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Strongly compact algebras associated with composition operators

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Dedicated to the memory of Nigel Kalton, 1946–2010.

Abstract. An algebra of bounded linear operators on a Hilbert space is called *strongly compact* whenever each of its bounded subsets is relatively compact in the strong operator topology. The concept is most commonly studied for two algebras associated with a single operator T: the algebra $\mathrm{alg}(T)$ generated by the operator, and the operator's commutant $\mathrm{com}(T)$. This paper focuses on the strong compactness of these two algebras when T is a composition operator induced on the Hardy space H^2 by a linear fractional self-map of the unit disc. In this setting, strong compactness is completely characterized for $\mathrm{alg}(T)$, and "almost" characterized for $\mathrm{com}(T)$, thus extending an investigation begun by Fernández-Valles and Lacruz [A spectral condition for strong compactness, *J. Adv. Res. Pure Math.* **3** (4) 2011, 50–60]. Along the way it becomes necessary to consider strong compactness for algebras associated with multipliers, adjoint composition operators, and even the Cesàro operator.

Contents

1. Introduction

An algebra of bounded linear operators on a Hilbert space is said to be *strongly compact* if, in the strong operator topology (the topology of pointwise convergence) every bounded subset of the algebra is relatively

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compact. This concept appears to have originated in the 1980 paper [16] of Lomonosov, where it is used to study the invariant subspace problem. About a decade later Marsalli [18] characterized the strongly compact self-adjoint subalgebras of operators on Hilbert space. Just recently Lacruz, Lomonosov, and Rodríguez-Piazza [14] set out many interesting results, examples, and counter-examples concerning strongly compact algebras, while Fernández-Valles and Lacruz [10] provided further examples and initiated the study of strong compactness for algebras connected with composition operators on the Hardy space H^2.

This paper continues the work of [10], with particular emphasis on composition operators induced by linear fractional self-maps of the unit disc. There results a complete characterization of strong compactness for the algebra generated by the operator, and an "almost complete" characterization for the commutant.[1] In the process, connections emerge with similar problems for multiplication operators, and even the Cesàro operator.

1.1. The operators. Although our primary concern here is with composition operators, there's no way to keep multiplication operators out of the discussion (see [24] for more on this phenomenon). Here is a formal introduction to both classes of operators.

Composition operators. Our setting will be the Hardy–Hilbert space H^2, which consists of functions holomorphic on the unit disc $\mathbb{U}$ with square-summable Maclaurin coefficient sequence. Each holomorphic function $\varphi : \mathbb{U} \to \mathbb{U}$ induces, at least on the space of *all* functions holomorphic on $\mathbb{U}$, a linear *composition operator* C_φ defined by the formula $C_\varphi f = f \circ \varphi$. The foundation for the study of composition operators is Littlewood's Subordination Principle, which implies that each composition operator restricts to a bounded operator on H^2; for more details see, for example, [23, Chapter 1].

The study of composition operators seeks to connect the function theoretic properties of the map φ with the operator theoretic properties of C_φ. In the present work we want to know how the properties of φ influence the strong compactness, or lack thereof, of both $\operatorname{alg}(C_\varphi)$, the algebra of operators on H^2 generated by C_φ and the identity (i.e., the collection of polynomials in C_φ) and $\operatorname{com}(C_\varphi)$, the algebra consisting of all operators on H^2 that commute with C_φ. The work below will focus mostly on the simplest composition operators: those induced by linear fractional self-maps of the unit disc. As is often the case when one studies composition operators, this apparently restrictive setting already furnishes a rich diversity of behavior.

[1]For composition-operator *aficionados:* the problem left open concerns a subclass of hyperbolic inducing maps, a typical one being $\varphi(z) = (1 + z)/2$.

Multiplication operators. In what follows, H^∞ denotes the space of bounded holomorphic functions on the open unit disc $\mathbb{U}$. For $b \in H^\infty$ write

$$\|b\|_\infty := \sup\{|b(z)| : |z| < 1\},$$

and let M_b denote the operator on H^2 of "multiplication by b":

$$(M_b f)(z) := b(z)f(z) \qquad (z \in \mathbb{U}, f \in H^2).$$

M_b is a bounded linear operator on H^2 of norm $\|b\|_\infty$ (see [8, Cor. 7.8, page 179] for this result in the more general setting of Toeplitz operators).

Multiplication operators on H^2 play an important role in the study of strong compactness for *commutants* of composition operators. Indeed, if $b \in H^\infty$ and $b \circ \varphi = b$, then M_b commutes with C_φ, hence $\mathrm{alg}(M_b) \subset \mathrm{com}(C_\varphi)$. Thus it becomes important to have information about which multiplication operators on H^2 generate strongly compact algebras.

1.2. Summary of main results. The new results obtained here concern composition operators on H^2 induced by linear fractional maps of $\mathbb{U}$ that fix a boundary point. These are necessarily either parabolic and hyperbolic (see below for more details); the table below summarizes the results, both new and "old" that are known for this situation. The shaded cells denote results obtained by Fernández-Valles and Lacruz in [10, Theorem 4.1]; the rest—with the exception of the cell marked "???"—denote new results to be proved below. The abbreviations in the table's first column are: P = "parabolic", H = "hyperbolic", A = "automorphic", NA = "nonautomorphic", SC = "strongly compact." In the bottom row, "$\mathbb{U}_e$" denotes the complement, in the Riemann sphere, of the closed unit disc, while the notation "???" means "Open Problem."

The rest of this paper proceeds as follows. The next section gathers up some prerequisites on strong compactness, linear fractional maps, and multiplication operators. In particular, §2.3 covers strong compactness for composition operators induced by linear fractional transformations with no fixed point on $\partial\mathbb{U}$ (the situation *not* covered by Table 1). To make the exposition more self contained, selected proofs will be included.

Section 3 discusses multiplication operators on H^2, proving some results needed for the "commutant column" of Table 1. The heart of the paper is §4 which establishes the results populating the unshaded entries in Table 1. That section also includes a brief discussion of the shaded entries—results of Fernández-Valles and Lacruz from [10]—as well as some results on composition operators induced by more general maps.

Section 5 takes up the notion of strong compactness for algebras associated with *adjoints* of composition operators and multipliers. Here results for the algebra generated by such an adjoint are definitive: The algebra generated by the adjoint of a multiplier is *always* strongly compact, while $\mathrm{alg}(C_\varphi^*)$ is strongly compact iff φ fixes a point of $\mathbb{U}$. For commutants, however, the situation is more interesting, and decisive results are yet to be found.

Type of φ	Fixed pt. position	alg(C_φ)	com(C_φ)	Example: $\varphi(z) =$	Where?
PA	$\partial\mathbb{U}$ only	SC	not SC	$\frac{(1+i)z-1}{z+(-1+i)}$	Thm. 4.1.1
PNA	$\partial\mathbb{U}$ only	SC	SC	$\frac{1}{2-z}$	Thm. 4.1.1
HA	$\partial\mathbb{U}$ only	SC	not SC	$\frac{1+2z}{2+z}$	Thm. 4.1.1
HNA	$\partial\mathbb{U}$ & $\mathbb{U}$	not SC	not SC	$\frac{z}{2-z}$	Thm. 4.1.2
	$\partial\mathbb{U}$ & $\mathbb{U}_e$	SC	???	$\frac{1+z}{2}$	Thm. 4.1.3

TABLE 1. *Strong compactness for* $\varphi \in \mathrm{LFT}(\mathbb{U})$ *with a fixed point on the unit circle. Shaded cells show results from* [10].

The paper closes in §6 with a discussion of the main problem left open:

> *If* φ *has a fixed point on* $\partial\mathbb{U}$ *and another one in* $\mathbb{U}_e$ *(example:* $\varphi(z) = (1+z)/2$*), is* $\mathrm{com}(C_\varphi)$ *strongly compact?*

The Cesàro operator enters the discussion, and this affords the opportunity to answer a question left open in [10] about strong compactness for the algebra it generates.

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2. Prerequisites

2.1. Strong compactness. We'll be working in the simplest nontrivial setting, where "operator" means "bounded linear operator on some separable Hilbert space," with the generic such space denoted by the symbol "H". For an operator T on H, denote by $\mathrm{alg}(T)$ the algebra generated by T and the identity operator (i.e., the collection of polynomials in T), and by $\mathrm{com}(T)$ the commutant of T—all those operators on H that commute with T. Since $\mathrm{alg}(T) \subset \mathrm{com}(T)$, strong compactness for $\mathrm{com}(T)$ implies the same for $\mathrm{alg}(T)$. It is easy to see that:

(i) For either of these algebras, strong compactness (or the lack thereof) is preserved when similarity transformations are applied to T.

(ii) Strong compactness for $\mathrm{alg}(T)$ implies the same for its closure in the operator norm. Here are a few more preliminary results—all well known— with selected proofs presented in the interest of expositional completeness.

Proposition 2.1.1. *If H is finite dimensional, then the algebra of all operators on H is strongly compact; otherwise it's not.*

Proof. If H is finite dimensional then so is the algebra of all its operators, so all the usual operator topologies thereon coincide with the Euclidean topology—in which the closed unit ball is compact. In the infinite dimensional case even the algebra of compact operators is not strongly compact. Indeed, let (e_n) be an orthonormal basis for H. Then the sequence of rank-one operators $T_n : x \to \langle e_1, x\rangle e_n$ converges to the zero-operator in the weak operator topology, but $\|T_n e_1\| = 1$ for each n. Consequently $(T_n e_1)$ has no subsequence that converges in H, hence the operator sequence (T_n) has no strongly convergent subsequence. □

For an algebra $\mathscr{A}$ of operators on H let $\mathscr{A}_1$ denote those operators in $\mathscr{A}$ of norm ≤ 1. For $x \in H$ let $\mathscr{A}_1 x := \{Ax : A \in \mathscr{A}_1\}$. A useful consequence of the Tychonoff Product Theorem, previously observed in [14, 18], is:

Proposition 2.1.2. *An algebra $\mathscr{A}$ of operators on H is strongly compact if and only if $\mathscr{A}_1 x$ is relatively compact in H for every vector x in a dense subset of H.*

This fact, along with the strong compactness of the algebra of all operators in the finite dimensional case, quickly yields the following useful sufficient condition, first noted in [14, Prop. 1, page 193]:

Proposition 2.1.3. *An algebra $\mathscr{A}$ of bounded operators on H is strongly compact whenever H contains a collection of finite dimensional subspaces with dense union, each of which is invariant for (every operator in) the algebra.*

Here is an illustration of the utility of Proposition 2.1.3; along with Corollary 3.1.2 of the next section, it shows that strong compactness (or lack thereof) need not survive the taking of adjoints. Following common practice we abuse notation by writing M_z for the operator of "multiplication by z", i.e., the operator M_b where $b(z) \equiv z$.

Proposition 2.1.4. $\mathrm{com}(M_z^*)$ is *strongly compact on H^2.*

Proof. It is well known that $\mathrm{com}(M_z) = \{M_f : f \in H^\infty\}$ (see [11, Problem 147, page 79], for example), so $\mathrm{com}(M_z^*) = \{M_f^* : f \in H^\infty\}$. It's also well known—and easy to check (see, e.g., [4, Prop. 2.3])—that for each $f \in H^\infty$ and $a \in \mathbb{U}$ we have $M_f^* K_a = \overline{f(a)}\, K_a$ where K_a is the H^2 reproducing kernel for the point a:

$$K_a(z) = \frac{1}{1 - \overline{a}z} \qquad (z \in \mathbb{U}). \tag{1}$$

Thus for each $a \in \mathbb{U}$ the one dimensional subspace $V_a := \operatorname{span}\{K_a\}$ is invariant for each operator in $\operatorname{com}(M_z^*)$. Since the set $\{K_a : a \in \mathbb{U}\}$ spans a dense subspace of H^2, the desired result follows from Proposition 2.1.3. □

For more information on strong compactness of algebras generated by adjoints of multiplication operators, see §5.2.

A particularly important special case of Proposition 2.1.3 involves eigenvectors. It appears to have first been proved by Marsalli [18], and plays a significant role in [10].

Corollary 2.1.5. *If an operator T has a densely spanning collection of eigenvectors then* $\operatorname{alg}(T)$ *is strongly compact. If, in addition, each of the corresponding eigenspaces has finite dimension, then* $\operatorname{com}(T)$ *is strongly compact.*

Proof. For λ an eigenvalue of T let E_λ denote the λ-eigenspace, i.e., the set of vectors x such that $Tx = \lambda x$. Our overarching assumption is that there is a set D of eigenvalues such that $\bigcup\{E_\lambda : \lambda \in D\}$ is dense in H. Each eigenvector of T spans a one dimensional $\operatorname{alg}(T)$-invariant subspace, hence, by Proposition 2.1.3, $\operatorname{alg}(T)$ is strongly compact. Furthermore each E_λ is invariant for $\operatorname{com}(T)$, hence if the eigenspace E_λ is finite dimensional for each $\lambda \in D$ then Proposition 2.1.3 also implies that $\operatorname{com}(T)$ is strongly compact. □

Remark. The proof of Proposition 2.1.3 (and therefore of Corollary 2.1.5) yields a bit more than advertised. It's easy to check that whenever an algebra of operators satisfies the hypotheses of the Proposition then so does its closure in the weak operator topology, hence that closure is also strongly compact. For example, since each of the positive results listed in Table 1 concerning strong compactness of algebras generated by individual composition operators C_φ is proved using Corollary 2.1.5, those results are actually true for the weak-operator closure of $\operatorname{alg}(C_\varphi)$.

On the other hand Lacruz and Rodríguez-Piazza have shown in [15] that strong compactness for the algebra generated by an operator—even a *normal* operator—does not always carry over to the weak-operator closure of that algebra.

While compact operators need not generate strongly compact algebras (see [14, Prop. 8, page 200] for a class of weighted shifts with this property), the situation is different for compact operators with dense range.

Proposition 2.1.6 ([14], Prop. 2, page 193)**.** *If a compact operator K has dense range, then* $\operatorname{com}(K)$ *is strongly compact.*

Proof. Let $\mathscr{C} = \mathrm{com}(K)$, and let $\mathscr{C}_1$ be the set of operators in $\mathscr{C}$ of norm ≤ 1. Then for $x \in H$:

$$\mathscr{C}_1(Kx) = K(\mathscr{C}_1\, x) = K\,(\text{a bounded subset of } H),$$

hence, by the compactness of the operator K the set $\mathscr{C}_1(Kx)$ is relatively compact in H. Thus each of the sets $\mathscr{C}_1\, y$ is relatively compact in H for a dense set of vectors y, namely—since K has dense range—those of the form $y = Kx$ for $x \in H$. This guarantees, by Proposition 2.1.2, that $\mathscr{C}$ is a strongly compact algebra. □

2.2. Linear fractional matters. We use the term *linear fractional transformation* (abbreviated "LFT") to denote a mapping of the Riemann sphere $\widehat{\mathbb{C}}$:

$$z \to \frac{az+b}{cz+d} \tag{2}$$

where the coefficients a, b, c, d are complex numbers with $ad - bc \neq 0$, this last condition guaranteeing that the mapping is nonconstant—in fact, a homeomorphism of $\widehat{\mathbb{C}}$—where in (2) the usual algebraic conventions apply to the point at infinity.

We will be particularly interested in $\mathrm{LFT}(\mathbb{U})$, the collection of linear fractional transformations that take the unit disc into itself. Within $\mathrm{LFT}(\mathbb{U})$ is the collection of linear fractional maps taking $\mathbb{U}$ *onto* itself. These turn out to be precisely the *conformal automorphisms* of $\mathbb{U}$, i.e., the univalent holomorphic maps taking $\mathbb{U}$ onto itself (see, e.g., [21, Theorem 12.6, page 255]); we'll use the notation $\mathrm{Aut}(\mathbb{U})$ to designate them.

Each LFT not the identity map has either one or two fixed points. Those with one fixed point are the *parabolic* ones, and each of these is easily seen—upon conjugation with an LFT that takes the fixed point to ∞—to be conjugate to a translation. From this it follows readily that if a parabolic LFT fixes a disc or halfplane then its fixed point must lie on the boundary of that disc or halfplane.

In the remaining case our LFT, let's call it Φ, has two fixed points. Upon conjugating with an LFT that takes one of these points to the origin and the other to ∞ we see that Φ is conjugate to a dilation $z \to \mu z$ for some complex number $\mu \neq 1$. This dilation parameter μ is called the *multiplier* of Φ (note, however, that $1/\mu$ has equal claim to being the multiplier). Φ is said to be *hyperbolic* if μ is positive; *elliptic* if $|\mu| = 1$, and *loxodromic* otherwise. It's easy to see that an elliptic transformation that preserves the unit disc must map that disc onto itself, i.e., it must be an automorphism. For more details on these matters see, e.g., [23, Chapter 0].

We will require the following simple result about composition operators induced by linear fractional self-maps of $\mathbb{U}$:

Proposition. *For each $\varphi \in \mathrm{LFT}(\mathbb{U})$ the operator C_φ has dense range.*

Proof. $\Delta := \varphi(\mathbb{U})$ is a disc contained in $\mathbb{U}$, so there is also a map $L(z) \equiv az + b$ in LFT($\mathbb{U}$) that takes $\mathbb{U}$ onto Δ. Thus we have the factorization $\varphi = L \circ \alpha$ where $\alpha := L^{-1} \circ \varphi$ is an automorphism of $\mathbb{U}$. This induces a factorization $C_\varphi = C_\alpha C_L$, where C_α is an invertible operator on H^2. Thus, to establish the density of the range of C_φ it's sufficient to do the same for the range of C_L. But this is easy: $\operatorname{ran} C_L$ contains every polynomial! Indeed, let p be a polynomial and set $q = (p - b)/a = L^{-1}(p)$. Then q is a polynomial, so belongs to H^2, and $C_L(q) = p$. Thus $p \in \operatorname{ran} C_L$, as desired. □

This result is a very special case of something far more general (cf. [3, Prop. 1.5, page 18]): *If φ is a holomorphic self-map taking $\mathbb{U}$ onto a domain bounded by a Jordan curve, then C_φ has dense range.* The result itself, along with Proposition 2.1.6 implies that: *Every compact linear fractionally induced composition operator has strongly compact commutant.* See Theorem 4.2.3 for a more general result.

It is well known that a linear fractional composition operator is compact if and only if the image of its inducing map has closure contained in $\mathbb{U}$. However the situation for general compact composition operators on H^2 is much more subtle. For more on this see, e.g., [22], or [23, Chapters 2 & 10].

2.3. Composition operators: first results. Here is a summary, taken mostly from [10] of results on strong compactness for composition operators on H^2 induced by linear fractional self-maps of $\mathbb{U}$ that have *no* fixed point on $\partial\mathbb{U}$.

Each such map φ has exactly two fixed points: one in $\mathbb{U}$ and the other in $\mathbb{U}_e$, the complement in the Riemann sphere of the closed unit disc. Upon conjugating φ by an appropriate conformal automorphism of $\mathbb{U}$ we arrive at an "interior-exterior" map whose interior fixed point is at the origin. Such a conjugation induces a similarity of composition operators which—as we have already noted—does not affect strong compactness (or lack thereof) for either $\operatorname{alg}(C_\varphi)$ or for $\operatorname{com}(C_\varphi)$. Thus we may without loss of generality, assume that $\varphi(0) = 0$.

Suppose first that φ is an automorphism, i.e., $\varphi(\mathbb{U}) = \mathbb{U}$. Then $\varphi(\partial\mathbb{U}) = \partial\mathbb{U}$, so by reflection $\varphi(\infty) = \infty$, hence φ is a rotation: $\varphi(z) \equiv \omega z$ for some unimodular complex number $\omega \neq 1$ (i.e., the original map was elliptic). In this case each monomial z^n is an eigenvector of C_φ with eigenvalue ω^n ($n = 0, 1, 2, \ldots$). Since these eigenvectors span a dense subspace of H^2 it follows from the first part of Corollary 2.1.5 that $\operatorname{alg}(C_\varphi)$ is strongly compact. If, in addition, ω is not a root of unity, then each ω^n-eigenspace is *simple*, so the second part of Corollary 2.1.5 guarantees that $\operatorname{com}(C_\varphi)$ is strongly compact.

If, however ω *is* a root of unity, say $\omega^N = 1$, then H^2 decomposes into the orthogonal direct sum of eigenspaces

$$E_k := \ker(C_\varphi - \omega^k I) = \overline{\operatorname{span}}\{z^{k+jN}\, j = 0, 1, 2, \ldots\}$$

with $C_\varphi = \omega^k I$ on E_k ($k = 0, 1, 2, \ldots N-1$). Thus for any operators $T_k : E_k \to E_k$ the operator

$$T_0 \oplus T_1 \oplus \cdots T_{N-1} : H^2 \to H^2$$

commutes with C_φ, hence the algebra of all such operators, which is not strongly compact, is contained in $\mathrm{com}(C_\varphi)$. Thus $\mathrm{com}(C_\varphi)$ is itself not strongly compact. In summary:

Proposition 2.3.1 ([10], Theorem 4.1)**.** *If φ is a linear fractional self-map of $\mathbb{U}$ that is elliptic, then* $\mathrm{alg}(C_\varphi)$ *is strongly compact. Furthermore:* $\mathrm{com}(C_\varphi)$ *is strongly compact if and only if the multiplier of φ is not a root of unity.*

What about nonelliptic interior-exterior linear fractional self-maps of $\mathbb{U}$? As we noted above, such a map cannot be surjective. The following proposition, partially treated in [10, Theorem 4.1], tells the story.

Proposition 2.3.2. *Suppose φ is a linear fractional self-map of $\mathbb{U}$ with fixed points in $\mathbb{U}$ and $\mathbb{U}_e$, but with $\varphi(\mathbb{U}) \neq \mathbb{U}$. Then* $\mathrm{com}(C_\varphi)$*, hence also* $\mathrm{alg}(C_\varphi)$*, is strongly compact.*

Proof. The key is to note that $\psi := \varphi \circ \varphi$, which is also a linear fractional self-map of $\mathbb{U}$, maps the closure of $\mathbb{U}$ into $\mathbb{U}$. Indeed, suppose this is not the case, i.e., that $|\psi(\omega)| = 1$ for some $\omega \in \partial\mathbb{U}$. Then both $\eta := \varphi(\omega)$ and $\varphi(\eta) = \psi(\omega)$ (which are distinct because φ has no fixed point on $\partial\mathbb{U}$) belong to $\partial\mathbb{U}$, so $\varphi(\partial\mathbb{U})$ is a circle in the closed unit disc that contains two distinct points of the unit circle, and so must be the whole unit circle, contradicting our assumption that $\varphi(\mathbb{U}) \neq \mathbb{U}$.

It follows that $C_\psi = C_\varphi^2$ is compact on H^2 (again, see e.g. [23, Chapter 2]), and by Proposition 2.2 it has dense range. Thus by Proposition 2.1.6 $\mathrm{com}(C_\varphi^2)$ is strongly compact, hence so is its subalgebra $\mathrm{com}(C_\varphi)$. □

Remark. An example of a map φ with the "interior-exterior" fixed point configuration of Proposition 2.3.2 for which the closure $\varphi(\mathbb{U})$ is *not* contained in $\mathbb{U}$ is $\varphi(z) = (1-z)/2$, which fixes the points $1/3$ and ∞, and for which $\varphi(-1) = 1$. (Note that we have here a loxodromic map that—contrary to what is stated in [10], first line of the proof of Theorem 4.1—is *not* conjugate, via automorphisms of $\mathbb{U}$ to a dilation.)

3. Multiplication operators

As pointed out at the end of §1.1, the study of strong compactness for commutants of composition operators on H^2 leads one naturally to consider the same question for algebras generated by multiplication operators.

3.1. Multiplier algebras that are not strongly compact. For $b \in H^\infty$ let

$$E(b) := \{\zeta \in \partial\mathbb{U} : |b(\zeta)| = \|b\|_\infty\}.$$

The notation here refers to the fact that, for almost every $\zeta \in \partial\mathbb{U}$, the function b has a radial limit, which we'll denote by $b(\zeta)$. Moreover, $\|b\|_\infty$ coincides with the essential supremum of the moduli of these radial limits (all measure-theoretic concepts being defined relative to Lebesgue arclength measure on the unit circle). With these ideas in hand, here's a necessary condition for strong compactness of $\mathrm{alg}(M_b)$; although nowhere near sufficient, it will be useful later on (see, for example, the proof of Theorem 4.1.1 below).

Theorem 3.1.1. *Suppose $b \in H^\infty$ is not constant, and that $\mathrm{alg}(M_b)$ is strongly compact on H^2. Then $E(b)$ has measure zero.*

Proof. Without loss of generality we may assume $\|b\|_\infty = 1$. For each positive integer n let $p_n(z) := z^n$. Since b is not constant, $b^n \to 0$ pointwise on $\mathbb{U}$, and therefore (since the sequence is bounded in H^2) also weakly in H^2. Since $\mathrm{alg}(M_b)$ is assumed to be strongly compact, and each of the operators $p_n(M_b) = M_{b^n}$ has norm 1 (since $\|M_{b^n}\| = \|b^n\|_\infty = 1$), the sequence of vectors $(p_n(M_b)1 : n \geq 0) = (b^n : n \geq 0)$ is relatively compact in H^2, and therefore must converge to 0 in the norm of that space. Thus, letting m denote normalized arclength measure on $\partial\mathbb{U}$:

$$0 = \lim_n \|p_n(M_b)1\|_2^2 = \lim_n \int_{\partial\mathbb{U}} |b|^{2n} dm \geq \lim_n \int_{E(b)} |b|^{2n} dm = m(E(b))$$

hence $m(E(b)) = 0$, as promised. □

Recall that in case $b(z) \equiv z$ we adopt the familiar abuse of notation, denoting M_b simply by M_z.

Corollary 3.1.2. $\mathrm{alg}(M_z)$ *is not strongly compact on H^2.*

The next example shows that the necessary condition $m(E(b)) = 0$ of Theorem 3.1.1 is far from sufficient for $\mathrm{alg}(M_b)$ to be relatively compact on H^2.

Example 3.1.3. *There exists $b \in H^\infty$ with $m(E(b)) = 0$, yet for which $\mathrm{alg}(M_b)$ is not strongly compact.*

Proof. Let $b(z) = (1 + z)/2$ and set $p(w) = b^{-1}(w) = 2w - 1$ $(z, w \in \mathbb{C})$. Then

$$p^n(M_b) = M_{p^n \circ b} = M_{(p \circ b)^n} = M_{z^n} \qquad (n = 0, 1, 2, \ldots)$$

in particular $p^n(M_b)$ is, for each index n, an operator in $\mathrm{alg}(M_b)$ of norm 1. We have already noted that the sequence $(p^n(M_b)1) = (z^n)$ has no subsequence convergent in H^2. Thus $\mathrm{alg}(M_b)$ is not strongly compact. □

This argument can be extended to provide a significant generalization of Theorem 3.1.1. In its statement, presented below as a sufficient condition for "non-strong-compactness", we will think of the bounded analytic function b as extended to a.e. point of the unit circle via nontangential limits, and will use the term "Jordan domain" to mean the (necessarily simply connected) domain interior to a Jordan curve.

Theorem 3.1.4. *Suppose $b \in H^\infty$, Ω is a Jordan domain that contains $b(\mathbb{U})$, and there is a subset E of $\partial\mathbb{U}$ having positive arc-length measure with $b(E) \subset \partial\Omega$. Then* $\mathrm{alg}(M_b)$ *is not strongly compact on H^2.*

Proof. The Riemann Mapping Theorem provides a univalent holomorphic map g taking Ω onto $\mathbb{U}$, and because Ω is a Jordan domain a theorem of Carathéodory (see [19, Theorem 2.6, page 24]) guarantees that g extends to a homeomorphism (which we'll still call g) taking the closure of Ω onto the closed unit disc. The function $g \circ b$ is a holomorphic self-map of $\mathbb{U}$ which, by the definition of E and the boundary-continuity of g, has radial limits of modulus one on E. Thus by Theorem 3.1.1, $\mathrm{alg}(M_{g\circ b})$ is not strongly compact.

Since Ω is a Jordan domain its closure has connected complement, so Mergelyan's Theorem [21, Theorem 20.5, page 390] guarantees that the polynomials are dense in $A(\Omega)$, the space of functions continuous on the closure of Ω and holomorphic on its interior (taken in the supremum norm). Thus there is a sequence of polynomials (p_n) that converges uniformly on Ω to g, and so $M_{b\circ g}$ is the limit, in the H^2-operator norm, of the sequence of operators $p_n(M_b) = M_{p_n\circ b}$, each of which belongs to $\mathrm{alg}(M_b)$. Thus $M_{b\circ g}$ also belongs to $\overline{\mathrm{alg}}(M_b)$, the norm-closure of $\mathrm{alg}(M_b)$, hence $\overline{\mathrm{alg}}(M_b)$ contains $\overline{\mathrm{alg}}(M_{g\circ b})$, which in turn contains $\mathrm{alg}(M_{g\circ b})$. We showed in the last paragraph that this latter algebra is not strongly compact, so neither is $\overline{\mathrm{alg}}(M_b)$. As we noted in §2.1, it's easy to check that an algebra of operators is strongly compact if and only if its operator-norm closure is strongly compact. Thus, as promised, $\mathrm{alg}(M_b)$ is not strongly compact. □

3.2. Multiplier algebras that *are* strongly compact. In contrast to the results above, there are nontrivial multiplication operators that *do* generate strongly compact algebras. The result below shows that to get a sufficient condition for this to happen, we need only add to the necessary condition of Theorem 3.1.1 one additional restriction.

Theorem 3.2.1. *Suppose $b \in H^\infty$ with $\|b\|_\infty = 1$, and that $|b| < 1$ a.e. on $\partial\mathbb{U}$. If, in addition, the closure of $b(\mathbb{U})$ contains $\partial\mathbb{U}$, then* $\mathrm{alg}(M_b)$ *is strongly compact on H^2.*

Proof. Fix a sequence of operators in the unit ball of $\mathrm{alg}(M_b)$. The n-th term of this sequence has the form $p_n(M_b)$ where p_n is a polynomial for which

$$1 \geq \|p_n(M_b)\| = \|M_{p_n\circ b}\| = \|p_n \circ b\|_\infty = \|p_n\|_\infty$$

with the last equality arising from the maximum principle and our hypothesis that the closure of $b(\mathbb{U})$ contains $\partial\mathbb{U}$.

Being uniformly bounded, the collection of polynomials (p_n) is a normal family on $\mathbb{U}$, hence there is a subsequence (p_{n_k}) that converges uniformly on compact subsets of $\mathbb{U}$ to a function f, bounded (by 1) and holomorphic on $\mathbb{U}$. Thus $p_{n_k} \circ b \to f \circ b$ uniformly on compact subsets of $\mathbb{U}$.

Fix $h \in H^2$ and observe that:

$$\|p_{n_k}(M_b)h - M_{f\circ b}\, h\| = \int_{\partial \mathbb{U}} |p_{n_k} \circ b - f \circ b|^2 \, |h|^2 \, dm.$$

Now a.e. on $\partial\mathbb{U}$ we have $|b| < 1$, hence the integrand on the right $\to 0$ a.e. on $\partial\mathbb{U}$. Since that integrand is bounded a.e. on $\partial\mathbb{U}$ by $4|h|^2$, the Dominated Convergence Theorem guarantees that $\|p_{n_k}(M_b)h - M_{f\circ b}\, h\|_2 \to 0$ as $n \to \infty$.

Thus the (closed) unit ball of $\mathrm{alg}(M_b)$ is sequentially relatively compact in the strong operator topology, hence—because H^2 is separable, making its bounded sets strongly metrizable—it is relatively compact in that topology. In other words, $\mathrm{alg}(M_b)$ is strongly compact. □

Here is an application of Theorem 3.2.1. For $\lambda \in \mathbb{R}$ set $f_\lambda(z)$ equal to the principal value of $(1-z)^{i\lambda}$. Note that f_λ is bounded on the unit disc. For future reference note that this example is not randomly chosen; it is, for each number s in the open unit interval, an eigenfunction for the composition operator C_{sz+1-s}, with corresponding eigenvalue $s^{i\lambda}$.

An exercise in conformal mapping shows that f_λ takes $\overline{\mathbb{U}}\backslash\{1\}$ (the closed unit disc with the singularity at 1 removed) onto the open annulus $\{1/\rho_\lambda < |w| < \rho_\lambda\}$, where $\rho_\lambda := e^{\lambda\pi/2}$. Thus $b_\lambda := e^{-\rho_\lambda} f_\lambda$ maps $\overline{\mathbb{U}}\backslash\{1\}$ onto an open annulus whose outer boundary is the unit circle. The function b_λ therefore satisfies the hypotheses of Theorem 3.2.1, and so its associated multiplier generates a strongly compact algebra, hence the same is true for f_λ. In summary:

Proposition 3.2.2. *For* $\lambda \in \mathbb{R}$ *and* $z \in \mathbb{U}$ *let* $f_\lambda(z) := (1-z)^{i\lambda}$. *Then* $f_\lambda \in H^\infty$ *and* $\mathrm{alg}(M_{f_\lambda})$ *is strongly compact on* H^2.

Remark. Suppose, more generally, that $F_\zeta(z) := (1-z)^\zeta$, and that $\mathrm{Re}\,\zeta \geq 0$, so $F_\zeta \in H^\infty$. We've just seen that $\mathrm{alg}(F_\zeta)$ is strongly compact whenever ζ is pure imaginary. What if ζ is *not* pure imaginary? In that case another exercise in conformal mapping shows that F_ζ satisfies the hypotheses of Theorem 3.1.4, hence $\mathrm{alg}(M_{F_\zeta})$ is *not* strongly compact!

Question. For which $f \in H^\infty$ is $\mathrm{alg}(M_f)$ strongly compact?

4. Main results

Having completed in §1 the discussion of strong compactness for composition operators induced by linear fractional maps that fix no point of the unit circle, and having laid in §2 and §3 the groundwork concerning strong compactness and multiplier algebras, we now give proofs for the results (both new and old) summarized in Table 1—the case where at least one fixed point lies on the circle. There follows a brief discussion of strong compactness in the "non-linear-fractional" setting.

4.1. Linear fractional maps with a fixed point on $\partial\mathbb{U}$.

Theorem 4.1.1 (All fixed points on $\partial\mathbb{U}$). *Suppose $\varphi \in \mathrm{LFT}(\mathbb{U})$ has all its fixed points on $\partial\mathbb{U}$, so is either parabolic or is a hyperbolic automorphism. Then* $\mathrm{alg}(C_\varphi)$ *is strongly compact. In addition:*

(i) *If φ is a parabolic nonautomorphism, then* $\mathrm{com}(C_\varphi)$ *is strongly compact* [10, Theorem 4.1].

(ii) *If φ is an automorphism (parabolic or hyperbolic), then* $\mathrm{com}(C_\varphi)$ *is* not *strongly compact.*

Proof. Suppose first that φ is parabolic so there's just one fixed point, which we may assume without loss of generality to be the point 1. The map $\tau(z) = (1+z)/(1-z)$ sends $\mathbb{U}$ to the open right half plane RHP, and the point 1 to ∞, hence $\Phi(w) = \tau \circ \varphi \circ \tau^{-1}$ is a translation that maps RHP to itself, and so has the form $\Phi(w) = w + a$ where the "translation parameter" a has nonnegative real part. Back in the unit disc this provides the formula

$$\varphi(z) = \frac{(2-a)z + a}{-az + (2+a)}\,. \tag{3}$$

Furthermore, for each $\lambda \geq 0$ the exponential $E_\lambda(w) := \exp(-\lambda w)$ is bounded on RHP, with $E_\lambda \circ \Phi = \exp(-\lambda a)E_\lambda$.

Thus, upon defining $e_\lambda := E_\lambda \circ \tau$, we have $e_\lambda \in H^\infty$ and $C_\varphi e_\lambda = \exp(-\lambda a)\, e_\lambda$, i.e., e_λ is an eigenvector of $C_\varphi\colon H^2 \to H^2$ corresponding to the eigenvalue $\exp(\lambda a)$. Note that

$$e_\lambda(z) = \exp\left(-\lambda\,\frac{1+z}{1-z}\right) = e_1(z)^\lambda \tag{4}$$

where e_1 is the "unit singular function." So e_λ is an inner function, and it is a folk-theorem that the collection $\{e_\lambda : \lambda \geq 0\}$ spans a dense subset of H^2 (see, e.g., [10, Lemma 5.1] for a proof). Thus in the parabolic case, whether automorphic or not, Corollary 2.1.5 insures that $\mathrm{alg}(C_\varphi)$ is strongly compact.

For $\mathrm{com}(C_\varphi)$ the situation is more subtle. By the argument given above, the point spectrum of C_φ contains the curve $\Gamma_a := \{\exp(-\lambda a) : \lambda \geq 0\}$ (in fact this curve is the *entire* point spectrum, but we do not need this fact here). However, as we'll now see, the character of the eigenspaces associated to the points of Γ_a differs dramatically when one passes from the nonautomorphic to the automorphic case.

The parabolic nonautomorphism case [10, Theorem 4.1]. If the parabolic map $\varphi \in \mathrm{LFT}(\mathbb{U})$ is not an automorphism, then its *alter ego* Φ is not an automorphism of the right half-plane. Thus $\Phi(w) = w + a$ with $\mathrm{Re}\, a > 0$, hence as λ traverses the nonnegative real axis, the eigenvalues $\exp(-\lambda a)$ that constitute the curve Γ_a traverse either the interval $(0, 1]$ (when a is real) or a spiral starting at the point 1 and converging to the origin, circling the origin infinitely often. More to the point for us: these eigenvalues are all distinct, and *each one has multiplicity one.*

In view of the importance of this (admittedly, well known) last assertion, it seems worthwhile to give a proof. Suppose $f \in H^2$ is an eigenfunction of C_φ for the eigenvalue $e^{-\lambda a}$. Our goal is to show that f is a constant multiple of e_λ.

To this end, for n a nonnegative integer let $z_n = \varphi_n(0)$, where φ_n denotes the n-th iterate of φ (φ_0 being the identity map on $\mathbb{U}$). Then for each n:

$$f(z_n) = C_\varphi^n f(0) = e^{-n\lambda a} f(0),$$

and similarly $e_\lambda(z_n) = e^{-n\lambda a} e_\lambda(0)$. Thus we have $f(z_n) = ce_\lambda(n)$ for each index n, where $c := f(0)/e_\lambda(0) = e^\lambda f(0)$.

Since the translation parameter of φ_n is na it follows from (3) that

$$\varphi_n(z) = \frac{(2-na)z + na}{-naz + (2+na)}$$

whereupon

$$z_n := \varphi_n(0) = \frac{na}{2+na}.$$

After some algebraic manipulation this formula yields

$$\lim_{n\to\infty} n(1-|z_n|^2) = \frac{2\,\mathrm{Re}\,a}{|a|^2} > 0$$

hence

$$\sum_{n=0}^{\infty}(1-|z_n|^2) = \infty\,. \tag{5}$$

Since both f and $c\,e_\lambda$ belong to H^2 (the former by hypothesis, and the latter by the boundedness of e_λ), the fact that both functions agree on the sequence $\{z_n\}$ implies, by (5), that they agree at every point of $\mathbb{U}$ (see, e.g., [9, Theorem 2.3, page 18] or [21, Theorem 15.23, pp. 311–312]), i.e., $f = c\,e_\lambda$, as desired.

In summary: C_φ has a densely spanning collection of eigenfunctions corresponding to eigenvalues of multiplicity one. Thus by Proposition 2.1.5, $\mathrm{com}(C_\varphi)$ is strongly compact.

The parabolic automorphic case. For this one the translation parameter a is purely imaginary: $a = i\alpha$ for some real α. Thus the curve Γ_a (which is still the point spectrum of C_φ) is the unit circle traversed infinitely often as λ traverses the positive real axis. In particular, if $\lambda = 2\pi/\alpha$ and $f = e_\lambda$, then $f \circ \varphi = f$. Since f belongs to H^∞ it induces a multiplier M_f on H^2, and

$$C_\varphi M_f = M_{f\circ\varphi} C_\varphi = M_f C_\varphi$$

i.e., $M_f \in \mathrm{com}(C_\varphi)$. Thus $\mathrm{alg}(M_f) \subset \mathrm{com}(C_\varphi)$. But f, being an inner function, assumes values of maximum modulus (namely 1) at almost every point of $\partial\mathbb{U}$, so by Theorem 3.1.1, $\mathrm{alg}(M_f)$ is not strongly compact, hence neither is $\mathrm{com}(C_\varphi)$.

The hyperbolic automorphic case. Suppose φ is hyperbolic and automorphic, so both its fixed points lie on $\partial\mathbb{U}$. We may suppose, after conjugation by an appropriate automorphism, that these fixed points are at ± 1, with -1 being the attractive one. The multiplier μ now lies in the open interval $(0,1)$. The linear fractional map τ we employed to understand the parabolic case now conjugates φ to the dilation $\Phi(w) = \mu w$. For each $\lambda \in \mathbb{C}$ the function $F_\lambda(w) = w^\lambda$ solves the functional equation $F \circ \Phi = \mu^\lambda F$. Thus, back in the unit disc, the function

$$f_\lambda(z) := \tau(z)^\lambda = \left(\frac{1+z}{1-z}\right)^\lambda \qquad (z \in \mathbb{U}) \tag{6}$$

is an eigenvector for C_φ, temporarily viewed as a linear transformation on the space of *all* functions holomorphic on $\mathbb{U}$, with corresponding eigenvalue μ^λ (here all complex powers are taken in the principal value sense). Since $f_\lambda \in H^2$ iff $|\operatorname{Re}\lambda| < 1/2$ the set $\{f_\lambda : |\operatorname{Re}\lambda| < 1/2\}$ is a collection of eigenfunctions for C_φ on H^2. Fernández-Valles and Lacruz [10, Lemma 4.4] have shown that this set spans a dense subspace of H^2. Thus, just as in the parabolic case, Proposition 2.1.5 guarantees that $\operatorname{alg}(C_\varphi)$ is strongly compact.

As for the commutant, note that the eigenvalue μ^λ has value 1 precisely when λ is an integer multiple of $2\pi i/\ln\mu$. For definiteness let's take $\lambda = 2\pi i/\ln\mu$. Then $f_\lambda \circ \varphi = f_\lambda$ so, as in the parabolic case, the multiplier M_{f_λ} commutes with C_φ, hence $\operatorname{alg}(M_{f_\lambda}) \subset \operatorname{com}(C_\varphi)$. A conformal mapping exercise shows that f_λ takes $\overline{\mathbb{U}}\backslash\{-1,1\}$ onto the annulus $\{w : 1/R \le |w| \le R\}$, where $R := \exp(-\pi^2/\ln\mu)$ is (since $0 < \mu < 1$) larger than 1 (in fact, f_λ is a covering map taking $\mathbb{U}$ onto the interior of that annulus). Furthermore f_λ takes the (open) upper semicircle of $\partial\mathbb{U}$ onto the outer boundary of the annulus, and the lower semicircle to the inner boundary. Thus $f = f_\lambda/R$ lies in the unit ball of H^∞ and has radial limits of modulus one on an arc of $\partial\mathbb{U}$, so upon turning once more to Theorem 3.1.1 we see that $\operatorname{alg}(M_f) = \operatorname{alg}(M_{f_\lambda})$ is not strongly compact, hence neither is $\operatorname{com}(C_\varphi)$. □

Let's now turn to the case of hyperbolic nonautomorphisms that have a fixed point on $\partial\mathbb{U}$. Whether the other fixed point (which cannot lie on the unit circle) is in $\mathbb{U}$ or $\mathbb{U}_e$ has huge consequences for the question of strong compactness for the commutant of, and the algebra generated by, the associated composition operator.

Theorem 4.1.2. *Suppose $\varphi \in \operatorname{LFT}(\mathbb{U})$ has a fixed point on $\partial\mathbb{U}$ and a second one in $\mathbb{U}$. Then $\operatorname{alg}(C_\varphi)$ is not strongly compact.*

Proof. We may, without loss of generality, assume the fixed points of φ are at 0 and 1, in which case the familiar change of variable $w = \tau(z) = (1+z)/(1-z)$ converts φ into a linear fractional map Φ of the right half-plane that fixes the origin and ∞, and so has the form $\Phi(w) = sw + (1-s)$

for some $0 < s < 1$. Upon pulling back to the unit disc via τ^{-1} we find that

$$\varphi(z) = \frac{sz}{1-(1-s)z}\,.$$

Let $H_0^2 := zH^2$, the space of functions in H^2 that vanish at the origin, and let $\langle 1\rangle$ denote the subspace of constant functions. Then $H^2 = \langle 1\rangle \oplus H_0^2$ and both summands are invariant for C_φ.

Let $\psi(z) = sz + (1-s)$. In [3, Proof of Theorem 2.8, pp. 35–36] Bourdon and I observed that the restriction of C_φ to H_0^2 has adjoint equal to $M_z(sC_\psi)M_{1/z}$ where C_ψ acts on H^2 and M_z is viewed as a unitary operator mapping H^2 onto H_0^2, with $M_{1/z}$ as its inverse (this can also be deduced from Cowen's Adjoint Formula, [5, Theorem 2]). Since the restriction of any composition operator to the subspace of constant functions is the identity map on that subspace, we see that C_φ^* is unitarily equivalent to $I_0 \oplus C_\psi$, where I_0 denotes the identity operator on the subspace $\langle 1\rangle$ of constant functions. Thus C_φ is unitarily equivalent to $I_0 \oplus C_\psi^*$.

We will see in Theorem 5.1.1 below that, because ψ fixes no point of the open unit disc, $\mathrm{alg}(C_\psi^*)$ is not strongly compact on H^2. At this point it is tempting to argue by contradiction: "If $\mathrm{alg}(I_0 \oplus C_\psi^*)$ *were* strongly compact, then the same would be true of $\mathrm{alg}(C_\psi^*)$, which we know is not true." Unfortunately strong compactness need not be inherited by direct summands (see [14, §3, Propositions 5 & 6, page 197]), so more care is required.

For ease of notation let $T := I_0 \oplus C_\psi^*$. To verify that $\mathrm{alg}(T)$ is not strongly compact it's enough to find a sequence of polynomials (p_n) with $\|p_n(T)\|$ bounded in n, yet for which the operator sequence $(p_n(T))$ has no strongly convergent subsequence. Since $\mathrm{alg}(C_\psi^*)$ is not strongly compact we already know there is a sequence (p_n) of polynomials that performs the same function for C_ψ^*. Since strong convergence of operators restricts to invariant subspaces it follows that $p_n(T) = p_n(I_0) \oplus p_n(C_\psi^*)$ does not have a strongly convergent subsequence, so all that remains to verify is that $\sup_n \|p_n(T)\| < \infty$.

For this, note that for any polynomial $p(z) = \sum_k a_k z^k$ we have

$$\|p(T)\| = \max\{\|p(I_0)\|, \|p(C_\psi^*)\|\}$$

where $p(I_0) = (\sum_k a_k)I_0$. Thus to complete the proof we need only show that

$$\left|\sum_k a_k\right| \le \|p(C_\psi^*)\|\,. \tag{7}$$

To this end, let "1" denote the constant function taking the value 1 everywhere on $\mathbb{U}$ and let "$\langle\ ,\ \rangle$" denote the inner product on H^2. Then, upon noting that every composition operator fixes 1, we have

$$\begin{aligned}\|p(C_\psi^*)\| \geq \left|\langle p(C_\psi^*)1,\, 1\rangle\right| &= \left|\sum_k a_k \langle C_\psi^{*k}1,\, 1\rangle\right| \\ &= \left|\sum_k a_k \langle 1,\, C_\psi^k 1\rangle\right| = \left|\sum_k a_k \langle 1,\, 1\rangle\right| \\ &= \left|\sum_k a_k\right|\end{aligned}$$

which completes the proof. □

Note that the last calculation establishes inequality (7) for *any* holomorphic self-map ψ of the unit disc.

Theorem 4.1.3. *Suppose $\varphi \in \mathrm{LFT}(\mathbb{U})$ has one fixed point on $\partial\mathbb{U}$ and the other in $\mathbb{U}_e$. Then $\mathrm{alg}(C_\varphi)$ is strongly compact.*

Proof. (a) Suppose first that φ has the special form

$$\varphi(z) = rz + (1-r) \qquad (z \in \mathbb{U}) \tag{8}$$

for some $0 < r < 1$. For each nonnegative integer n let

$$f_n(z) = (1-z)^n \qquad (z \in \mathbb{U})$$

and note that $f_n \in H^2$ with $C_\varphi f_n = r^n f_n$, i.e., f_n is an eigenvector for C_φ. It's easy to see that the linear span of this collection of eigenvectors contains each of the monomials $1, z, z^2, \ldots$, and so is dense in H^2. Thus, by Corollary 2.1.5, $\mathrm{alg}(C_\varphi)$ is strongly compact.

(b) Suppose now that φ is any linear fractional self-map of $\mathbb{U}$ with a fixed point on $\partial\mathbb{U}$ and another one in $\mathbb{U}_e$. Since strong compactness for the algebra generated by an operator is invariant under similarity we may without loss of generality assume—upon conjugating by an appropriate rotation if necessary—that the boundary fixed point is at 1. An appropriate conformal automorphism of $\mathbb{U}$ that fixes the point 1 will take the other fixed point to ∞. (*Proof:* Replace $\mathbb{U}$ and $\mathbb{U}_e$, respectively, by the right and left half-planes RHP and LHP via the map $\tau(z) = (1+z)/(1-z)$. Then our fixed point at 1 for φ corresponds to a fixed point for the corresponding half-plane map Φ at ∞, with the other fixed point lying somewhere in the LHP. A pure imaginary translation will take that LHP fixed point to the negative real axis, after which a positive dilation moves the new point to -1. The result is a conformal automorphism of RHP that fixes ∞ and takes the fixed point in the LHP to -1. Back in the unit disc this produces the required automorphism that takes the exterior fixed point of φ to ∞ and leaves unchanged the fixed point at 1.)

Claim: The map φ produced by this normalization has the form (8). Indeed, since φ now fixes the points 1 and ∞ it must have the form $\varphi(z) = rz + s$ for some complex numbers r and s. Since $\varphi(1) = 1$ and $\varphi(\partial\mathbb{U}) \subset \overline{\mathbb{U}}$, we have $r + s = 1$, and for every real θ: $|re^{i\theta} + s| \leq 1$. Upon choosing θ so that $re^{i\theta}$ is a positive multiple of s, we see from this last inequality that $|r| + |s| \leq 1$ which, along with the first equation, says that r and s must both be positive. Thus φ has the form (8), as desired, and the proof is complete. □

This last result sets the stage for the most vexing question of all; it appears to be unresolved even for the composition operator induced by the map $\varphi(z) = (1 + z)/2$.

Question 4.1.4. Under the hypotheses of Theorem 4.1.3 (φ hyperbolic with attractive fixed point on $\partial\mathbb{U}$ and repulsive one in $\mathbb{U}_e$), is $\mathrm{com}(C_\varphi)$ strongly compact?

Remark. We saw in Theorem 4.1.1 that for composition operators induced by parabolic and hyperbolic automorphisms the commutants are *not* strongly compact. The argument that established this does not work in the situation of Question 4.1.4. To see what goes wrong, let $\varphi(z) = sz + (1 - s)$ for some $0 < s < 1$ and consider the full collection of eigenvectors

$$g_\lambda(z) := (1 - z)^\lambda \qquad (z \in \mathbb{U})$$

for $C_\varphi : \mathrm{Hol}\,(\mathbb{U}) \to \mathrm{Hol}\,(\mathbb{U})$. These belong to H^2 precisely when $\mathrm{Re}\,\lambda > -1/2$. We have $C_\varphi g_\lambda = s^\lambda g_\lambda$ (which shows that all the points of the punctured disc $\{0 < |z| < 1/\sqrt{s}\}$ belong to the point spectrum of C_φ—in fact, this is the entire point spectrum, and the spectrum of C_φ is the closure of this disc [6, Theorem 3(iv), page 862]).

What's important for us is the fact that $s^\lambda = 1$ iff $\lambda = 2\pi ik/\ln s$ for some integer k, so there is a countable family $F_k = g_{2\pi ik/\ln r}$ of eigenvectors of C_φ for the eigenvalue 1. Unfortunately, Proposition 3.2.2 tells us that each of the algebras $\mathrm{alg}(M_{F_k})$ *is* strongly compact, so these algebras give no information about the possibility of strong compactness for the containing algebra $\mathrm{com}(C_\varphi)$.

Note that the difference between this hyperbolic nonautomorphic case and the hyperbolic automorphic one is that here the eigenfunctions take $\partial\mathbb{U}\backslash\{1\}$ into the image of $\mathbb{U}$, whereas in the automorphic case boundaries are preserved.

4.2. "Non-linear-fractional" composition operators. For the rest of this section the symbol φ will denote an arbitrary holomorphic self-map of $\mathbb{U}$. Here are two results, which, although rather special, lead nevertheless to interesting questions.

To set the stage let R_φ denote the set of points of $\partial\mathbb{U}$ at which φ has a radial limit, and extend φ by radial limits to $\mathbb{U} \cup R_\varphi$. Denote by φ_n the n-th iterate of φ

$$\varphi_n = \varphi \circ \varphi \circ \cdots \circ \varphi \qquad (n \text{ times})$$

and view φ_n to be extended, as above, to the union of $\mathbb{U}$ and a set of full measure on the unit circle.

Theorem 4.2.1 (Necessary condition for strong compactness)**.** *Suppose* φ *is not an automorphism, and has a fixed point* $p \in \mathbb{U}$*. If* $\operatorname{alg}(C_\varphi)$ *is strongly compact then* $\varphi_n(\zeta) \to p$ *for a.e.* $\zeta \in \partial\mathbb{U}$*.*

Proof. As usual we may, without loss of generality, assume that $p = 0$. Thus $C_\varphi^n = C_{\varphi_n}$ has norm 1 for each n. Also $\varphi_n \to 0$ uniformly on compact subsets of $\mathbb{U}$ and so (because $\|\varphi_n\| \le 1$ for each n) weakly in H^2. Let $p_n(z) \equiv z^n$ and $f(z) \equiv z$. Then $\|p_n(C_\varphi)\| = 1$ for each n, so—because we are assuming that $\operatorname{alg}(C_\varphi)$ is strongly compact—the set $\{p_n(C_\varphi)f = \varphi_n : n \ge 0\}$ is relatively compact in H^2. This, along with the above-mentioned weak convergence, implies that $\|\varphi_n\| \to 0$ as $n \to \infty$. In particular, there is a subsequence of (φ_n) that converges a.e. on $\partial\mathbb{U}$ to 0, so for each $\zeta \in \partial\mathbb{U}$ that belongs to this a.e. convergence set we have $\varphi_m(\zeta) \in \mathbb{U}$ for some m, and so $\varphi_n(\zeta) \to 0$ as $n \to \infty$. □

Corollary 4.2.2. *Suppose* φ *is a nonautomorphic inner function that fixes a point of* $\mathbb{U}$*. Then* $\operatorname{alg}(C_\varphi)$ *is not strongly compact.*

Proof. A theorem of Lindelöf (see [23, page 163] for example) shows that $|\varphi_n| = 1$ a.e. on $\partial\mathbb{U}$, hence the result follows from Theorem 4.2.1. □

Question. Does the conclusion of the last corollary (or, for that matter, the last theorem) continue to hold if φ is no longer assumed to have an interior fixed point?

In case φ has no interior fixed point the Denjoy–Wolff theorem asserts that there is a unique boundary point ω that plays the role of a fixed point in the sense that $\varphi_n \to \omega$ uniformly on compact subsets of $\mathbb{U}$, and also in the sense of radial limits: $\varphi(\omega) = \omega$. The problem in generalizing the proof of Theorem 4.2.1 to this situation is that now $\|C_\varphi^n\| \to \infty$ (see [23, page 163] for example).

For commutants of composition operators with general symbols the result below shows, for example, that the map

$$\varphi(z) = \frac{1 - \sqrt{1 - z^2}}{z} \tag{9}$$

which maps the unit disc univalently onto a "lens-shaped" subdomain with vertices at ± 1, induces a composition operator on H^2 with strongly compact commutant. The point here is that in this case C_φ is compact (see, for example, [23, Chapter 2]), and has dense range (see below).

Theorem 4.2.3. *If* C_φ *is compact and* φ *maps* $\mathbb{U}$ *univalently onto a Jordan domain, then* $\operatorname{com}(C_\varphi)$ *is strongly compact.*

For the proof—whose details I omit—one observes that, thanks to the theorems of Mergelyan and Carathéodory (see proof of Theorem 3.1.4), the polynomials in φ can be shown to be dense in H^2, i.e., C_φ has dense range.

The result then follows from Proposition 2.1.6. The application to the composition operator induced by the mapping (9) then follows from the fact that this operator is compact (see, for example, [24, Chapter 2]).

The "Jordan-domain condition" can be weakened: It's enough to demand that the polynomials should be dense in $A(K)$, the algebra of functions continuous on K and analytic on its interior $\varphi(\mathbb{U})$, or even just in the Hardy space $H^2(\varphi(\mathbb{U}))$ (see [9, Chapter 10, page 168] for the precise definition of this space). However I do not know if these density requirements can be removed completely. More precisely:

Question. Suppose C_φ is a compact composition operator. Is $\mathrm{com}(C_\varphi)$, or even just $\mathrm{alg}(C_\varphi)$, strongly compact?

To put this question in context, recall (paragraph preceding Proposition 2.1.6) that there are weighted shifts on ℓ^2 that are compact, yet generate algebras—and hence commutants—that are not strongly compact.

5. Adjoints

The previous sections show that for multipliers and composition operators the study of strong compactness of both algebras and commutants leads to interesting results and intriguing questions. The same is true for adjoints.

5.1. Composition operator adjoints. Here (finally) is a definitive result; when examined in the light of Table 1 it shows that, within the class of composition operators, neither the generated algebra nor the commutant need pass on the property of strong compactness (or lack thereof) to the corresponding algebras generated by the adjoint.

Theorem 5.1.1. *For a holomorphic self-map φ of $\mathbb{U}$: $\mathrm{alg}(C_\varphi^*)$ is strongly compact iff φ fixes a point of $\mathbb{U}$.*

Proof. Suppose φ fixes a point of $\mathbb{U}$. By the usual similarity argument we may assume this fixed point is the origin. Then, with respect to the orthonormal basis $\{z^n\}_0^\infty$ for H^2, the matrix of C_φ is lower triangular, hence the matrix of C_φ^* is upper triangular. Thus the subspace of polynomials of degree $\leq n$ is, for $n = 0, 1, 2, \ldots$, invariant for C_φ^*, so by Proposition 2.1.3, $\mathrm{alg}(C_\varphi^*)$ is strongly compact.

For the converse, suppose φ fixes *no* point of $\mathbb{U}$. We wish to show that $\mathrm{alg}(C_\varphi^*)$ is not strongly compact. Let $z_n = \varphi_n(0)$, where φ_n denotes the n-th iterate of φ. Then (see [23, pp. 16–17], for example)

$$\|C_{\varphi_n}\| \leq \sqrt{\frac{1+|z_n|}{1-|z_n|}} < \frac{2}{\sqrt{1-|z_n|^2}} = 2\|K_{z_n}\| \tag{10}$$

where K_{z_n} is the reproducing kernel for the point z_n, as defined by Equation (1) in the proof of Proposition 2.1.4. For $n = 0, 1, 2, \ldots$ let $p_n(z) = \frac{z^n}{2\|K_{z_n}\|}$, so by (10):

$$\|p_n(C_\varphi^*)\| = \|p_n(C_\varphi)^*\| = \|p_n(C_\varphi)\| \le 1 \tag{11}$$

i.e., the set $\{p_n(C_\varphi^*)\}_0^\infty$ lies in the unit ball of operators on H^2.

Consider now the sequence $(p_n(C_\varphi^{*n})1)_0^\infty$ of vectors in H^2. Upon noting that $K_0 \equiv 1$, we see that

$$p_n(C_\varphi^*)1 = \frac{C_{\varphi_n}^* K_0}{2\|K_{z_n}\|} = \frac{K_{z_n}}{2\|K_{z_n}\|} \tag{12}$$

where the last equality uses the fact that $C_\varphi^* K_a = K_{\varphi(a)}$ for each $a \in \mathbb{U}$ (see, e.g., [23, §3.4, page 43]). Thus

$$\|p_n(C_\varphi^*)1\| = \frac{1}{2} \qquad (n = 0, 1, 2, \ldots). \tag{13}$$

We are assuming that φ has no fixed point in $\mathbb{U}$, so the Denjoy-Wolff Theorem (see [23, Chapter 4] for example) insures that $|z_n| \to 1-$ as $n \to \infty$. Thus by (12), as $n \to \infty$,

$$(p_n(C_\varphi^*)1)(z) = \frac{1}{2}\frac{\sqrt{1-|z_n|^2}}{1-\overline{z_n}z} \to 0$$

where the limit is uniform on compact subsets of $\mathbb{U}$. It follows from this and (11) that $p_n(C_\varphi^*)1 \to 0$ weakly in H^2.

Now if $\mathrm{alg}(C_\varphi^*)$ were strongly compact then, in view of (11) the set $\{p_n(C_\varphi^*)1\}$ would be relatively compact in H^2, hence the corresponding sequence, being weakly convergent to zero, would have to be norm convergent to zero. But this contradicts (13) above; thus $\mathrm{alg}(C_\varphi^*)$ is not strongly compact. □

What about $\mathrm{com}(C_\varphi^*)$? If it is strongly compact, then so is $\mathrm{alg}(C_\varphi^*)$, so by the result above φ must have a fixed point in $\mathbb{U}$. However *the converse is false:* the map $\varphi(z) \equiv -z$, which fixes the origin, induces a self-adjoint composition operator on H^2 whose commutant is, by Proposition 2.3.1, *not* strongly compact.

In case C_φ (equivalently C_φ^*) is compact then all subtlety concerning the strong compactness of $\mathrm{com}(C_\varphi^*)$ vanishes, as shown by the following result, which serves as a sort of companion to Theorem 4.2.3.

Corollary 5.1.2. *Suppose φ is a holomorphic self-map of $\mathbb{U}$ for which C_φ is compact. Then $\mathrm{com}(C_\varphi^*)$ is strongly compact iff φ is not constant.*

Proof. Suppose φ is not constant. Then, $\varphi(\mathbb{U})$ is a nonvoid open set, so C_φ is one-to-one, hence its adjoint has dense range. This, along with the compactness of C_φ^*, yields—thanks to Proposition 2.1.6—the strong compactness of $\mathrm{com}(C_\varphi^*)$.

If, conversely, φ *is* identically constant, then by our standard similarity argument we may assume this constant is zero. Now suppose ψ is any holomorphic self-map of $\mathbb{U}$ for which $\psi(0) = 0$. Then for each $z \in \mathbb{U}$ we have $\varphi(\psi(z)) = 0 = \psi(0) = \psi(\varphi(z))$, i.e., $\varphi \circ \psi = \psi \circ \varphi$. Thus:

If $\varphi \equiv 0$ then $\mathrm{com}(C_\varphi)$ contains C_ψ for any holomorphic self-map ψ with $\psi(0) = 0$.

In particular, $\mathrm{com}(C_\varphi) \supset \mathrm{alg}(C_\psi)$ for $\psi(z) = z/(2 - z)$. By Theorem 4.1.2, $\mathrm{alg}(C_\psi)$ is not strongly compact, hence neither is $\mathrm{com}(C_\varphi) = \mathrm{com}(C_\varphi^*)$. □

5.2. Multiplication operator adjoints. For adjoints of multiplication operators on H^2 the situation is similar to that for composition operators: definitive results for the generated algebra, and open questions for the commutant.

Proposition 5.2.1. *$\mathrm{alg}(M_f^*)$ is strongly compact for each $f \in H^\infty$.*

Proof. We saw in the course of proving Proposition 2.1.4 that the H^2 reproducing kernels form a set of eigenvectors for M_f^* with dense linear span. Thus $\mathrm{alg}(M_f^*)$ is strongly compact by Corollary 2.1.5. □

The situation for *commutants* of adjoints of multiplication operators is more interesting. First of all, they're not all strongly compact. For an interesting class of examples with this property, recall that the Koebe Uniformization Theorem (see, e.g., [1, Ch. 10], especially Theorem 10-4, pp. 150–151) asserts that every plane domain Ω that omits two or more points is the image of the unit disc by a *covering map,* i.e., a map $f : \mathbb{U} \to \Omega$ with the property that each point of Ω has an open neighborhood V that is *evenly covered* in the sense that each component of $f^{-1}(V)$ is homeomorphic, via f, to V (see, for example, [1, §9.2]). We have already encountered examples of covering maps: The eigenfunctions (4) for composition operators induced by parabolic automorphisms (fixing the point 1) are covering maps of $\mathbb{U}\backslash\{0\}$, while for operators induced by hyperbolic automorphisms (fixing the points ±1) the eigenfunctions f_λ given by (6), with λ imaginary, are covering maps of origin-centered annuli.

Theorem 5.2.2. *Suppose Ω is a bounded plane domain that is not simply connected, and let $f : \mathbb{U} \to \Omega$ be a covering map. Then $\mathrm{com}(M_f^*)$ is not strongly compact.*

Proof. The covering map f has a nontrivial subgroup Γ of $\mathrm{Aut}(\mathbb{U})$ such that

$$f \circ \gamma = f \qquad (\gamma \in \Gamma).$$

(These are called the "covering transformations" or "deck transformations" associated with f). Furthermore, except for the identity, no such transformation has a fixed point in $\mathbb{U}$ (see [1, §9.5], for example). Our hypothesis that Ω is not simply connected guarantees (indeed, is equivalent to) the fact that Γ consist of more than the identity map.

In summary: M_f commutes with some composition operator C_γ, where $\gamma \in \mathrm{LFT}(\mathbb{U})$ has no fixed point in $\mathbb{U}$. Thus C_γ^*, which by Theorem 5.1.1

generates a non-strongly-compact algebra, commutes with M_f^*. It follows that $\mathrm{com}(M_f^*)$ contains the non-strongly-compact algebra $\mathrm{alg}(C_\gamma^*)$, so is itself is not strongly compact. □

Note that for a covering map f taking the unit disc onto a non-simply-connected domain, each point of $f(\mathbb{U})$ is the image of infinitely many points of $\mathbb{U}$. This suggests that we explore the effect of imposing some kind of restriction on the degree of covering. To this end let's say that a function f holomorphic on $\mathbb{U}$ *finitely covers* a point $w \in f(\mathbb{U})$ if the fiber $f^{-1}(\{w\})$ over the image point w has just finitely many points. Let $n(w)$ denote the number of points in this fiber, counting multiplicities. Say f *finitely covers* a subset E of $f(\mathbb{U})$ if it finitely covers each point of E.

Deddens and Wong [7, Corollary 4] (see also [4, Corollary 4.6]) have shown that if $f \in H^\infty$ *singly* covers an open subset of $f(\mathbb{U})$ (i.e., if $n(w) = 1$ for each w in that open subset) then $\mathrm{com}(M_f) = \mathrm{com}(M_z)$, hence in this special case $\mathrm{com}(M_f^*) = \mathrm{com}(M_z^*)$ and so, by Corollary 2.1.4, $\mathrm{com}(M_f^*)$ is strongly compact. Think of this as a prototype for the following result:

Theorem 5.2.3. *If $f \in H^\infty$ finitely covers a subset of $f(\mathbb{U})$ having positive logarithmic capacity, then $\mathrm{com}(M_f^*)$ is strongly compact.*

Proof. The "Rudin–Frostman Theorem" [20] guarantees that for all $\alpha \in \mathbb{U}$ outside a set of capacity zero, the function $f - \alpha$ has no "singular factor," i.e.,

$$f - \alpha = B_\alpha F_\alpha \tag{14}$$

where B_α is a Blaschke product and F_α an outer function.

We are assuming that f finitely covers a subset E of $f(\mathbb{U})$ that has positive logarithmic capacity, so we may assume that (14) holds for every $\alpha \in E$, in which case (since the outer factor F_α never takes the value zero) B_α is a finite Blaschke product:

$$B_\alpha(z) = \omega \prod_{\beta \in f^{-1}(\{\alpha\})} \frac{z - \beta}{1 - \overline{\beta} z} \tag{15}$$

where ω is a constant of modulus one.

Employing the notation $\overline{S}$ for the norm-closure of a subset S of H^2, we have for each $\alpha \in E$:

$$\overline{\mathrm{ran}\,(M_f - \alpha I)} = \overline{M_{f-\alpha}(H^2)} = \overline{M_{B_\alpha} M_{F_\alpha}(H^2)} = M_{B_\alpha} \overline{M_{F_\alpha}(H^2)} = B_\alpha H^2$$

where the second equality comes from the fact that M_{B_α} is bounded below on H^2 (in fact it is an isometry), and the third from the outer-ness of F_α. Thus the closure of the range of $M_f - \alpha I$ has codimension $n(\alpha) < \infty$, so its orthogonal complement, $V_\alpha := \ker(M_f^* - \overline{\alpha} I)$ has dimension $n(\alpha)$. (The argument of this paragraph is a standard one.)

I claim that the union of the M_f^*-eigenspaces V_α, as α ranges over E, spans a dense subset of H^2. Indeed, each V_α contains the set of reproducing kernels

$\{K_a : a \in f^{-1}(\{\alpha\})\}$. Since E, being of positive capacity, is uncountable, so is $f^{-1}(E)$, hence $\cup_{\alpha\in E} V_\alpha$ contains reproducing kernels for an uncountable set of points of $\mathbb{U}$. These points have a limit point in $\mathbb{U}$, so the corresponding reproducing kernels, and therefore the union of the subspaces V_α, spans a dense subspace of H^2.

In summary: M_f^* has a densely spanning collection of finite dimensional eigenspaces, so by Corollary 2.1.5, $\mathrm{com}(M_f^*)$ is strongly compact. $\square$

Question. For which $f \in H^\infty$ is $\mathrm{com}(M_f^*)$ strongly compact?

6. Connections with the Cesàro operator

The (discrete) Cesàro operator C_0 is the operator on ℓ^2 that sends a sequence to its sequence of averages. More specifically, if $x = (\zeta_n)_0^\infty \in \ell^2$, then

$$C_0\, x = \left(\zeta_0,\ \frac{\zeta_0+\zeta_1}{2},\ \frac{\zeta_0+\zeta_1+\zeta_2}{3},\ \cdots\right).$$

Hardy, Littlewood, and Polya [12, Chapter IX] provided a proof that C_0 is a bounded operator on ℓ^2, while much later Brown, Halmos, and Shields [2] gave a simpler proof, and initiated the operator-theoretic study of C_0, determining its norm and spectrum.

6.1. The commutant of the Cesàro operator. Deddens observed [6] that for each $0 < s < 1$ the adjoint of C_0 commutes with the composition operator $C_{sz+(1-s)}$. Thus C_0 itself commutes with $C^*_{sz+(1-s)}$, and so

$$\mathrm{alg}(C^*_{sz+(1-s)}) \subset \mathrm{com}(C_0).$$

We know from Theorem 5.1.1 that $\mathrm{alg}(C^*_{sz+(1-s)})$ is not strongly compact, hence neither is $\mathrm{com}(C_0)$. In [10, Problem 2.3] Fernández-Valles and Lacruz ask if, nevertheless, $\mathrm{alg}(C_0)$ might be strongly compact.

Theorem. $\mathrm{alg}(C_0)$ *is not strongly compact.*

Proof. Extending work of [2], Shields and Wallen showed in [25, §2] that $I - C_0$ is unitarily equivalent to M_z acting, not on H^2, but on a different Hilbert space $\mathcal{H}$ of functions analytic on $\mathbb{U}$ that contains the polynomials as a dense subspace, and on which point evaluations are continuous. They showed, moreover, that for every $f \in H^\infty$ the multiplication operator M_f acts on $\mathcal{H}$, with $\|M_f\| = \|f\|_\infty$. Thus if $\mathrm{alg}(C_0) = \mathrm{alg}(I - C_0)$ were strongly compact on ℓ^2 then the same would be true of $\mathrm{alg}(M_z)$ on the Shields–Wallen space $\mathcal{H}$, hence the set of vectors $\{M_z^n 1\}_0^\infty = \{z^n\}_0^\infty$ would be relatively compact in $\mathcal{H}$. If this were true then we would know from the continuity of point evaluations, and the fact that $z^n \to 0$ pointwise on $\mathbb{U}$, that $\|z^n\|_{\mathcal{H}} \to 0$.

Now in [13] Kriete and Trutt, answering a question posed in [25], showed that C_0 is subnormal. To do this they exhibited a Borel probability measure

μ on $\mathbb{U}$ for which the Hilbert space $\mathcal{H}$ is the closure in $L^2(\mu)$ of the polynomials. The Kriete-Trutt measure μ, while a bit complicated to describe explicitly, is supported on the union of the circles

$$\Gamma_k := \left\{ z : \left| z - \frac{k}{k+1} \right| = \frac{1}{k+1} \right\} \qquad (k = 0, 1, 2, \ldots).$$

Furthermore on Γ_k this measure has total variation $2^{-(k+1)}$ and is mutually absolutely continuous with respect to arclength measure. Note in particular that $\Gamma_0 = \partial\mathbb{U}$, and $\mu(\partial\mathbb{U}) = 1/2$. Thus for each n:

$$\|z^n\|_{\mathcal{H}}^2 = \int_{\mathbb{U}} |z^n|^2\, d\mu \geq \int_{\partial\mathbb{U}} |z^n|^2\, d\mu = \mu(\partial\mathbb{U}) > 0$$

so the sequence $(z^n) = (M_z^n 1)$ does not converge to zero in the norm of $\mathcal{H}$. This shows that $\operatorname{alg}(M_z)$, acting on $\mathcal{H}$, is not strongly compact, and therefore neither is $\operatorname{alg}(C_0)$, acting on ℓ^2. □

6.2. A possible approach to Question 4.1.4. It may be possible to exploit the connection between composition operators and C_0 to settle Question 4.1.4 about the possible strong compactness of $\operatorname{com}(C_\varphi)$ where $\varphi \in \operatorname{LFT}(\mathbb{U})$ is hyperbolic with its attractive fixed point on $\partial\mathbb{U}$ and its repulsive one in $\mathbb{U}_e$. The idea would be to use Deddens' observation that for $\varphi(z) = sz + (1-s)$, $0 < s < 1$ (the "model" for the class of maps we're talking about here), we have $C_\varphi = G(I - C_0^*)$, where

$$G(z) := s^{\frac{z}{1-z}} = e^{\gamma} \exp\left(\gamma \frac{z+1}{z-1}\right),$$

with $\gamma = \frac{1}{2}\log\frac{1}{s}$. Thus G is a scalar multiple of the positive power γ of the unit singular function [6, page 863, "Proof of 3(iv)"], and C_φ is unitarily equivalent to M_G^* acting on $\mathcal{H}$.

Now G is a covering map for a punctured disc so, by Theorem 5.2.2, $\operatorname{com}(M_G^*)$ is not strongly compact for H^2. This gives some hope that the same might be true for $\mathcal{H}$ as well, but unfortunately the proof for H^2 does not work in this more complicated setting. To see what goes wrong, recall that the proof for H^2 depended upon showing that if $\psi \in \operatorname{Aut}(\mathbb{U})$ is parabolic, then $\operatorname{alg}(C_\psi^*)$ is *not* strongly compact on H^2. The problem here is that C_ψ^*, while still a bounded operator on $\mathscr{H}$, *now generates an algebra that* is *strongly compact on that space.*

Indeed, the parabolic automorphism ψ on $\mathbb{U}$ is conformally equivalent, for some $\lambda \in \mathbb{R}$, to a translation map $T(w) = w - i\lambda$ acting on the half-plane $\mathbb{P} := \{w : \operatorname{Re} w > -1/2\}$. Now C_ψ, acting on $\mathscr{H}$, is unitarily equivalent to the composition operator C_T acting on the Newton space $N^2(\mathbb{P})$ (see [13, Remark 2, page 224]), and MacDonald and Rosenthal observe that this operator is unitarily equivalent to M_f^* acting on H^2, where $f(z) := (1-z)^{i\lambda}$ (see [17], Theorem 3.4 ff., bottom of page 2526). Thus C_ψ^*, acting on $\mathscr{H}$ is unitarily equivalent to M_f acting on H^2. We know (Proposition 3.2.2) that $\mathrm{alg}(M_f)$ *is* strongly compact, hence the same is true of $\mathrm{alg}(C_\psi^*)$.

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