WHAT YOUR CALCULUS BOOK DOESN'T TELL YOU ABOUT TAYLOR SERIES

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Abstract. These notes survey some surprising aspects of Taylor series that aren't usually included in calculus (or even advanced calculus) classes. Some of these, especially the existence of "smooth transition functions" are often assumed to be known once you start taking graduate courses.

1. What your calculus book DOES tell you

Our setting is the class $C^\infty(\mathbb{R})$ of infinitely differentiable functions on the real line $\mathbb{R}$, often referred to as the class of *smooth* functions on $\mathbb{R}$. For $x_0 \in \mathbb{R}$ fixed, each $f \in C^\infty(\mathbb{R})$ has a *Taylor series* "centered at x_0." This is the "formal" (meaning: "it may or may not converge") series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x-x_0)^3 + \cdots . \tag{1}$$

The N-th partial sum of the above Taylor series has, at the point x_0, the same value as f, and also the same derivatives up through order N. Thus it seems that this sequence of partial sums should do a good job, at least for values of x near x_0, of approximating $f(x)$. The error committed in approximating f by such a partial sum is given by the following beautiful result, often called "Taylor's theorem with Lagrange's form of the remainder," or more descriptively, the "Generalized Mean-Value Theorem" (cf. [1, pp. 542–545], for example).

Theorem 1.1. *Suppose $f \in C^\infty(\mathbb{R})$, $x_0 \in \mathbb{R}$, and N is any non-negative integer. Then for each $x \in \mathbb{R}$ there exists a point $\overline{x}$ between x_0 and x such that*

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \frac{f^{(N+1)}(\overline{x})}{(N+1)!}(x-x_0)^{N+1} .$$

This is the theorem we most often use in Calculus to show that the Taylor series of many commonly occurring functions converge rapidly to those functions. For example it easily

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provides the series representations below for the sine and exponential functions, valid for all $x \in \mathbb{R}$:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad \text{and} \quad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \ .$$

2. Something your calculus book doesn't tell you

The success our calculus books have in representing commonly occurring functions by convergent Taylor series might lead one to assume that *any* smooth function is so represented. However this is simply not true, as shown spectacularly by the following Fundamental Example:

$$f_0(x) := \begin{cases} e^{-1/x^2} & \text{if } x > 0, \\ 0 & \text{if } x \le 0. \end{cases} \tag{2}$$

Clearly f_0 is infinitely differentiable at every point of $\mathbb{R}\backslash\{0\}$.

Exercise 2.1.[1] Show that, in fact, $f_0 \in C^\infty(\mathbb{R})$ *with* $f_0^{(n)}(0) = 0$ *for every non-negative integer* n.

Thus the Taylor series of f_0 with center $x_0 = 0$ has all coefficients equal to zero, so it converges pointwise on the real line, not to the function f_0, but to the function *identically equal to* 0. So we've learned the following sobering lesson:

> *Even if the Taylor series of a smooth function f converges at every point of the real line, it need not converge anywhere (except at its center) to f.*

Henceforth I'll use the term "Taylor series" to denote "Taylor series with center at $x_0 = 0$," (commonly known as "Maclaurin series").

Here's something else that our Fundamental Example shows:

> *Different functions in $C^\infty(\mathbb{R})$ can have the same Taylor series!*

For example, f_0 and the function identically zero have the same Taylor series.

Exercise 2.2. Give some more examples of different functions having the same Taylor series. For example, find a function different from $\sin x$ that has the same Taylor series as $\sin x$.

Exercise 2.3. Show that the set of functions in $C^\infty(\mathbb{R})$ that have Taylor series $\equiv 0$ forms an infinite dimensional subspace of $C^\infty(\mathbb{R})$.

[1]... which every student of mathematics should do at least once!

Hint: For $\lambda > 0$ let $g_\lambda(x) = f(\lambda x)$. Show that the dilates $\{g_\lambda : \lambda > 0\}$ form a linearly independent subset of the subspace in question.

3. Smooth transition functions

Counterexamples frequently have no other function than to be counterexamples. The Fundamental Example of the last section is different: we'll see in this section that it's the basis for constructing vitally important smooth "transition functions," i.e., smooth approximations to characteristic functions of intervals. These will be crucial to the work of the next section.

Continuing with the notation f_0 for the Fundamental Example of §2, note that the function f_1 defined by

$$f_1(x) = \frac{f_0(x)}{f_0(x) + f_0(1-x)}$$

belongs to $C^\infty(\mathbb{R})$, with:

$$f_1(x) \begin{cases} = 0 & \text{if } x \le 0, \\ \in (0,1) & \text{if } 0 < x < 1, \\ = 1 & \text{if } x \ge 1. \end{cases}$$

Thus f_1 makes a smooth transition, on the interval $[0,1]$ from the value 0 at all points to the left of that interval to the value 1 at all points to the right. Next, convince yourself that the function $x \to f_1(1-x)$ makes a similar smooth transition from the value 1 at all points the left of $[0,1]$ to the value 0 to all points at the right.

Thus the function b (for "bump") defined on $\mathbb{R}$ by

$$b(x) = f_1(x+1)f_1(1-x) \qquad (x \in \mathbb{R}).$$

is smooth, with

$$b(x) \begin{cases} = 0 & \text{if } |x| \ge 2, \\ \in (0,1) & \text{if } 1 < |x| < 2, \\ = 1 & \text{if } |x| \le 1. \end{cases}$$

The functions we've constructed have the graphs shown in Figure 1 below:

Exercise 3.1. Modify the above construction of the smooth bump function b to produce, for each $a > 1$, a similar bump function b_a that makes its transitions on the intervals $(-a-1, -a)$ and $(1, a)$.

Exercise 3.2. Use dilations of the functions b_a of the previous exercise to produce "smooth approximations to the characteristic function" of an arbitrary finite interval $[\alpha, \beta]$. More precisely, given $(\gamma, \delta) \supset [\alpha, \beta]$ show that there exists a smooth function b that's identically zero off $[\gamma, \delta]$, identically one on $[\alpha, \beta]$, and elsewhere has values strictly between 0 and 1.

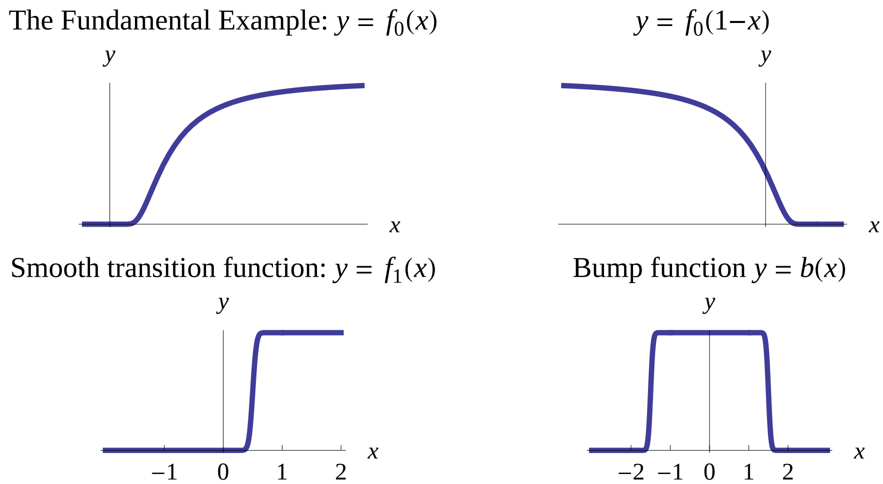

FIGURE 1. *The Fundamental Example and its offspring*

4. WHICH SERIES ARE TAYLOR SERIES?

We saw in §2 that a Taylor series may converge, but not to its "parent function" (Exercise 2.1), and Exercise 2.3 implies that every Taylor series has an "infinite dimensional hyperplane" of parent functions. What other kinds of tricks might smooth functions and their Taylor series play on us? Can the Taylor series of a smooth function diverge "everywhere" (here "everywhere" means "everywhere except at its center")? For example, is the "everywhere" divergent series $\sum_{n=0}^{\infty} n!\, x^n$ the Taylor series of some smooth function? The answer to all such questions is "yes"; as shown dramatically by the following theorem.

Theorem 4.1 (Borel's Theorem)**.** *Given any sequence $(a_n)_0^\infty$ of real numbers, the power series $\sum_{n=0}^{\infty} a_n\, x^n$ is the Taylor series of some function in $C^\infty(\mathbb{R})$.*

Given a real sequence $(b_n)_0^\infty$, if we apply Borel's theorem with $a_n = n!b_n$ then we obtain a function $f \in C^\infty(\mathbb{R})$ with $f^{(n)}(0) = b_n$. Thus, any real sequence can be the "derivative sequence" (at a fixed point) for some smooth function.

Proof of Borel's theorem

(a) *Prologue.* Should it happen that the series $\sum_{n=0}^{\infty} a_n\, x^n$ converges in some interval about the origin, then we're done. The function f to which the series converges is infinitely differentiable in that interval, and has the series as its Taylor series (something your calculus book *does* tell you). To get from this a function in $C^\infty(\mathbb{R})$, suppose the interval of absolute convergence of the series is $(-a, a)$, and define the function f_1 on the real line by

$$f_1(x) = \begin{cases} f(x)\, b(\frac{2x}{a}) & \text{if } |x| \le a/2, \\ 0 & \text{if } |x| \ge a, \end{cases}$$

where b is the bump function of the previous section. Recall that $b \in C^\infty(\mathbb{R})$, it's $\equiv 1$ on the interval $[-1,1]$, it's $\equiv 0$ outside of $(-2,2)$, and it takes values strictly between 0 and 1 otherwise. Thus $b(\frac{2x}{a})$ is smooth on $\mathbb{R}$, $\equiv 1$ on $[-a/2, a/2]$, and $\equiv 0$ outside of $(-a,a)$. Thus f_1 coincides with f on the interval $(-a/2, a/2))$ and one checks easily that it is smooth on all of $\mathbb{R}$.

But in general, our series will not converge anywhere except at the origin: think of $\sum_{n=0}^\infty n!\, x^n$ as a typical example. What to do then?

(b) *Bumping along.* This time the bump function b will play a more essential role. We begin by fixing a strictly increasing sequence $(\lambda_n)_0^\infty$ of positive numbers—about which more will be said later—and just writing down the formal series

$$f(x) \;:=\; \sum_{n=0}^{\infty} a_n\, x^n\, b(\lambda_n x) \tag{3}$$

To see that the series on the right actually converges for every real number x, thus legitimizing the definition of the function f, note that for a given non-zero x, the terms of the series for which $\lambda_n|x| \geq 2$ are (thanks to b) all zero. Since $\lambda_n \nearrow \infty$, this happens for all but a finite number of terms of the series. Thus, for each $x \neq 0$ the series on the right-hand side of (3) just has finitely many non-zero terms, and therefore converges trivially. If $x = 0$, then all terms on the right-hand side of (3) are zero, except possibly for the leading one, $f(0) = a_0$. Thus the series in (3) converges for each real x, and defines a function f on the whole real line.

(c) *Differentiating without fear.* We want to show that the function f defined above is smooth and has our series $\sum_{n=0}^\infty a_n x^n$ as its Taylor series. For this we need to impose further restrictions on the sequence (λ_n). To see what these need to be, let's just proceed "formally" for a while, assuming that f is infinitely differentiable, and that we can pass differentiation through the summation sign.[2]

To ease the task of differentiating the individual terms of the series in (3), let's use the notation f_n for the n-th one of these, and rewrite it as follows:

$$f_n(x) \;:=\; a_n\, x^n\, b(\lambda_n x) \;=\; \frac{a_n}{\lambda_n^n}\, \psi_n(\lambda_n x) \qquad (x \in \mathbb{R}), \tag{4}$$

where

$$\psi_n(x) \;:=\; x^n b(x) \;=\; \begin{cases} x^n & \text{if } |x| \leq 1, \\ 0 & \text{if } |x| \geq 2. \end{cases} \tag{5}$$

[2] *Exercise:* we *can* do this for differentiation evaluated at any point $\neq 0$. Why? Why does the case of differentiation at the origin require more work?

Since $\psi_n(x) \equiv x^n$ for $|x| \leq 1$ we see that for $|x| < 1$:

$$\psi_n^{(k)}(x) \;=\; \frac{d^k}{dx^k}x^n \;=\; \begin{cases} k!\,x^{n-k} & \text{if } k \leq n, \\ 0 & \text{if } k > n. \end{cases} \tag{6}$$

In particular

$$\psi_n^{(k)}(0) \;=\; \begin{cases} 0 & \text{if } k \neq n, \\ n! & \text{if } k = n, \end{cases} \tag{7}$$

and therefore, upon applying the Chain Rule to the last expression in (4) for f_n:

$$f_n^{(k)}(0) \;=\; \frac{a_n}{\lambda_n^{n-k}}\,\psi_n^{(k)}(0) \;=\; \begin{cases} 0 & \text{if } n \neq k, \\ a_k\,k! & \text{if } n = k \end{cases} \tag{8}$$

Thus if we are allowed to interchange differentiation-at-the-origin with possibly infinite summation, we'll have, for each non-negative integer k

$$f^{(k)}(0) \;=\; (\sum_{n=0}^{\infty} f_n)^{(k)}(0) \;=\; \sum_{n=0}^{\infty} f_n^{(k)}(0) = k!\,a_k\,,$$

i.e., $a_k = \frac{f^{(k)}(0)}{k!}$, so our original series $\sum_{n=0}^{\infty} a_k\,x^k$ is indeed the Taylor series of f.

(d) *Legitimizing our differentiational escapade.* The following useful theorem of advanced calculus tells us when we can differentiate through an infinite summation (see [3, Th. 7.17, pp. 152–153], and the remark at the bottom of page 153, for example):

Theorem 4.2. *Suppose $[a, b]$ is a real interval on which, for each positive integer n, is defined a differentiable function f_n. Suppose further that the series of derivatives $\sum_{n=0}^{\infty} f'_n$ converges uniformly on $[a, b]$, and that the numerical series $\sum_{n=0}^{\infty} f_n(a)$ (no derivative here) also converges. Then the series $\sum_{n=0}^{\infty} f_n$ converges uniformly on $[a, b]$ to a differentiable function f, and*

$$f'(x) \;=\; \sum_{n=0}^{\infty} f'_n(x) \qquad (x \in [a, b]).$$

This theorem can be used repeatedly to show that if, on $[a, b]$ each f_n is infinitely differentiable, with $\sum_{n=0}^{\infty} f^{(k)}$ converging uniformly for each non-negative integer k, then $\sum_{n=0}^{\infty} f_n$ is infinitely differentiable on $[a, b]$, with k-th derivative equal to $\sum_{n=0}^{\infty} f_n^{(k)}$. This is exactly what we need to see how to choose the sequence (λ_n) in (3) to legitimize our previous arguments.

Recall from (3), (4), and (5) that we have defined our function f like this:

$$f(x) = \sum_{n=0}^{\infty} f_n(x) \quad \text{where} \quad f_n(x) \;=\; a_n\,x^n\,b_n(\lambda_n x) \;=\; \frac{a_n}{\lambda_n^n}\,\psi_n(\lambda_n x)$$

where $\psi_n(x) := x^n b(x)$. Thus for all non-negative integers k and n,

$$f_n^{(k)}(x) = \frac{a_n}{\lambda_n^{n-k}} \psi_n^{(k)}(\lambda_n x) \qquad (x \in \mathbb{R}). \tag{9}$$

For a real-valued function g on $\mathbb{R}$, let's use the notation $\|g\| := \sup_{x\in\mathbb{R}} |g(x)|$. Thus by (9):

$$\|f_n^{(k)}\| \;=\; \frac{|a_n|}{\lambda_n^{n-k}} \|\psi_n^{(k)}\| \qquad (n, k = 0, 1, 2, \ldots). \tag{10}$$

Note that for each n and k the function $\psi_n^{(k)}$ is continuous on the real line and non-zero only on the interval $[-2, 2]$. Since continuous functions are bounded on compact sets, $\|\psi_n^{(k)}\| < \infty$ for each n and k, hence the same is true for $\|f_n^{(k)}\|$. This tells us how to choose the sequence (λ_n) to make our previous differentiation argument work: Choose λ_n so that, as before, $\lambda_n \nearrow \infty$, but now in addition require:

$$\lambda_n \;\geq\; \max\{2^n, |a_n|, \|\psi_n\|, \|\psi_n'\|, \ldots, \|\psi_n^{(n)}\|\} \qquad (n = 0, 1, 2, \ldots), \tag{11}$$

and proceed as follows:

$$\begin{aligned}
\sum_{n=0}^{\infty} \|f_n^{(k)}\| \;&=\; \sum_{n=0}^{\infty} \frac{|a_n|}{\lambda_n^{n-k}} \|\psi_n^{(k)}\| \\
&=\; \sum_{n=0}^{k+1} \frac{|a_n|}{\lambda_n^{n-k}} \|\psi_n^{(k)}\| \;+\; \sum_{n=k+2}^{\infty} \underbrace{\underbrace{\frac{1}{\lambda^{n-k-2}}}_{\leq 1/2^{n-k-2}} \underbrace{\frac{|a_n|}{\lambda_n}}_{\leq 1} \underbrace{\frac{\|\psi_n^{(k)}\|}{\lambda_n}}_{\leq 1}}_{\text{by choice of } \lambda_n\text{'s}} \\
&\leq\; \sum_{n=0}^{k+1} \frac{|a_n|}{\lambda_n^{n-k}} \|\psi_n^{(k)}\| \;+\; \sum_{n=0}^{\infty} \frac{1}{2^n} \\
&<\; \infty.
\end{aligned}$$

Thus each series $\sum_{n=0}^{\infty} f_n$ converges, by the Weierstrass M-test (see [3, Thm. 7.10, page 148], for example), uniformly on the real line, thus justifying our previous differentiation arguments, and completing the proof of Borel's theorem. □

5. Taylor series and asymptotic expansions

5.1. **In the footsteps of Euler.** In the first few pages of his little monograph [2], Erdélyi introduces us to the formal power series:

$$\sum_{n=0}^{\infty}(-1)^n\, n!\, x^n \quad \text{i.e.,} \quad 1 \;-\; x \;+\; 2!\,x^2 \;-\; 3!\,x^3 \;+\; 4!\,x^4 \;+\; \cdots\;, \tag{12}$$

which, although it converges at no point other than the origin, nevertheless attracted the attention of Euler, circa 1754. Its divergence notwithstanding, Borel's theorem (Theorem 4.1) tells us that this series represents some function in $C^\infty(\mathbb{R})$ (and therefore many functions, cf. Exercises 2.1–2.3 above). Can we write one down in some kind of closed form?

In attempting to answer this question we're aided by Euler's integral formula for the factorial function:

$$n! \;=\; \int_0^\infty e^{-t}\, t^n\, dt \qquad (n = 0, 1, 2, \ldots), \tag{13}$$

which suggests the following calculation, whose last two steps (marked by the quasi-equality symbol " = ") make no sense whatsoever:

$$\begin{aligned}
\sum_{n=0}^{\infty}(-1)^n\, n!\, x^n \quad &= \quad \sum_{n=0}^{\infty}(-1)^n \int_{t=0}^\infty e^{-t}(xt)^n\, dt \\
\text{“ = ”} \quad & \int_{t=0}^\infty e^{-t}\left(\sum_{n=0}^{\infty}(-xt)^n\right) dt \\
\text{“ = ”} \quad & \int_{t=0}^\infty \frac{e^{-t}}{1+xt}\, dt\ .
\end{aligned}$$

Note that second line above involves the interchange of sum and integral in a series that converges nowhere (except at the origin), while the third one suffers from the problem that the series expansion under the integral sign can only be summed for $t < 1/x$. Nevertheless, the end result is a perfectly respectable function

$$E(x) := \int_{t=0}^\infty \frac{e^{-t}}{1+xt}\, dt \qquad (x \ge 0), \tag{14}$$

which we hope might be a "parent" for the power series (12). In fact, it's exactly that: by repeatedly differentiating "under the integral sign" (justified by the exponential convergence of the integrand to zero as $t \to \infty$, see [3, Theorem 9.24 & Example 9.43, pp. 236–238]), we obtain

$$E^{(n)}(x) \;=\; (-1)^n\, n! \int_{t=0}^\infty \frac{t^n\, e^{-t}}{(1+xt)^{n+1}}\, dt \qquad (n = 0, 1, 2, \ldots), \tag{15}$$

where we've used (13) for the last equality. Thus

$$(-1)^n\, n! \;=\; \frac{E^{(n)}(0)}{n!} \qquad (n = 0, 1, 2, \ldots),$$

hence the power series (12) is, indeed, the Taylor series for the function E.

But is this useful? Can the series (12) tell us anything useful about the function E? Surprisingly, it can. To see what this is, note that for each $x \geq 0$:

$$|E^{(n)}(x)| \;\leq\; n! \int_{t=0}^{\infty} t^n \, e^{-x} \, dt \;=\; (n!)^2 \qquad (n = 0, 1, 2, \ldots).$$

so by Taylor's theorem with Lagrange's form of the remainder (Theorem 1.1) we see that for each non-negative integer N:

$$(16) \qquad E(x) \;=\; \sum_{n=0}^{N} (-1)^n \, n! \, x^n \;+\; R_N(x) \quad \text{where} \quad |R_N(x)| \;\leq\; (N+1)! \, x^{N+1} \;.$$

The point here is that, although for fixed x the error estimate in (16) becomes large as N increases, for *fixed* N of "moderate size" it is "small" for "small x." To illustrate: upon taking $N = 2$ in (16) we see that for $0 \leq x < 0.1$,

$$E(x) \;:=\; \int_{t=0}^{\infty} \frac{e^{-t}}{1+xt} \, dt \;\approx\; 1 \;-\; x \;+\; 2\,x^2$$

with absolute error $\leq 6\,x^3 \leq .006$.

5.2. **Taylor series and asymptotic expansions.** Our conclusion (16) about the connection between Euler's series (12) and the integral $E(x)$ can be formalized in the following way. Suppose S is a subset of the real line having the point x_0 as a limit point. We allow the possibility that x_0 may be ∞ or $-\infty$. We'll call a sequence $(\varphi_n)_0^\infty$ of real-valued functions on S an "asymptotic sequence as $x \to x_0$" provided that

$$\varphi_{n+1}(x) \;=\; \mathrm{o}(\varphi_n(x)) \quad \text{as} \quad x \to x_0.$$

For such a sequence of functions we call any formal series of the form $\sum_{n=0}^{\infty} a_n \, \varphi_n(x)$, with real coefficients a_n, an "asymptotic series" (as $x \to x_0$). We say such a series is an "asymptotic expansion" (as $x \to x_0$) of a function $f \colon S \to \mathbb{R}$ provided that, for each non-negative integer N:

$$f(x) \;-\; \sum_{n=0}^{N} a_n \, \varphi_n(x) \;=\; \mathrm{o}(\varphi_n(x)) \quad \text{as} \quad x \to x_0.$$

When this happens we write $f(x) \sim \sum_{n=0}^{\infty} a_n \, x^n$ as $x \to x_0$.

Example: The monomial sequence $((x - x_0)^n)_0^\infty$ is an asymptotic sequence as $x \to x_0$, hence any power series $\sum_{n=0}^{\infty} a_n \, (x - x_0)^n$ is an asymptotic series as $x \to x_0$. The qualitative content of our work on the connection between Euler's series (12) and the integral $E(x)$ can be recast in this language as follows:

$$\int_{t=0}^{\infty} \frac{e^{-t}}{1+xt} \, dt \;\sim\; \sum_{n=0}^{\infty} (-1)^n \, n! \, x^n \quad \text{as} \quad x \to 0+ \;.$$

The situation described above is, in fact, quite general.

Theorem 5.1. *Suppose I is an open interval of the real line centered at x_0, and $f \in C^\infty(I)$. Then:*

(a) $f(x) \sim \displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$ *as* $x \to x_0$.

(b) *If* $f(x) \sim \displaystyle\sum_{n=0}^{\infty} a_n (x-x_0)^n$ *as* $x \to x_0$, *then* $a_n = \dfrac{f^{(n)}(x_0)}{n!}$ *for each non-negative integer* n.

In other words:

Each Taylor series is the unique asymptotic expansion of its parent function.

Proof. (a) We appeal once more to Theorem 1.1, Taylor's theorem with the Lagrange form of remainder. Fix a non-negative integer N. Then by Theorem 1.1, for each $x \in I$ there exists $\overline{x}$ between x and x_0 such that

$$\begin{aligned} f(x) \; - \; \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \; &= \; \frac{f^{(N+1)}(\overline{x})}{(N+1)!}(x-x_0)^{N+1} \\ &= \; \mathrm{O}\big((x-x_0)^{N+1}\big) \; = \; \mathrm{o}\big((x-x_0)^{N}\big) \quad \text{as} \quad x \to x_0. \end{aligned}$$

Thus the Taylor series of f, center at x_0, is an asymptotic expansion of f as $x \to x_0$.

(b) Suppose $f \sim \sum_{n=0}^{\infty} a_n (x-x_0)^n$ as $x \to x_0$. We wish to show that $a_n = f^{(n)}(x_0)/n!$ for each non-negative integer n. By our hypothesis, $f(x) - a_0 = \mathrm{o}(1)$ as $x \to x_0$, so $f(x_0) = a_0$ by the continuity of f.

We proceed by induction. Suppose the desired result is true for indices $< n$, i.e., $a_k = f^{(k)}(x_0)/k!$ for $0 \le k \le n-1$. Then for all $x \in I$:

$$\begin{aligned} f(x) - \sum_{k=1}^{n} a_k (x-x_0)^k \; &= \; \left(f(x) \; - \; \sum_{k=1}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \right) \; - \; a_n (x-x_0)^n \\ &= \; \frac{f^{(n)}(\overline{x})}{n!}(x-x_0)^n \; - \; a_n (x-x_0)^n \end{aligned}$$

where in the first line we used the induction hypothesis, and in the last line Theorem 1.1 with $N = n-1$, which guarantees the existence of the point $\overline{x}$ between x and x_0. By our hypothesis that $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is an asymptotic expansion of $f(x)$, we know that as $x \to x_0$:

$$f(x) \; - \; \sum_{k=1}^{n} a_k (x-x_0)^k \; = \; \mathrm{o}((x-x_0)^n),$$

so upon dividing both sides of the result of the calculation above by $(x-x_0)^n$ we see that

$$\frac{f^{(n)}(\overline{x})}{n!} \; - \; a_n \; = \; \mathrm{o}(1) \quad \text{as} \quad x \to x_0.$$

By the continuity of $f^{(n)}$ this implies that $f^{(n)}(x_0)/n! = a_n$, as desired. □

Exercise 5.2. Use the formula

$$\sum_{n=0}^{N} y^n \;=\; \frac{1-y^{N+1}}{1-y} \qquad (y \in \mathbb{R}\backslash\{1\})$$

to derive the estimate (16), and therefore prove, without appealing to either Advanced Calculus or Taylor's theorem, that the series (12) furnishes an asymptotic expansion for the function E. Then use Theorem 5.1(b) to conclude that this series is the Taylor series for E (center at the origin).

5.3. **The error function.** In this section we'll study a classical special function from the point of view of both Taylor and asymptotic expansions. The object of our attention is the "error function"

$$\operatorname{erf}(x) \;:=\; \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt\ , \tag{17}$$

where, now and for the rest of this section, we assume that x is non-negative.

Closely related to erf is the "complementary error function"

$$\operatorname{erfc}(x) \;:=\; \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt \;=\; 1 \;-\; \operatorname{erf}(x) \tag{18}$$

the last equality coming from something we learn in calculus: $\int_0^\infty e^{-t^2}\, dt = \sqrt{\pi}/2$.

Clearly erf $\in C^\infty(\mathbb{R})$, and it's easy to show that the Taylor series for erf (center at the origin) converges to $\operatorname{erf}(x)$ for every x: indeed

$$\operatorname{erf}(x) \;=\; \int_0^x \left(\sum_{n=0}^\infty \frac{(-t^2)^n}{n!}\right) dt \;=\; \sum_{n=0}^\infty \frac{(-1)^n}{n!} \int_0^x t^{2n}\, dt\ ,$$

where the interchange of integration and summation is justified by the uniform convergence of the series on the interval of integration. Thus, for all x:

$$\operatorname{erf}(x) \;=\; \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)(n!)}\, x^{2n+1}\ . \tag{19}$$

The Taylor series (19) gives excellent approximations to the error function for small values of x, but is not so good when x is large, as illustrated by Figure 2 below, wherein the series has been truncated at the power x^{25}; the dashed curve is the graph of the error function.

However for large values of x we will see below that the error function has following asymptotic expansion:

$$\operatorname{erf}(x) \;\sim\; 1 \;-\; \frac{e^{-x^2}}{\sqrt{\pi}} \sum_{n=0}^\infty (-1)^{n+1}\, \frac{(2n-1)!!}{2^n}\, \frac{1}{x^{n+1}} \qquad (\text{as } x \to \infty), \tag{20}$$

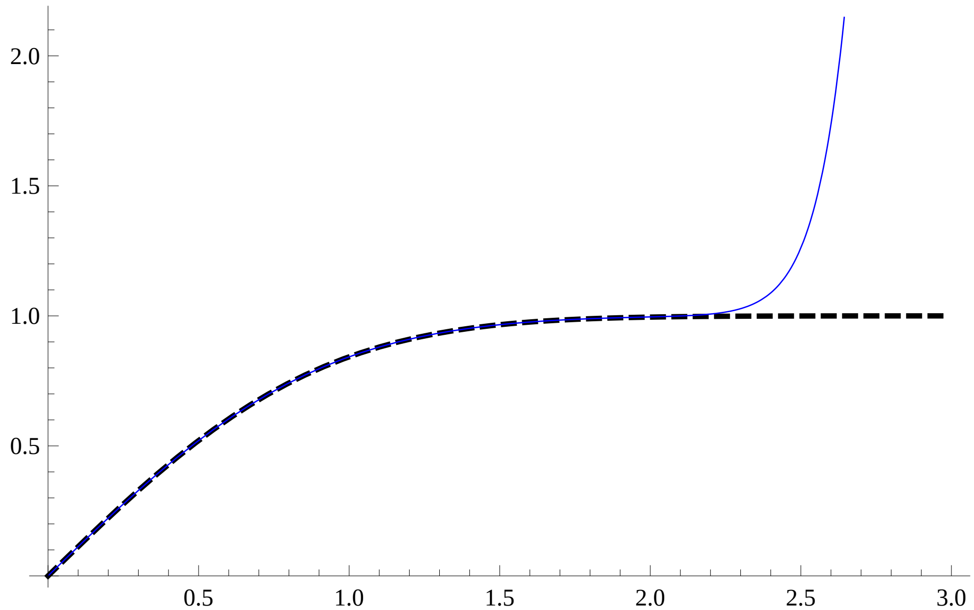

FIGURE 2. *Approximating the error function by Taylor series*

where by "$(2n-1)!!$" we mean "1" of $n=0$ or 1, and "the product of the odd integers from 1 through $2n-1$" if $n>1$. Clearly the series on the right-hand side of (20) converges for *no* value of x. However, that is not the point: the asymptotic expansion (20) implies, for example, that

$$\operatorname{erf}(x) \;=\; 1 \;-\; \frac{e^{-x^2}}{\sqrt{\pi}x} \;+\; \mathrm{o}\left(\frac{1}{x}\right) \qquad (\text{as } x \to \infty),$$

which suggests that if we can—say in the course of deriving (20)—be more precise about what "little-oh" means here, we might be able to get a good approximation to the error function for large values of x.

This is just what happens; to see how it goes, make the change of variable $s = x^2$ in the integral defining the complementary error function to obtain:

$$\operatorname{erfc}(x) \;=\; \frac{1}{\sqrt{\pi}} \int_{x^2}^{\infty} s^{-1/2}\, e^{-t}\, dt \qquad (x \geq 0). \tag{21}$$

Then fix $x>0$ and integrate by parts:

$$\begin{aligned} \operatorname{erfc}(x) \;&=\; \frac{1}{\sqrt{\pi}} \int_{x^2}^{\infty} s^{-1/2}\, \frac{d}{dt}(-e^{-t})\, dt \\ &=\; \frac{1}{\sqrt{\pi}}\, \frac{e^{-x^2}}{x} \;-\; \frac{1}{2\sqrt{\pi}} \int_{x^2}^{\infty} s^{-3/2}\, e^{-s}\, ds\ . \end{aligned}$$

The integral in the last line above is non-negative and $\leq x^{-3}\int_{x^2}^{\infty} e^{-s}\, ds = e^{-x^2}/x^3$. Thus for $x>0$: $\operatorname{erfc}(x)$ is over-estimated by $e^{-x^2}/(\sqrt{\pi}\, x)$, but by no more than $e^{-x^2}/(2\sqrt{\pi}x^3)$, hence $\operatorname{erf}(x)$ is under-estimated by $1 - e^{-x^2}/(\sqrt{\pi}\, x)$, but by no more than the same amount. In particular,

$$0 \;<\; \mathrm{erf}\,(x) \;-\; \left(1 \;-\; \frac{e^{-x^2}}{\sqrt{\pi}\,x}\right) \;<\; 5\times 10^{-5} \quad \text{whenever} \quad x \geq 2.5.$$

This is shown in Figure 3 below, again with the dashed curve representing the graph of the error function and the solid curve the graph of the approximation $y = 1 - e^{-x^2}/(\sqrt{\pi}\,x)$.

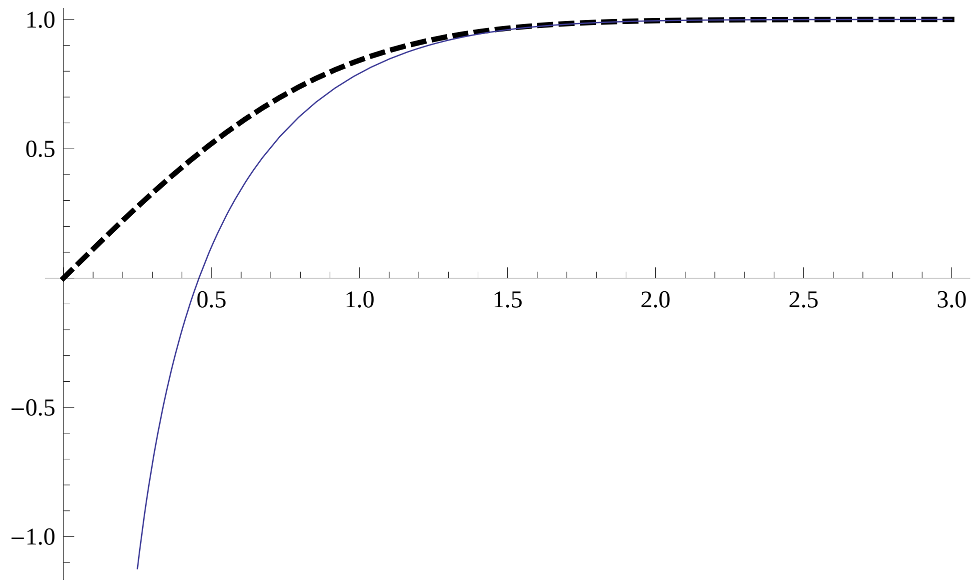

FIGURE 3. *Approximating the error function by first two terms of the asymptotic series* (20)

To derive the full asymptotic expansion (20) we continue the above argument, at each state using integration by parts on the error term. This is best organized as follows: Let, for n a non-negative integer,

$$F_n(x) = \int_{x^2}^{\infty} s^{-(n+1/2)}\,e^{-s}\,ds \qquad (x \geq 0),$$

noting that $F_0(x) = \sqrt{\pi}\,\mathrm{erfc}\,(x)$ is the integral we just analyzed. Integration by parts argument now shows that for each n:

$$F_n(x) = \frac{e^{-x^2}}{x^{2n+1}} \;-\; \left(n + \frac{1}{2}\right) F_{n+1}(x) \tag{22}$$

Upon iterating this result we find, upon setting $\varphi_n(x) = e^{-x^2/x^{2n+1}}$, that

$$F_0(x) \;=\; \sum_{n=0^N} (-1)^n\,\frac{(2n-1)!!}{2^n}\,\varphi_{2n+1}(x) \;+\; (-1)^{N+1}\,\frac{(2N+1)!!}{2^{N+1}}\,F_{N+1}(x) \tag{23}$$

Now

$$0 \leq F_{N+1}(x) \;:=\; \int_{x^2}^{\infty} s^{-(N+3/2)}\,e^{-s}\,ds \;\leq\; x^{-(2N+3)} \int_{x^2}^{\infty} e^{-s}\,ds \;=\; \varphi_{2N+3}(x)$$

so, because for each n: $\varphi_{n+1}(x) = \mathrm{o}(\varphi_n(x))$ as $x \to \infty$, this completes the proof that (20) is an asymptotic expansion for $\operatorname{erf}(x)$. $\square$

References

[1] Robert Ellis and Denny Gulick, *Calculus with Analytic Geometry,* Second edition, Harcort Brace Jovanovich, 1982.

[2] A. Erdélyi, *Asymptotic Expansions,* Office of Naval Research, Technical Report #3, 1956. Reprinted by Dover Publications, Inc., 2010.

[3] Walter Rudin, *Principles of Mathematical Analysis,* Third edition, McGraw-Hill 1976.

[4] Wikipedia, *Non-analytic smooth function,* available at `http://en.wikipedia.org/wiki/Non-analytic_smooth_function`

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