Groups of Operators and the Midpoint Rule

Introduction: We learn to sum geometric sums $S_n = a + ar + ar^2 + ar^{n-1} = \frac{a-ar^n}{1-r}$ early in algebra. The trick is to "shift" the sum S_n by multiplying by r, then subtract from S_n. The trick is simple enough to be performed mentally.

The shifting trick works for sums of **groups of operators**. Furthermore, groups of operators lend themselves to a "midpoint rule" which is simple and powerful. An example we've seen several times in seminar is the Dirichlet kernel.

We'll discuss how the notion of a "group of operators" arises from the study of initial value problems, give several examples, and derive the Dirichlet kernel and Faulhaber's identities.

Groups of Operators: Groups of operators arise when solving autonomous Initial Value Problems (IVPs) of the form

$$\begin{cases} \dfrac{d}{dt}u(t) = F(u(t)), \\ \quad u(0) = u_0. \end{cases} \qquad (IVP)$$

It is a theorem that if $F(\cdot)$ is at all well-behaved (Lipschitz continuous, for example), then there is a unique solution to (IVP). Joel gave a "fixed point" proof in seminar which appears in Chapter 3 of his book[1]. Note that his proof is valid for non-autonomous IVPs, too.

Let $S(t)$ denote the solution operator. By "solution operator", we mean that, for each t, $S(t)$ is a mapping which takes the initial value u_0 to the (unique) solution of (IVP) at time t:

$$S(t)u_0 = u(t), \quad \text{where } u(t) \text{ satisfies } (IVP).$$

The hardest part of this definition is making sense of the notation. Some examples will, perhaps, help.

Example 1: Naturally, the simplest examples are those in which F is a linear operator. The simplest linear operators are the mappings $F : R \to R$ defined by

$$F(x) = ax.$$

In words, F is the operator "multiply by (the constant value) a". (If you like, take $a = -2$ so that the differential equation is $\frac{d}{dt}u(t) = -2u(t)$.)

The solution to (IVP) in this case is

$$u(t) = e^{at}u_0.$$

In fact, the exponential function *exists* to solve (IVP) in this case.

The solution operator is

$$S(t) = \text{ "multiply by } e^{at}\text{"},$$

which we abbreviate as $S(t) = e^{at}$. For each t, the solution operator $S(t)$ is linear (because both differentiation and $F(\cdot)$ are linear).

Example 2: The simplest nonlinear operator $F(\cdot)$ is surely the square:

$$F(x) = x^2.$$

[1] Joel H. Shapiro, *A Fixed-Point Farrago*, Springer, 2016, p. 33ff.

The differential equation in (IVP) reads

$$\frac{d}{dt}u(t) = u^2(t),$$

which we may solve by "separating variables":

$$\begin{aligned} \frac{\frac{d}{dt}u(t)}{u^2(t)} &= 1, \quad \text{so, by the Chain Rule,} \\ \frac{d}{dt}\frac{-1}{u(t)} &= 1, \quad \text{so} \\ \frac{1}{u(t)} &= c - t \end{aligned}$$

for some constant of integration c. To satisfy the initial condition in (IVP), select $c = \frac{1}{u_0}$. Then

$$u(t) = \frac{u_0}{1 - tu_0}.$$

The solution operator maps initial values u_0 (which are just numbers) to

$$S(t)u_0 = \frac{u_0}{1 - tu_0}.$$

Note that $S(t)$ is not a linear operator unless $t = 0$. For example, if α is a scalar $\neq 1$, then

$$S(t)(\alpha u_0) = \frac{\alpha u_0}{1 - t\alpha u_0} \neq \alpha\frac{u_0}{1 - tu_0} = \alpha S(t)(u_0).$$

That's no surprise because F isn't linear. It's also no surprise that $S(t)u_0$ is not defined for all times t.

The exponential operator e^{at} from Example 1 enjoys what we will call the "group property":

$$e^{a(t_1+t_2)} = e^{at_1}e^{at_2}.$$

The t_i belong to the group of real (or complex) numbers under addition. The exponentials e^{at_i} belong to the group of positive real (or non-zero complex) numbers under multiplication. In the language of algebra, the "group property" above says that the exponential function is a group homomorphism from the first group to the second.

The important theorem for us is that the "group property" holds for *all* solution operators S. In symbols, it says

Theorem 1: (modulo technical details ...)

$$S(t_2 + t_1) = S(t_2)S(t_1). \qquad (GP)$$

The algebraic meaning is:

1. The numbers t_1 and t_2 belong to $(R, +)$, the group of real numbers under addition (or, if you prefer, to $(C, +)$, the group of complex numbers under addition).
2. The operators $S(t_1+t_2)$, $S(t_1)$, and $S(t_2)$ belong to the group of operators $(X, \circ)$, where X is a collection of mappings and $\circ$ is the group operation "composition".
3. The mapping $t \mapsto S(t)$ is a (group) homomorphism from $(R, +)$ (or $(C, +)$) to $(X, \circ)$. That's what (GP) says.

The point of all of our group notation is to be able to write (GP).

Proof: As always, we prove theorems about operators by letting them act on operands. In other words, we prove that (modulo technical details)

$$S(t_2 + t_1)u_0 = S(t_2)S(t_1)u_0$$

for any choice of u_0. The left side is the solution to (IVP) at time $t_2 + t_1$. The rightmost term,

$$S(t_1)u_0,$$

is the solution to (IVP) at time t_1. Consequently, the (whole) right side,

$$S(t_2) \circ S(t_1)u_0,$$

is the solution to the initial value problem

$$\begin{cases} \dfrac{d}{dt}u(t) = F(u(t)), \\ u(0) = S(t_1)u_0 \end{cases}$$

at time t_2. The theorem is nothing more than the observation that

> Starting at u_0 and following the solution $u(t)$ of (IVP) out to time $t_1 + t_2$ is the same as starting at u_0 and following the solution out to time t_1, then resetting the clock to $t = 0$, starting at $u(t_1) = S(t_1)u_0$ and following the solution out to t_2. The differential equation is autonomous, so "resetting the clock" does not change it. Since the solutions to IVPs are unique, the two solutions are the same.

That's the proof. The end of the first leg of the journey, $u(t_1)$, matches up exactly with the initial point of (IVP) for the second leg. Since the differential equation is autonomous, and since solutions are unique, the Group Property (GP) holds. ////

Example 2, cont.: The solution operator found in Example 2 satisfies

$$\begin{aligned} S(t_2)S(t_1)u_0 &= S(t_2)\frac{u_0}{1 - t_1 u_0} \\ &= \frac{\frac{u_0}{1-t_1 u_0}}{1 - t_2 \frac{u_0}{1-t_1 u_0}} \\ &= \frac{u_0}{1 - (t_1 + t_2)u_0} \\ &= S(t_2 + t_1)u_0, \end{aligned}$$

as promised by Theorem 1.

Remark: The proof is valid only if the solution $S(t)u_0$ exists for all times t between 0, t_1, t_2, and $t_1 + t_2$. Example 2 illustrates that solutions may only exist locally, so the theorem is only true locally. That's why the statement of the theorem starts, "(modulo technical details ...)".

Aside: Theorem 1 does not *directly* cover non-autonomous differential equations. The simplest example is surely

$$\begin{cases} \dfrac{d}{dt}u(t) = t \\ u(0) = u_0. \end{cases}$$

The right side of the differential equation is linear in u (the mapping $f(u) = 0 \cdot u$ is linear), but depends explicitly on t, so the (IVP) is non-autonomous. The solution is

$$u(t) = u_0 + \frac{t^2}{2}.$$

The solution operator

$$S(t)u_0 = u_0 + \frac{t^2}{2}$$

does not satisfy the Group Property (GP):

$$\begin{aligned} S(t_2)S(t_1)u_0 &= S(t_2)\left(u_0 + \frac{t_1^2}{2}\right) \\ &= \left(u_0 + \frac{t_1^2}{2}\right) + \frac{t_2^2}{2} \\ &\neq u_0 + \frac{(t_2+t_1)^2}{2} = S(t_2+t_1)u_0. \end{aligned}$$

So Theorem 1 doesn't *directly* hold.

We can, however, turn the non-autonomous IVP into an autonomous one, at the cost of an extra dimension. Let $v(t) = t$ and consider the IVP

$$\begin{cases} \dfrac{d}{dt}\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} v \\ 1 \end{bmatrix}, \\ \begin{bmatrix} u(0) \\ v(0) \end{bmatrix} = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}, \end{cases}$$

where we will eventually take $v_0 = 0$. This system *is* autonomous. The solution operator

$$S(t)\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} u_0 + \frac{(v_0+t)^2 - v_0^2}{2} \\ v_0 + t \end{bmatrix}$$

does enjoy the group property:

$$\begin{aligned} S(t_2)S(t_1)\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} &= S(t_2)\begin{bmatrix} u_0 + \frac{(v_0+t_1)^2 - v_0^2}{2} \\ v_0 + t_1 \end{bmatrix} \\ &= \begin{bmatrix} \left(u_0 + \frac{(v_0+t_1)^2 - v_0^2}{2}\right) + \frac{((v_0+t_1)+t_2)^2 - (v_0+t_1)^2}{2} \\ (v_0+t_1) + t_2 \end{bmatrix} \\ &= \begin{bmatrix} u_0 + \frac{(v_0+t_1+t_2)^2 - v_0^2}{2} \\ v_0 + (t_1+t_2) \end{bmatrix} \\ &= S(t_2+t_1)\begin{bmatrix} u_0 \\ v_0 \end{bmatrix}. \end{aligned}$$

So we've traded a (scalar) non-autonomous IVP for a (vector) autonomous IVP and recovered the Group Property (GP).

Examples of Linear Groups: Linear IVPs in several dimensions are especially interesting. The operator F becomes "multiply by the matrix A" instead of just "multiply by the scalar a" in Example 1. The important result is that the power series

$$e^{tA} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots, \qquad (PS)$$

where I is the identity matrix, converges for all values of t, real or complex. Since

$$\begin{aligned} Ae^{tA} &= A + tA^2 + \frac{t^2}{2!}A^3 + \frac{t^3}{3!}A^4 + \cdots \\ &= \frac{d}{dt}\underbrace{\left(I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots\right)}_{e^{tA}}, \end{aligned}$$

and since $e^{0A} = I$, the solution operator $S(t)$ is represented by the power series. (Nonlinear operators don't have this power series representation because the operator doesn't "distribute" across the sum the way "multiply by A" does.) We'll use the exponential notation e^{At} in place of the "generic" solution operator $S(t)$ in the examples that follow.

Example 3: Diagonal matrices are easiest:

$$e^{Dt} = I + tD + \frac{t^2}{2}D^2 + \frac{t^3}{3!}D^3 + \cdots,$$

where each of the powers of D is diagonal. If, for example,

$$D = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix},$$

then

$$D^k = \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix}$$

so

$$e^{Dt} = \begin{bmatrix} e^{at} & 0 \\ 0 & e^{bt} \end{bmatrix}.$$

The exponential of a diagonal matrix is the diagonal matrix of exponentials. Linear algebra texts that cover this material advocate diagonalizing a general matrix (or, if necessary, reducing it to Jordan Canonical Form) so that the exponentiation is easy. Diagonalization is (usually) computationally expensive, however, and we can sometimes find the exponential without resorting to it.

Example 4: An important pair of examples is

$$\begin{cases} \dfrac{d}{dt}\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} 0 & \pm 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} u(t) \\ v(t) \end{bmatrix}, \\ \begin{bmatrix} u(0) \\ v(0) \end{bmatrix} = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}. \end{cases}$$

Check that, with $A = \begin{bmatrix} 0 & \pm 1 \\ 1 & 0 \end{bmatrix}$,

$$A^2 = \pm I,$$

so the power series splits into terms with I and terms with A:

$$\begin{aligned} e^{tA} &= I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \frac{t^4}{4!}A^4 + \cdots \\ &= I + tA \pm \frac{t^2}{2!}I^2 \pm \frac{t^3}{3!}A + \frac{t^4}{4!}I + \cdots, \\ &= \left(1 \pm \frac{t^2}{2!} + \frac{t^4}{4!}\right) I + \left(t \pm \frac{t^3}{3!} + \frac{t^5}{5!}\right) A. \end{aligned}$$

If all the "$\pm$" are "+", then the power series multiplying I is $\cosh(t)$ and the power series multiplying A is $\sinh(t)$. If all the "$\pm$" are "$-$", then the power series multiplying I is $\cos(t)$ and the power series multiplying A is $\sin(t)$. We have therefore shown that

$$e^{t\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} = \begin{bmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{bmatrix} \quad \text{and} \quad e^{t\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}} = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}$$

The group property (GP) in these cases are the "sum-to-product" rules for the hyperbolic and trig functions. For example, the trigonometric version is

$$\begin{aligned} e^{(t_1+t_2)A} &= \begin{bmatrix} \cos(t_1+t_2) & -\sin(t_1+t_2) \\ \sin(t_1+t_2) & \cos(t_1+t_2) \end{bmatrix} \\ &= e^{t_1 A} e^{t_2 A} \\ &= \begin{bmatrix} \cos(t_1) & -\sin(t_1) \\ \sin(t_1) & \cos(t_1) \end{bmatrix} \begin{bmatrix} \cos(t_2) & -\sin(t_2) \\ \sin(t_2) & \cos(t_2) \end{bmatrix} \\ &= \begin{bmatrix} \cos(t_1)\cos(t_2) - \sin(t_1)\sin(t_2) & -\sin(t_1)\cos(t_2) - \cos(t_1)\sin(t_2) \\ \sin(t_1)\cos(t_2) + \cos(t_1)\sin(t_2) & \cos(t_1)\cos(t_2) - \sin(t_1)\sin(t_2) \end{bmatrix}. \end{aligned}$$

The two trig identities follow by equating the components of the two matrices. The hyperbolic identities are analogous.

Example 5: The matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is called *nilpotent* because its powers are eventually the zero matrix:

$$A^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad A^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad A^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = A^5 = \cdots.$$

The terms of the power series (PS) are eventually zero for nilpotent matrices, so e^{tA} is just a polynomial in A. In this example,

$$e^{tA} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 = \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \frac{t^3}{3!} \\ 0 & 1 & t & \frac{t^2}{2!} \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since we spend quite a bit of time in real and complex analysis emphasizing the difference between exponentials and polynomials, it is an ironic twist that the exponential e^{tA} *is* polynomial in t.

Group Property (GP) in this example is called the Binomial Theorem:

$$\begin{aligned} e^{(t_1+t_2)A} = \begin{bmatrix} 1 & (t_1+t_2) & \frac{(t_1+t_2)^2}{2!} & \frac{(t_1+t_2)^3}{3!} \\ 0 & 1 & (t_1+t_2) & \frac{(t_1+t_2)^2}{2!} \\ 0 & 0 & 1 & (t_1+t_2) \\ 0 & 0 & 0 & 1 \end{bmatrix} &= e^{t_1 A} e^{t_2 A} \\ &= \begin{bmatrix} 1 & t_1 & \frac{t_1^2}{2!} & \frac{t_1^3}{3!} \\ 0 & 1 & t_1 & \frac{t_1^2}{2!} \\ 0 & 0 & 1 & t_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & t_2 & \frac{t_2^2}{2!} & \frac{t_2^3}{3!} \\ 0 & 1 & t_2 & \frac{t_2^2}{2!} \\ 0 & 0 & 1 & t_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & t_1+t_2 & \frac{t_1^2+2t_1t_2+t_2^2}{2!} & \frac{t_1^3+3t_1^2t_2+3t_1t_2^2+t_2^3}{3!} \\ 0 & 1 & t_1+t_2 & \frac{t_1^2+2t_1t_2+t_2^2}{2!} \\ 0 & 0 & 1 & t_1+t_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

For example, the top right corner entries say $(t_1+t_2)^3 = t_1^3 + 3t_1^2 t_2 + 3t_1 t_2^2 + t_2^3$.

The Midpoint Rule: The group property (GP) simplifies sums of exponentials the form

$$S_n = e^{t_0 A} + e^{(t_0+h)A} + e^{(t_0+2h)A} + e^{(t_0+3h)A} + \cdots + e^{(t_0+(n-1)h)A} + e^{(t_0+nh)A}. \qquad (S_n)$$

S_n is a geometric sum, which means we can write it in closed form. The usual trick is to use the group property to shift all the terms by h and subtract, giving

$$\begin{array}{rl} e^{hA}S_n = & \phantom{e^{t_0 A} +{}} e^{(t_0+h)A} + e^{(t_0+2h)A} + e^{(t_0+3h)A} + \cdots + e^{(t_0+nh)A} + e^{(t_0+(n+1)h)A} \\ S_n = & e^{t_0 A} + e^{(t_0+h)A} + e^{(t_0+2h)A} + e^{(t_0+3h)A} + \cdots + e^{(t_0+nh)A} \\ \hline \left(e^{hA} - I\right) S_n = & -e^{t_0 A} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + e^{(t_0+(n+1)h)A} \end{array}$$

so that

$$\left(e^{hA} - I\right) S_n = e^{(t_0+(n+1)h)A} - e^{t_0 A}. \qquad (GS_1)$$

This computation is very visual, so it is easy to apply without making an error. It collapses the sum, and we need only invert the matrix $\left(e^{hA} - I\right)$ (if possible) to find S_n. This computation suffers from an asymmetry, however. It shifts forward in time, but not backward, then subtracts. Part of the point of the group theory is that we should not favor forward-in-time computations any more than backward-in-time. There is a more symmetric way to collapse the sum which is just as visual and just as easy to apply: shift both forward and backward by $\frac{h}{2}$ — *half* a step forward and *half* a step backward — and subtract:

$$\begin{array}{rl} e^{\frac{h}{2}A}S_n = & \phantom{e^{(t_0-\frac{1}{2}h)A} +{}} e^{\left(t_0+\frac{h}{2}\right)A} + e^{\left(t_0+\frac{3h}{2}\right)A} + e^{\left(t_0+\frac{5h}{2}\right)A} + \cdots + e^{\left(t_0+n-\frac{h}{2}\right)A} + e^{\left(t_0+n+\frac{h}{2}\right)A} \\ e^{\frac{-h}{2}A}S_n = & e^{\left(t_0-\frac{1}{2}h\right)A} + e^{\left(t_0+\frac{h}{2}\right)A} + e^{\left(t_0+\frac{3h}{2}\right)A} + e^{\left(t_0+\frac{5h}{2}\right)A} + \cdots + e^{\left(t_0+n-\frac{h}{2}\right)A} \\ \hline \left(e^{\frac{h}{2}A} - e^{-\frac{h}{2}A}\right) S_n = & -e^{\left(t_0-\frac{1}{2}h\right)A} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + e^{\left(t_0+n+\frac{h}{2}\right)A} \end{array}$$

so that

$$\left(e^{\frac{h}{2}A} - e^{-\frac{h}{2}A}\right) S_n = e^{\left(t_0+n+\frac{h}{2}\right)A} - e^{\left(t_0-\frac{1}{2}h\right)A}. \qquad (GS_2)$$

One of the advantages of this symmetric formulation is that the matrix $\left(e^{\frac{h}{2}A} - e^{-\frac{h}{2}A}\right)$ on the left is often easier to invert than the matrix $\left(e^{hA} - I\right)$ from Equation (GS_1). Furthermore, we can "see" the symmetry on the right if we use the group property to factor the right as

$$\left(e^{\frac{h}{2}A} - e^{-\frac{h}{2}A}\right) S_n = e^{\left(t_0+\frac{nh}{2}\right)A}\left(e^{\frac{(n+1)h}{2}A} - e^{-\frac{(n+1)h}{2}A}\right). \qquad (GS_3)$$

The argument

$$t_0 + \frac{nh}{2} = \bar{t}$$

in the first exponent on the right is the *midpoint* of all the arguments. This computation is therefore called the "midpoint rule" for exponential (= geometric) sums. We can formalize the rule by noting that

$$\frac{e^{xA} - e^{-xA}}{2} = \sinh(xA) = xA + \frac{x^3}{3!}A^3 + \frac{x^5}{5!}A^5 + \cdots.$$

Then the midpoint rule is just

$$\sinh\left(\frac{h}{2}A\right) S_n = e^{\bar{t}A}\sinh\left(\frac{(n+1)h}{2}A\right). \qquad (MP_1)$$

All of the matrices in Equation (MP_1) are power series in the matrix A, so they all commute. If the matrix on the left is invertible, then

$$S_n = e^{\bar{t}A} \sinh\left(\frac{h}{2}A\right)^{-1} \sinh\left(\frac{(n+1)h}{2}A\right). \qquad (MP_2)$$

The right side consists of a term which depends only on $\bar{t}$, the midpoint of the arguments, and two terms involving hyperbolic sines of A, which are independent of $\bar{t}$. The hyperbolic sines are odd in A, which sometimes simplifies the computations.

Remark: I don't advocate memorizing or even employing Equations (MP_1) or (MP_2) directly. The point of the discussion is that the half-step-forward, half-step-backward difference leading to Equation (GS_2) is at least as easy as the usual whole-step-forward, no-step-backward difference leading to Equation (GS_1).

Remark: If you choose to use the formula, it is worth remembering that the $n+1$ on the right is the number of summands.

Remark: The group property has the interesting effect of separating the sum into a term that depends only on the midpoint $\bar{t}$ of the arguments, and a term that depends only on the number $(n+1)$ and spacing (h) of summands.

Example 6: The complex Dirichlet kernel: A real- or complex-valued function $f(t)$ defined on $[0,1]$ has (complex) Fourier series

$$f(t) \sim \sum_{k=-\infty}^{\infty} c_k e^{2\pi ikt},$$

where the coefficients

$$c_k = \int_0^1 e^{-2\pi iks} f(s)\, ds.$$

The motivation for the theory is the desire to express an "arbitrary" function f in terms of (complex) exponentials — trig functions — $e^{2\pi ikt}$. The "$\sim$" means that the series may not converge in any sense. To study the convergence of the doubly-infinite series, we might study the partial sums

$$\begin{aligned}
\sigma_{n_1,n_2}(t) &= \sum_{k=-n_1}^{n_2} c_k e^{2\pi ikt} \\
&= \sum_{k=-n_1}^{n_2} \left(\int_0^1 e^{-2\pi iks} f(s)\, ds\right) e^{2\pi ikt} \\
&= \int_0^1 \left[\sum_{k=-n_1}^{n_2} e^{2\pi ik(t-s)}\right] f(s)\, ds
\end{aligned}$$

The sum in square brackets is of the form (S_n) with $A = 2\pi i(t-s)$ and $h = 1$. Since A is invertible (at least, when $t \neq s$), Equation (MP_2) says

$$\begin{aligned}
\sum_{k=-n_1}^{n_2} e^{2\pi ik(t-s)} &= e^{2\pi i\frac{n_2-n_1}{2}(t-s)} \frac{\sinh\left(\frac{n_2+n_1+1}{2} 2\pi i(t-s)\right)}{\sinh\left(\frac{1}{2} 2\pi i(t-s)\right)} \\
&= e^{2\pi i\frac{n_2-n_1}{2}(t-s)} \frac{\sin\left((N+1)\pi(t-s)\right)}{\sin\left(\pi(t-s)\right)},
\end{aligned}$$

where $N+1 = n_2+n_1+1$ is the number of summands, and we have used the identity $\sinh(iz) = \sin(z)$. The ratio of sines is real-valued, and is the **Dirichlet** kernel. As promised, the sum splits into a factor involving the exponential at the midpoint $\bar{t} = 2\pi i\frac{n_2-n_1}{2}(t-s)$ and a factor (the Dirichlet kernel) independent of the difference $n_2 - n_1$.

Remark: Many kernels are functions involved in convolutions. The convolution $f * g$ is

$$(f * g)(t) = \int f(t-s)g(s)\,ds.$$

Convolutions are called **filters** in the engineering community.

Remark: The **frequency** of $e^{2\pi ikt}$ is k. If $n_1 = n_2 = n$, then σ_n keeps all the terms of the series with absolute frequency $\leq n$ and omits all the terms of absolute frequency $> n$. In the engineering community, such an operator is called a **low-pass filter**: it permits the passage of (genuine) frequencies below the cutoff n, and blocks the passage of the frequencies above the cutoff. Analog engineers build such filters out of electric components like resistors and capacitors. As Robert mentioned in his talk on Spread Spectrum, cutting off the high frequencies is a necessary first step to avoid **aliasing** in digital signals.

Example 7: The real Dirichlet kernel: Complex Fourier series are popular because the arithmetic is slightly easier than it is for real Fourier series. How much easier? Not much, if we use matrix exponentials. It might be worth using real Fourier series if only to avoid discussions about the meaning of "negative frequencies" (such as $-n_1$ in the example above). (Honestly, negative frequency exponentials go clockwise around the unit circle, while positive frequency exponentials go counterclockwise — don't let these conversations go on too long!)

A real-valued function $f(t)$ defined on $[0,1]$ has real Fourier series

$$f(t) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(2\pi kt) + b_k \sin(2\pi kt),$$

where the coefficients

$$a_k = 2\int_0^1 \cos(2\pi ks)\, f(s)\,ds,$$
$$b_k = 2\int_0^1 \sin(2\pi ks)\, f(s)\,ds.$$

The partial sums (= low-pass filters) are therefore

$$\begin{aligned}
\sigma_n(t) &= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos(2\pi kt) + b_k \sin(2\pi kt) \\
&= \int_0^1 f(s)\,ds + \sum_{k=1}^{n} 2\int_0^1 \cos(2\pi ks)\, f(s)\,ds \cos(2\pi kt) + 2\int_0^1 \sin(2\pi ks)\, f(s)\,ds \sin(2\pi kt) \\
&= \int_0^1 \left[1 + 2\sum_{k=1}^{n} \cos(2\pi ks)\cos(2\pi kt) + \sin(2\pi ks)\sin(2\pi kt)\right] f(s)\,ds \\
&= \int_0^1 \left[1 + 2\sum_{k=1}^{n} \cos(2\pi k(t-s))\right] f(s)\,ds
\end{aligned} \qquad (\sigma_n)$$

(Note that the trig identity in the last step is the group property (GP) from Example 4.) We therefore want to find the sum

$$\sum_{k=1}^{n} \cos(2\pi k(t-s)).$$

By itself, this sum is not of the form (S_n). We could, however, embed the sum into a matrix that *is* of the form (S_n):

$$S_n = \sum_{k=1}^{n} \begin{bmatrix} \cos(2\pi k(t-s)) & -\sin(2\pi k(t-s)) \\ \sin(2\pi k(t-s)) & \cos(2\pi k(t-s)) \end{bmatrix} = \sum_{k=1}^{n} e^{khA},$$

where

$$h = 2\pi(t-s) \qquad \text{and} \qquad A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

To find S_n, shift forward by half a step, then shift backward by half a step, then subtract:

$$\begin{aligned} e^{\frac{h}{2}A}S_n &= \phantom{e^{\left(1-\frac{1}{2}\right)hA} + {}} e^{\left(1+\frac{1}{2}\right)hA} + e^{\left(2+\frac{1}{2}\right)hA} + \cdots + e^{\left(n-\frac{1}{2}\right)hA} + e^{\left(n+\frac{1}{2}\right)hA} \\ e^{\frac{-h}{2}A}S_n &= e^{\left(1-\frac{1}{2}\right)hA} + e^{\left(1+\frac{1}{2}\right)hA} + e^{\left(2+\frac{1}{2}\right)hA} + \cdots + e^{\left(n-\frac{1}{2}\right)hA} \\ \hline \left(e^{\frac{h}{2}A} - e^{-\frac{h}{2}A}\right)S_n &= -e^{\frac{1}{2}hA} \qquad\qquad\qquad\qquad\qquad\qquad\qquad + e^{\left(n+\frac{1}{2}\right)hA}. \end{aligned}$$

That arithmetic is easy to "see" because it is visual.

Factor the exponential with the midpoint $\bar{t} = \frac{\frac{1}{2}+n+\frac{1}{2}}{2} = \frac{n+1}{2}$ out of the difference on the right:

$$\left(e^{\frac{h}{2}A} - e^{-\frac{h}{2}A}\right)S_n = e^{\frac{n+1}{2}hA}\left(e^{\frac{n}{2}hA} - e^{-\frac{n}{2}hA}\right)$$

The difference of exponentials is of the form

$$\left(e^{rA} - e^{-rA}\right) = \begin{bmatrix} \cos(r) & -\sin(r) \\ \sin(r) & \cos(r) \end{bmatrix} - \begin{bmatrix} \cos(-r) & -\sin(-r) \\ \sin(-r) & \cos(-r) \end{bmatrix} = 2\sin(r)\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = 2\sin(r)A.$$

Note that this matrix is simpler than the matrix $I - e^{rA}$ we would have gotten from the "usual" geometric sum (GS_1). We are left with

$$2\sin\left(\frac{h}{2}\right)AS_n = e^{\frac{n+1}{2}hA}2\sin\left(\frac{nh}{2}\right)A.$$

Every matrix in this equation commutes with A, and A is invertible, so

$$S_n = e^{\frac{n+1}{2}hA}\frac{\sin\left(\frac{nh}{2}\right)}{\sin\left(\frac{h}{2}\right)}.$$

We've seen the ratio of sines before. It is the complex Dirichlet kernel.

In summary, we've found that

$$\begin{bmatrix} \sum_{k=1}^{n}\cos(kh) & -\sum_{k=1}^{n}\sin(kh) \\ \sum_{k=1}^{n}\sin(kh) & \sum_{k=1}^{n}\cos(kh) \end{bmatrix} = S_n = \begin{bmatrix} \cos\left(\frac{n+1}{2}h\right) & -\sin\left(\frac{n+1}{2}h\right) \\ \sin\left(\frac{n+1}{2}h\right) & \cos\left(\frac{n+1}{2}h\right) \end{bmatrix}\frac{\sin\left(\frac{nh}{2}\right)}{\sin\left(\frac{h}{2}\right)},$$

where $h = 2\pi(t-s)$. Looking at the first column, we see that there are two sums of interest:

$$\begin{aligned} \sum_{k=1}^{n}\cos(kh)) &= \cos\left(\frac{n+1}{2}h\right)\frac{\sin\left(\frac{nh}{2}\right)}{\sin\left(\frac{h}{2}\right)} = \frac{\sin\left(\frac{2n+1}{2}h\right) - \sin\left(\frac{1}{2}h\right)}{2\sin\left(\frac{h}{2}\right)}, \quad \text{and} \\ \sum_{k=1}^{n}\sin(kh) &= \sin\left(\frac{n+1}{2}h\right)\frac{\sin\left(\frac{nh}{2}\right)}{\sin\left(\frac{h}{2}\right)} = \frac{\cos\left(\frac{h}{2}\right) - \cos\left(\frac{2n+1}{2}h\right)}{2\sin\left(\frac{h}{2}\right)}. \end{aligned}$$

The first of these is useful in the sum we originally sought in Equation (σ_n):

$$1 + 2\sum_{k=1}^{n}\cos(2\pi k(t-s)) = \frac{\sin((2n+1)\pi(t-s))}{\sin(\pi(t-s))} \equiv D_n(t-s),$$

the **real Dirichlet kernel**. The corresponding second sum is

$$2\sum_{k=1}^{n}\sin(2\pi k(t-s)) = \frac{\cos(\pi(t-s)) - \cos((2n+1)\pi(t-s))}{\sin(\pi(t-s))} \equiv \widetilde{D}_n(t-s),$$

the **conjugate Dirichlet kernel**. The mapping

$$f(x) \sim \frac{a_0}{2} + \sum_k a_k \cos(2\pi kx) + b_k \sin(2\pi kx) \to \frac{a_0}{2} + \sum_k a_k \sin(2\pi kx) - b_k \cos(2\pi kx) \sim \widehat{f}(x)$$

is the **Riesz transform** that Joel talked about in seminar at the beginning of the year. In the same way that

$$S_n = D_n * f,$$

the conjugate function satisfies

$$\widehat{S}_n = \widehat{D}_n * f.$$

The Riesz transform is also called the **Hilbert** transform. The point is that, while we worked (a bit) harder on real Fourier series, we learned two results.

Example 8: Bernoulli and Faulhaber: I'm fond of the symmetric Summation (GS_2), but there are historical reasons to use the "usual" Summation (GS_1) in Example 5. In that example,

$$S_n = \sum_{k=0}^{n} \begin{bmatrix} 1 & k & \frac{k^2}{2!} & \frac{k^3}{3!} \\ 0 & 1 & k & \frac{k^2}{2!} \\ 0 & 0 & 1 & k \\ 0 & 0 & 0 & 1 \end{bmatrix} = \sum_{k=0}^{n} e^{kA} \qquad \text{where} \qquad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Shift forward a full step (with no backward step), and take the difference (or use (GS_1) with $t_0 = 0$):

$$\left(e^A - I\right) S_n = e^{(n+1)A} - I. \tag{B_1}$$

Both differences are of the form

$$e^{tA} - I = \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \frac{t^3}{3!} \\ 0 & 1 & t & \frac{t^2}{2!} \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & t & \frac{t^2}{2!} & \frac{t^3}{3!} \\ 0 & 0 & t & \frac{t^2}{2!} \\ 0 & 0 & 0 & t \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The hitch here is that $e^A - I$ is not invertible, so we can't simply solve Equation (B_1) for S_n. We should not give up, though! We have

$$\begin{aligned} e^{tA} - I &= tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots \\ &= tA\left(I + \frac{t}{2!}A + \frac{t^2}{3!}A^2 + \cdots\right), \end{aligned}$$

and the matrix in parentheses *is* invertible (at least, for small t). Denote that matrix by

$$\frac{e^{tA} - I}{A} = I + \frac{t}{2!}A + \frac{t^2}{3!}A^2 + \cdots.$$

Then the left side of Equation (B_1) factors, giving

$$\left(\frac{e^A - I}{A}\right) AS_n = e^{(n+1)A} - I.$$

We conclude that

$$AS_n = \left(\frac{e^A - I}{A}\right)^{-1} \left(e^{(n+1)A} - I\right). \tag{B_2}$$

We have not found S_n, but we *have* found AS_n, which in Example 5 is very nearly as informative.

For the 4×4 matrix in Example 5, A^j is zero for $j \geq 4$, so

$$\frac{e^A - I}{A} = I + \frac{1}{2!}A + \frac{1}{3!}A^2 + \frac{1}{4!}A^3 = \begin{bmatrix} 1 & \frac{1}{2!} & \frac{1}{3!} & \frac{1}{4!} \\ 0 & 1 & \frac{1}{2!} & \frac{1}{3!} \\ 0 & 0 & 1 & \frac{1}{2!} \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

so, by inspection,

$$\left(\frac{e^A - I}{A}\right)^{-1} = \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{12} & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{12} \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The triangular structure of $\frac{e^A - I}{A}$ makes it straightforward (if tedious) to invert.

Equation (B_2) reads

$$AS_n = \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{12} & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{12} \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & n+1 & \frac{(n+1)^2}{2!} & \frac{(n+1)^3}{3!} \\ 0 & 0 & n+1 & \frac{(n+1)^2}{2!} \\ 0 & 0 & 0 & n+1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & n+1 & \frac{n(n+1)}{2} & \frac{2n^3+3n^2+n}{12} \\ 0 & 0 & n+1 & \frac{n(n+1)}{2} \\ 0 & 0 & 0 & n+1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (B_3)$$

Since AS_n is, visually, S_n with every row shifted up by one, we have the identity

$$\begin{bmatrix} 0 & 1 & k & \frac{k^2}{2!} \\ 0 & 0 & 1 & k \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & n+1 & \frac{n(n+1)}{2} & \frac{2n^3+3n^2+n}{12} \\ 0 & 0 & n+1 & \frac{n(n+1)}{2} \\ 0 & 0 & 0 & n+1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

from which we deduce the summation formulas

$$\sum_{k=0}^{n} 1 = n+1 \qquad \sum_{k=0}^{n} k = \frac{n(n+1)}{2} \qquad \sum_{k=0}^{n} k^2 = \frac{2n^3+3n^2+1}{6}.$$

It's easy to see how to extend Example 5 to find summation formulas of the form $\sum_{k=0}^{n} k^p$ for any natural number p. We could stop here, but that would be a shame.

The scalar version of Equation (B_2) has been thoroughly investigated. The scalar function $\frac{t}{e^t-1}$ has power series representation

$$\frac{t}{e^t - 1} = \sum_{\ell=0}^{\infty} B_\ell \frac{t^\ell}{\ell!},$$

where B_ℓ is the ℓ^{th} **Bernoulli number**[2]. Likewise, the scalar function

$$\frac{te^{xt}}{e^t - 1} = \sum_{\ell=0}^{\infty} B_\ell(x) \frac{t^\ell}{\ell!}$$

is the **generating function** for the Bernoulli polynomials $B_\ell(x)$[3]. In effect, the generating function defines the polynomials $B_n(x)$ as the coefficients of the power series representing the scalar function on the left. As

[2] Abramowitz, M. and Stegun, I. A. (Eds.), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Ch. 23 (Bernoulli and Euler Polynomials), 4th printing, National Bureau of Standards Applied Mathematics Series 55, 1964, p. 804.

[3] Abramowitz, M. and Stegun, I. A. (Eds.), *op cit*, p. 804.

with most named functions, mathematicians used to look up Bernoulli polynomials and numbers in tables[4], and we now simply ask our favorite computer algebra system for them. To illustrate our example, I looked up

$$\begin{aligned} B_0(x) &= 1 \\ B_1(x) &= -\frac{1}{2} + x \\ B_2(x) &= \frac{1}{6} - x + x^2 \\ B_3(x) &= \frac{1}{2}x - \frac{3}{2}x^2 + x^3 \end{aligned}$$

The definitions imply that $B_\ell = B_\ell(0)$. We therefore recognize that

$$\frac{t}{e^t - 1}\left(e^{xt} - 1\right) = \sum_{\ell=0}^{\infty} \left(B_\ell(x) - B_\ell\right) \frac{t^\ell}{\ell!}.$$

The corresponding matrix identity applies to Equation (B_2):

$$AS_n = \left(\frac{e^A - I}{A}\right)^{-1} \left(e^{(n+1)A} - I\right) = \sum_{\ell=0}^{\infty} \frac{B_\ell(n+1) - B_\ell(0)}{\ell!} A^\ell. \qquad (B_4)$$

Equation (B_4) is valid whenever the power series converge. For nilpotent matrices like the one in Example 5, convergence is guaranteed. Since A^4 and higher powers are all zero in this example, the first 4 Bernoulli polynomials above suffice to compute

$$\begin{aligned} \sum_{\ell=0}^{\infty} \frac{B_\ell(n+1) - B_\ell}{\ell!} A^\ell &= (n+1)A + \left(\frac{-(n+1) + (n+1)^2}{2!}\right) A^2 + \left(\frac{\frac{(n+1)}{2} - \frac{3}{2}(n+1)^2 + (n+1)^3}{3!}\right) A^3 \\ &= \begin{bmatrix} 0 & n+1 & \frac{n(n+1)}{2} & \frac{2n^3+3n^2+n}{12} \\ 0 & 0 & n+1 & \frac{n(n+1)}{2} \\ 0 & 0 & 0 & n+1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

The result agrees with the computation we made in Equation (B_3).

Sums of powers $\sum k^p$ expressed in terms of Bernoulli functions are called **Faulhaber's** identity.

[4] Abramowitz, M. and Stegun, I. A. (Eds.), *op cit*, p. 809.