aligned}$$

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Since Y is Banach, it follows that the series $\sum_{n=1}^{\infty} f_n \oplus y_n$ converges for $\| \; \|_*$ to a bounded linear operator on X. This completes the proof.

Operators on Banach Spaces

Let X and Y be normed spaces and $T : X \to Y$ is a bounded linear operator.

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T is *nuclear* iff $T \in B(X, Y)$ and $\exists$ sequence $(f_n)_{n\geq 1}$ in X^*, $\exists$ sequence $(y_n)_{n\geq 1}$ in Y such that $\sum_{n=1}^{\infty} \|f_n\|_* \|y_n\| < \infty$ and the series $\sum_{n=1}^{\infty} f_n \oplus y_n$ converges for $\| \ \|_*$ to T.

Relationship between Fi (X, Y) and Nu (X, Y)

Theorem

Let X and Y be normed spaces on $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

$$
\begin{aligned}
(1) Fi(X,Y) &= \left\{ \sum_{k=1}^{n} f_k \oplus y_k : n \in \mathbb{N}, f_k \in X^* \text{and } y_k \in Y, k = 1, \ldots n \right\} \\
&= \text{subspace generated by} \{ f \oplus y : f \in X^* \text{and} y \in Y \}.
\end{aligned}
$$

(2) $\mathrm{Fi}(X, Y) \subset \mathrm{Nu}(X, Y)$.

Proof. Let $M = \left\{ \sum_{k=1}^{n} f_k \oplus y_k : n \in \mathbb{N}, f_k \in X^* \text{and} y_k \in Y, k = 1, \dots n \right\}$ and let lin(E) be the subspace generated by $E \equiv \{f \oplus y : f \in X^* \text{and} y \in Y\}$.

1a) Clearly $M \subset lin(E)$.

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1a) Clearly $M \subset lin(E)$.

1b) If $\alpha \in \mathbb{F}, f \in X^*$ and $y \in Y$, one has

$$\alpha(f \oplus y) = (\alpha f) \oplus y = f \oplus (\alpha y) \in E.$$

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1a) Clearly $M \subset lin(E)$.

1b) If $\alpha \in \mathbb{F}, f \in X^*$ and $y \in Y$, one has

$$\alpha(f \oplus y) = (\alpha f) \oplus y = f \oplus (\alpha y) \in E.$$

1c) Let $h \in \text{lin}(E)$. $\exists\ n \in \mathbb{N}\ \exists\ \alpha_k \in \mathbb{F}\ \exists\ f_k \in X^*\ \exists\ y_k \in Y$ for $k = 1, \ldots, n$ such that

$$h = \sum_{k=1}^{n} \alpha_k (f_k \oplus y_k).$$

$\forall k \in \{1, \ldots, n\}, g_k \equiv \alpha_k f_k \in X^*$ and by 1b) $\alpha_k(f_k \oplus y_k) = g_k \oplus y_k$.

It follows that

$$h = \sum_{k=1}^{n} \alpha_k (f_k \oplus y_k) = \sum_{k=1}^{n} g_k \oplus y_k \in M.$$

Hence $\text{lin}(E) \subset M$ and $M = \text{lin}(E)$.

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1d) Let $g \in M$. Then $g = \sum_{k=1}^{n} f_k \oplus y_k$ with $n \in \mathbb{N}, f_k \in X^*$ and $y_k \in Y$ for $k = 1, \dots, n$.
$\forall k \in \{1, \dots, n\}$ we know by Lemma that $f_k \oplus y_k \in Fi(X, Y)$ with $\dim(f_k \oplus y_k)[X] \leq 1$.

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$\forall k \in \{1, \ldots, n\}$ we know by Lemma that $f_k \oplus y_k \in Fi(X, Y)$ with $\dim(f_k \oplus y_k)[X] \leq 1$.
Therefore

$$\text{dim}g[X] \leq \sum_{k=1}^{n} \dim(f_k \oplus y_k)[X] \leq n < \infty.$$

Clearly $g \in B(X, Y)$.Thus $g \in Fi(X, Y)$ and $M \subset Fi(X, Y)$.

1e) Let $T \in Fi(X,Y)$. Then $T \in B(X,Y)$ and $\dim T[X]$ is finite.
If $\dim T[X] = 0$, then $T = 0$. Taking $y = 0 \in Y$ and $f \in X^*$ arbitrary,one gets $T = f \oplus y \in M$.

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Suppose $\dim\ T[X] = n \in \mathbb{N}$ and let $\{y_1, y_2, \ldots, y_n\}$ be a basis of $T[X]$.
Define $V : \mathbb{F}^n \to T[X]$ by the formula

$$V(\alpha_1, \alpha_2, \ldots, \alpha_n) = \sum_{k=1}^{n} \alpha_k y_k.$$

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$$V(\alpha_1, \alpha_2, \ldots, \alpha_n) = \sum_{k=1}^{n} \alpha_k y_k.$$

Then $V \in B(\mathbb{F}^n, T[X])$, V is bijective, and if $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{F}$, one has $V^{-1}(\sum_{k=1}^{n} \alpha_k y_k) = (\alpha_1, \ldots, \alpha_n)$.

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If $j \in \{1, \ldots, n\}$, let $S_j : T[X] \to \mathbb{F}$, $S_j = pr_j \circ V^{-1}$, where pr_j is the jth projection of $\mathbb{F}^n$ on $\mathbb{F}$.

Then $S_j \in (T[X])^*$, and by Hahn-Banach Theorem, $\exists T_j \in Y^*$ such that

$$T_j|_{T[X]} = S_j \text{ and } \|T_j\|_* = \|S_j\|_*, \forall j = 1, \ldots, n.$$

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$\forall j \in \{1, \ldots, n\}$, $f_j \equiv T_j \circ T \in X^*$, and so $\sum_{k=1}^n f_k \oplus y_k \in M$.
We show that $T = \sum_{k=1}^n f_k \oplus y_k$.

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We show that $T = \sum_{k=1}^n f_k \oplus y_k$.

If $x \in X$ then $T(x) \in T[X]$ and since $\{y_1, \ldots, y_n\}$ is a base of $T[X]$, $\exists\ \alpha_1, \ldots, \alpha_n \in \mathbb{F}$ such that $T(x) = \sum_{k=1}^n \alpha_k y_k$.
We then have

$$\left(\sum_{k=1}^n f_k \oplus y_k\right)(x) = \sum_{k=1}^n f_k(x) y_k = \sum_{k=1}^n (T_k \circ T)(x) y_k$$

$$= \sum_{k=1}^n T_k(T(x)) y_k = \sum_{k=1}^n S_k(T(x)) y_k = \sum_{k=1}^n \left(pr_k \circ V^{-1}\right)(T(x)) y_k$$

$$= \sum_{k=1}^{n} pr_k \left(V^{-1} \left(\sum_{i=1}^{n} \alpha_1 y_i \right) y_k \right) = \sum_{k=1}^{n} pr_k(\alpha_1, \alpha_2, \ldots, \alpha_n) y_k$$

$$= \sum_{k=1}^{n} \alpha_k y_k = T(x)$$

Thus $\mathrm{Fi}(X, Y) \subset M$ and $\mathrm{Fi}(X, Y) = M$.

(2) Let $f \in Fi(X,Y)$. By (1) $\exists n \in \mathbb{N}\ \exists f_k \in X^*\ \exists y_k \in Y$, $k = 1, \dots, n$ such that $f = \sum_{k=1}^{n} f_k \oplus y_k$. If $k \in \mathbb{N}$ with $k > n$, define f_k the null form on X and $y_k = 0 \in Y$.
Hence $f_k \in X^*$ and $y_k \in Y$ with $f_k \oplus y_k = 0$.
We then have: $f_k \in X^*$ and $y_k \in Y \forall k \in \mathbb{N}$,
$\sum_{k=1}^{\infty} \|f_k\|_* \|y_k\| = \sum_{k=1}^{n} \|f_k\|_* \|y_k\| < \infty$ and

$$\sum_{k=1}^{\infty} f_k \oplus y_k = \sum_{k=1}^{n} f_k \oplus y_k = f \in B(X,Y).$$

This means by definition that $f \in Nu(X,Y)$.
Hence $Fi(X,Y) \subset Nu(X,Y)$.

Theorem

(1) If X is a normed space and Y is Banach then $Nu(X, Y) \subset co(X, Y)$ and so $Fi(X, Y) \subset Nu(X, Y) \subset co(X, Y) \subset B(X, Y)$.
(2) If X and Y are Banach such that either X or Y is finite dimensional then $Fi(X, Y) = Nu(X, Y) = co(X, Y) = B(X, Y)$.

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Proof. 1a) We have $Fi(X, Y) \subset Nu(X, Y)$ and $co(X, Y) \subset B(X, Y)$ without the requirement Y is Banach.

1b) Let $T \in Nu(X, Y)$. $\exists (f_n)$ in X^* $\exists (y_n)$ in Y such that $\sum_{n=1}^{\infty} \|f_n\|_* \|y_n\| < \infty$ and $(\sum_{k=1}^{n} f_k \oplus y_k)_{n \geq 1}$ converges in operator norm $\|\|_*$ to $T \in B(X, Y)$.

$\forall n \in \mathbb{N}, \ T_n = \sum_{k=1}^{n} f_k \oplus y_k \in Fi(X, Y)$.
Since $(T_n)_{n \geq 1}$ converges to T and Y is Banach, we know $T \in co(X, Y)$.
Thus $Nu(X, Y) \subset B(X, Y)$ when Y is Banach.

(2) If dim $X < \infty$ then $f \in B(X, Y) \Rightarrow \dim f[X] \leq \dim X < \infty$
$\Rightarrow f \in Fi(X, Y)$.
If dim $Y < \infty$ then $f \in B(X, Y) \Rightarrow \dim f[X] \leq \dim Y < \infty \Rightarrow$
$f \in Fi(X, Y)$.
Thus dim $X < \infty$ or dim $Y < \infty \Rightarrow B(X, Y) \subset Fi(X.Y)$.
Thus by (1), $Fi(X, Y) = Nu(X, Y) = co(X, Y) = B(X, Y)$.

Thank you