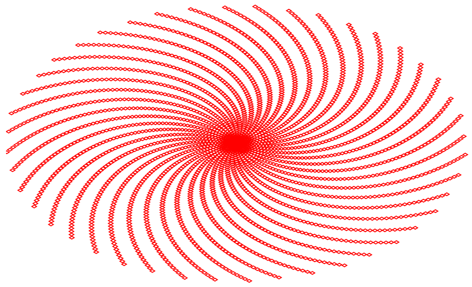
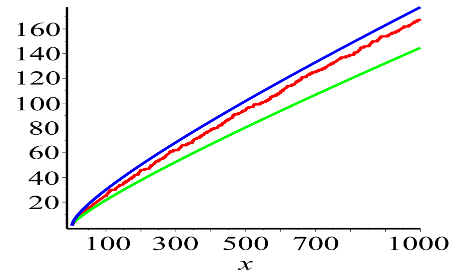
Technion, Haifa, Israel, Nov 2022





PRIMES!
An Informal Story

Based on the Lecture Notes:
An Introduction to Number Theory,
http://web.pdx.edu/~veerman/0_mainfile.pdf
(work in progress).

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SUMMARY:

* We start with a brief look at *rings* and *fields* and discuss the consequences of allowing (or disallowing) division.

* Legendre guesses the *prime number theorem* (PNT) in 1808 [11]. Almost there in 1850: Chebyshev's Theorem [4]: Stirling and $\binom{2n}{n}$.

* In 1859, Riemann rocks the boat. Need complex analytic functions [13]. Finally proved in 1896 independently by de la Vallée-Poussin [8] and Hadamard [10]: The PNT. Our version heavily relies on later proofs by Newman [12] and Zagier [16].

* In 1837, Dirichlet [6] complicates life: how many primes occur in sequences of the form $\{n + mi\}_{i=1}^{\infty}$ (*arithmetic progressions*): PNT Arithm. Progr. The complete solution uses PNT [19].

* The continued fraction expansion as the most natural way to approximate real numbers. How well do the rational approximants approximate reals is of fundamental importance in many areas of mathematics (and physics). Here used as a tool to visualize PNT and PNT Arithm Prog.

OUTLINE:
The headings of this talk are color-coded as follows:

Rings and Fields

The Riemann Hypothesis

Continued Fractions

The Prime Number Theorem

Primes in Arithmetic Progressions

Extra Material

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RINGS AND FIELDS[1]

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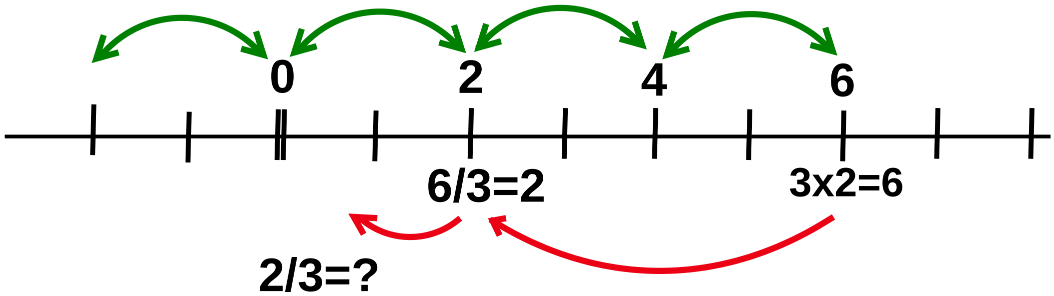
[1]We leave out ideals and marriage counseling.

An Uneasy Marriage: $+$ and $\times$



Can add 2 and subtract 2 equally easily at will.
Can multiply at will: $3 \cdot 2$, but its inverse is touchy (at best).

Definition[2]. Field: $\times$ and $+$ and inverses work (except 0^{-1}).
Ring: same, except no multiplicative inverses.

$\mathbb{Z}$ is a ring. $\mathbb{Q}$ and $\mathbb{R}$ are fields. Polynomials with coefficients in $\mathbb{R}$ also form a ring.

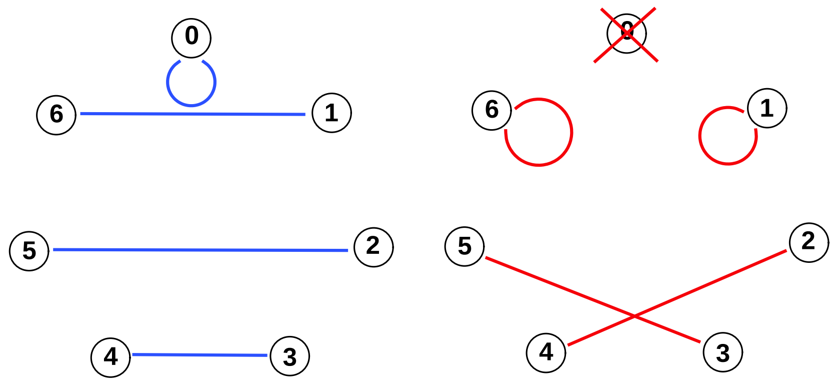
Definition. In a ring we have the following:
The zero is the (unique) additive identity.
A unit is an element with multiplicative inverse.
An irreducible[3], an element ($\neq 0$) *not* a product of 2 non-units.
A prime, p such that $p \mid ab$ implies $p \mid a$ or $p \mid b$.

What do we have in $\mathbb{Z}$??
The zero is 0, the units are ± 1, and primes and irreducibles are the same.

[2] Grossly simplified.
[3] Outside number theory, this is often used as defn of a prime! In $\mathbb{Z}$, the two notions coincide.

First a Field, Then a Ring

Z mod 7:

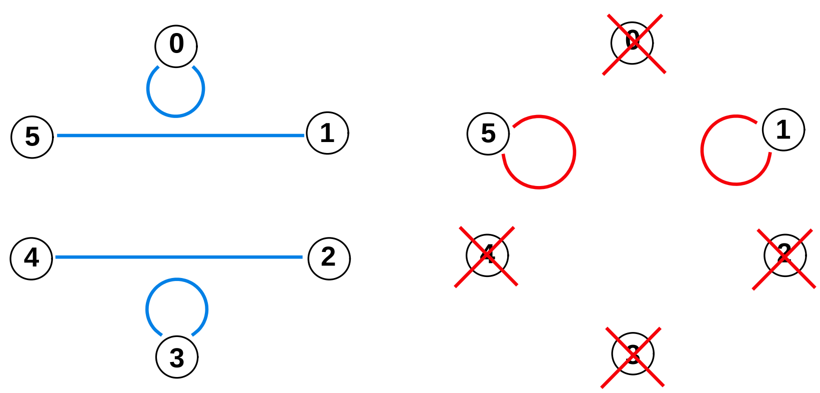


Left, the relation additive inverse; right, multiplicative inverse.

Zero: 0. **Units: all others.**

Example: $3 \cdot 4^{-1} =_7 3 \cdot 2 =_7 6$. And $6 \cdot 4 =_7 3$.

Z mod 6:



Zero: 0. **Units: 1 and 5.**

. Irred's: none. **Primes: 2, 3, 4.**

Example: $2 \mid ab$. Then in $\mathbb{Z}$: $2 \mid 6m + ab$ where $ab \in \{0, 2, 4\}$. So either a or b is even. Thus 2 divides a or b. So 2 prime.

But: $2 =_6 4 \cdot 2$. Thus 2 is reducible.

The Two Sides of the Coin

Absence of multiplicative inverses $\Longrightarrow$
Some numbers may not have *any* (non-trivial) divisors.
This lack of division also brings us to the study of primes.
A very complex problem with many open conjectures.

Still, for all a and b in $\mathbb{Z}$, there are q (quotient) and r(remainder):

$$a = qb + r \quad |r| < b\,.$$

This is the **division algorithm**.
You end up doing (abstract) **algebra**.

To avoid Prime Problems, legislate division!
For every p and q in $\mathbb{Z}$, define "formal" quotient "$p : q$".
Now we get a *field*, the **field of fractions** of $\mathbb{Z}$.
This field is in fact $\mathbb{Q}$.

Other questions arise: we still cannot solve for x in $x^2 = 2$.
Need to take limits, deal with different infinities...
You end up doing **analysis**.

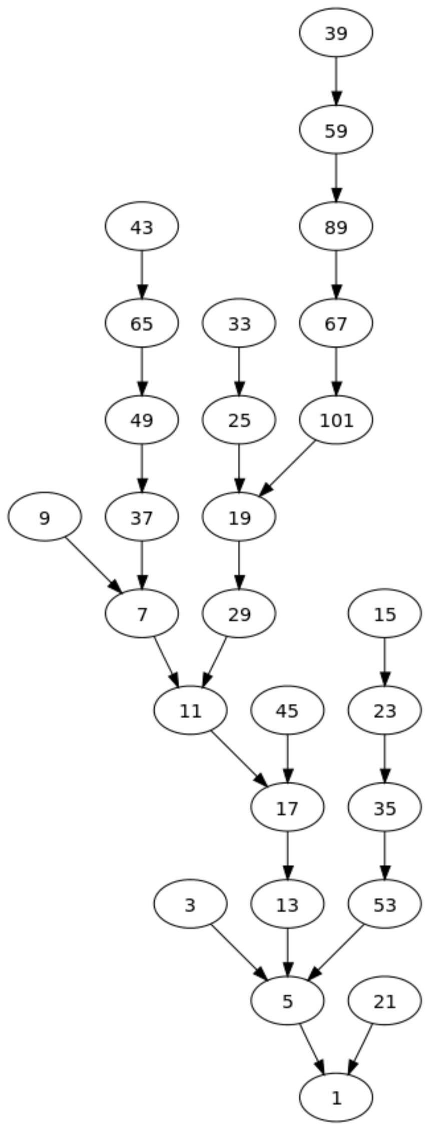
A whole other *can of worms*.

Open Problems: 3n+1

Definition. Define $C : \mathbb{N} \to \mathbb{N}$ as follows:

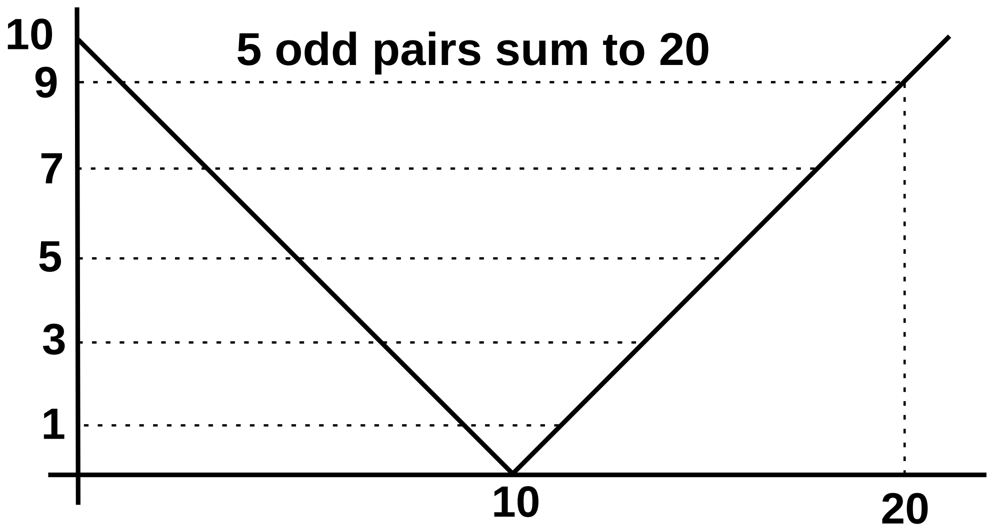
$$C(n) := \begin{cases} n/2 & \text{if } n \text{ even.} \\ \dfrac{3n+1}{2} & \text{if } n \text{ odd.} \end{cases}$$

Conjecture. For all n, eventually $2 \to 1 \to 2 \to \cdots$.
Dividing all "evens" by 2, we get (from wikipedia), see [3]:



Open Problems: Goldbach's Conjecture

Conjecture (1742). Every even number greater than two is the sum of two primes, see [17].



An odd number *cannot* be the sum of 2 primes (there is only 1 even prime). However, for $2n$ even:

$$\begin{cases} 4 = 2 + 2\,; \quad 6 = 3 + 3\,; \\ 8 = 3 + 5\,; \quad 10 = 5 + 5 = 3 + 7\,; \;\cdots \end{cases}$$

If primes are independent, then chance of both $n - m$ and $n + m$ being prime is[4] roughly $1/\ln(n-m)\ln(n+m)$. So [17]

$$\sum_{m=1}^{n-1} (\ln(n-m)\ln(n+m))^{-1} \approx n/2(\ln n)^2$$

is the "expected" number of Goldbach pairs for $2n$.

But they are not. See above.

[4] We'll see this in a bit.

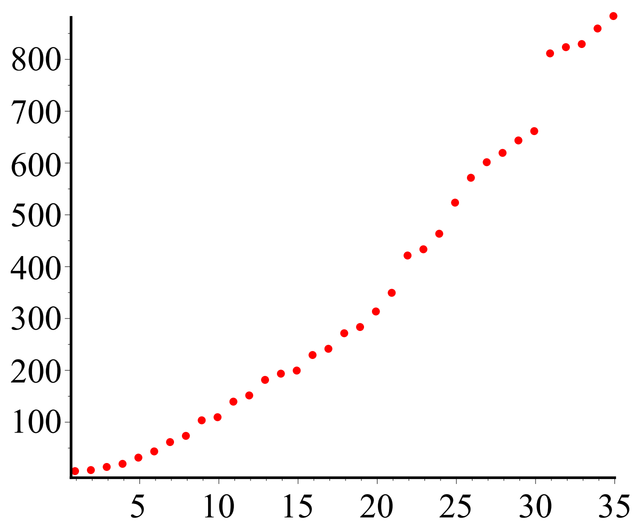
Open Problems: Twin Prime Conjecture

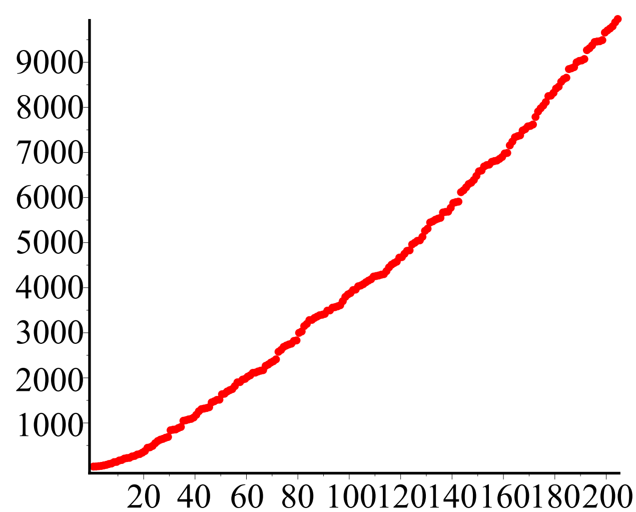
Conjecture. There are infinitely many primes p such that $p+2$ is prime.

Note: there is only 1 pair of primes that differ by 1, nl: (2,3). Here are some twin prime pairs.

$$(3,5),\ (5,7),\ (11,13),\ (17,19),\ (29,31),\ (41,43),\ (59,61),\ \cdots .$$

Also: 5 is the only number to occur in two twin prime pairs[5].





Lowest member of twin prime pair less than 1000 (left) and less than (10000) (right).

Generalized Conjecture. For any $k > 0$, there are infinitely many pairs of primes p such that $p + 2k$ is prime.

Toy Idea. The above curve looks superficially like cx^p where $p \approx 4/3$. Check numerically via scaling arguments.

[5] Because every 6 consecutive integers contain three evens and an odd multiple of 3.

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RIEMANN'S SUSPICION

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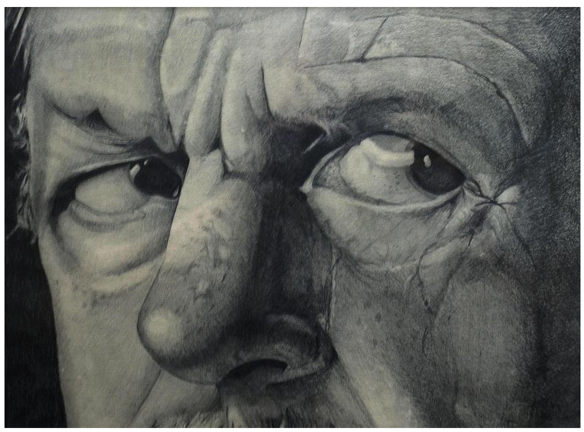


Drawing by Tim Ernst.

The Riemann Hypothesis

Examples of unique analytic continuation.

1. Unique anal. cont'n of $\sum z^n$ is $1/(1-z)$.

2. $\sum \dfrac{z^n}{n!}$ equals e^z, is already analytic everywhere.

Definition. For $z \in \mathbb{C}$ the Riemann zeta function $\zeta(1+z)$ is the unique analytic continuation of $\sum_{n=1}^{\infty} n^{-1-z}$.

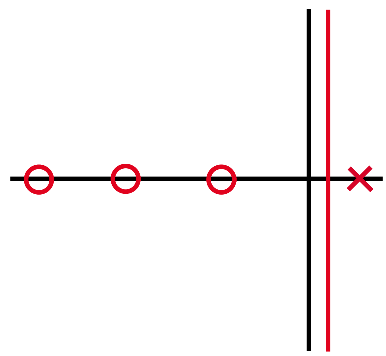
Riemann: "crazy" assertion that locus of zeroes of analytic cont'n of the zeta fn yields information about primes in $\mathbb{N}$.
Or: complex anal. to solve a 'discrete' problem !

Approximate Riemann Hypothesis:
Zeroes of $\zeta(1+z)$ lie on $\mathrm{Re}(z) \leq -1/2$.

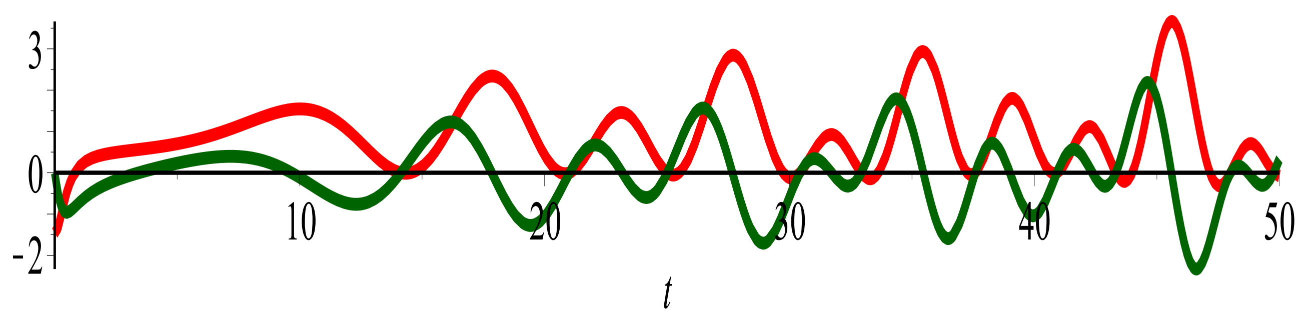
RH remains most important unsolved prob of math.!

More on the Riemann Hypothesis

$\zeta(z)$ has 1 pole: $\propto (z-1)^{-1}$ near 1. Well-defined for all $z \neq 1$.
Trivial zeroes: $z = -2, -4, -6$, etc. Easy to prove:



Full version of Riemann Hypothesis.
All non-trivial zeroes of $\zeta(z)$ have real part 1/2.
Or:
All non-trivial zeroes of $\zeta(1+z)$ have real part -1/2.
Red: $\mathrm{Re}\,\zeta(1/2+it)$, **green**: $\mathrm{Im}\,\zeta(1/2+it)$



From wikipedia *Riemann Hypothesis*, August 18, 2022: "The consensus of the survey articles ([2, 5, 14]) is that the evidence for it is strong but not overwhelming, so that while it is probably true, there is reasonable doubt."

Dyson: reduces RH to classifying 1D quasi-crystals [7].

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CONTINUED FRACTIONS

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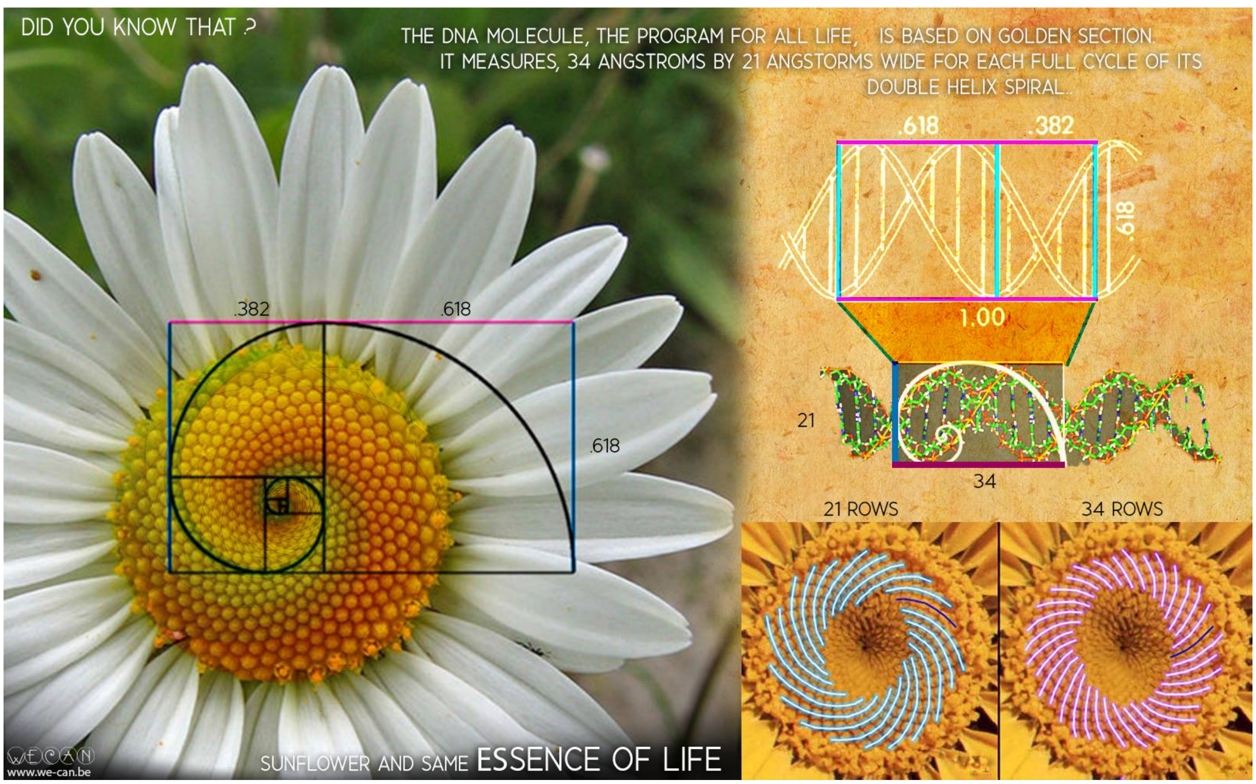


The Division Algorithm

Division Algorithm. $r_i < r_{i-1}$ positive integers.
There exist quotient q_i and remainder r_{i+1} such that

$$r_{i-1} = r_i q_i + r_{i+1} \quad \text{and} \quad r_{i+1} \in \{0, \cdots, r_i - 1\}$$

Now think of r_i/r_{i-1} as rational numbers in $(0, 1)$.

$$\frac{r_{i+1}}{r_i} = \frac{1}{r_i/r_{i-1}} - q_i$$

Note: q_i is integer part of $\frac{1}{r_i/r_{i-1}}$. Denote integer part by $\lfloor \cdot \rfloor$.

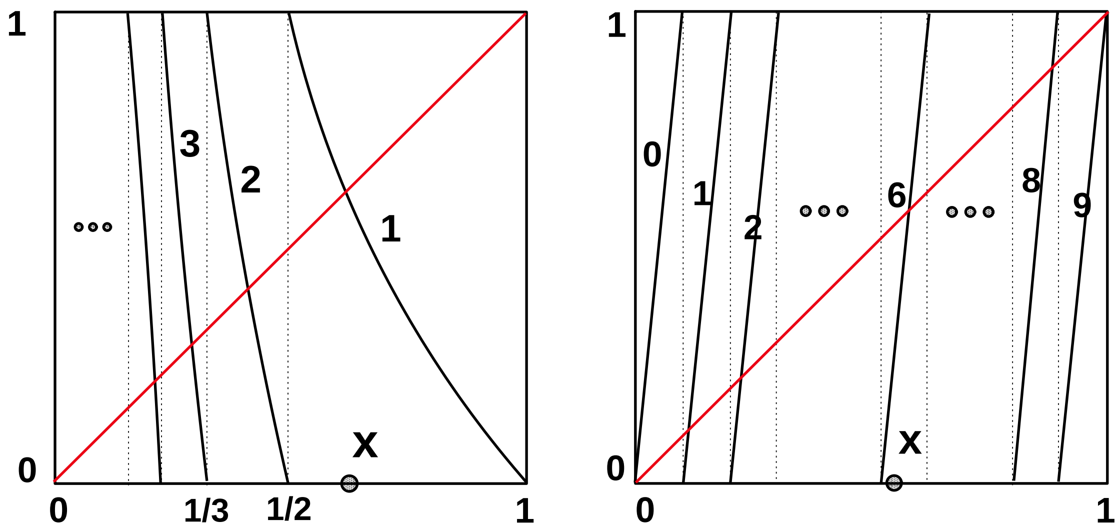
Why restrict to rationals? So:
Gauss Map. For simplicity $x \in (0, 1)$

$$T := x \to \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor$$

(Taking limits, we are in analysis-land.)

Continued Fractions

T is Gauss map and **T is times 10 mod 1 map:**



Cont'd fr'n expansion. Branches $b_k : I_k \to [0,1]$ (onto).
If $x \in I_k$, then first digit of expansion: $x = [k, \cdots]$.

Example. Golden mean $x = 0.5\left(\sqrt{5}-1\right) = .618\cdots$.
Gauss map: $x = [1, \cdots]$. Times 10: $x = [6, \cdots]$.

Cont'd fr'n exp'n. Label branches T^2 so that $TI_{k,\ell} = I_\ell$.
If $x \in I_{k,\ell}$, then $x = [k, \ell, \cdots]$.

Example. G'ss map: $x = [1, 1, \cdots]$. T's 10: $x = [6, 1, \cdots]$.

The 2nd convergent to $x = [k, \ell, \cdots]$ is:

$$p_2/q_2 := [k, \ell] := \text{ zero of the } b_{k,\ell} \text{ branch}.$$

Example. G'ss map: $[1, 1] = 1/2$. T's 10: $[6, 1] = 0.61$.

The Convergents ...

The location of these zeros is easily computed. So:
For the **Gauss map**

$$\frac{p_n}{q_n} \stackrel{\text{def}}{\equiv} [a_1, a_2, \cdots, a_n] = \frac{1}{a_1 + \frac{1}{a_2 + \cdots \frac{1}{a_n}}} \ .$$

For the **times 10 map**

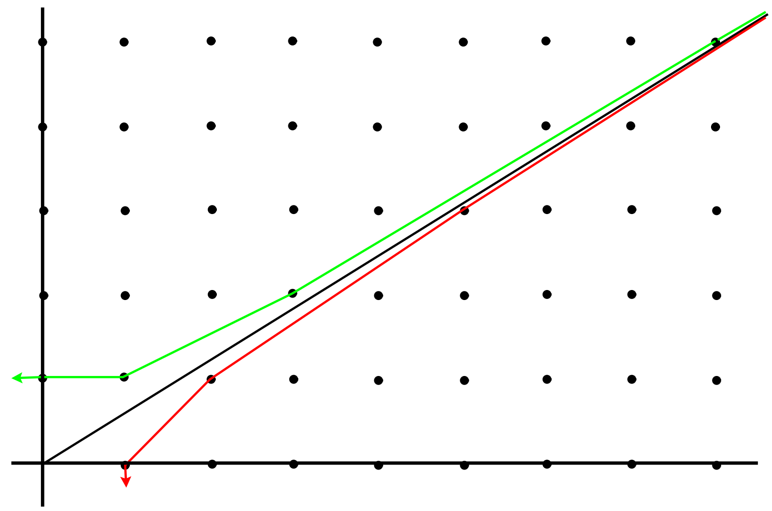
$$\frac{p_n}{q_n} \stackrel{\text{def}}{\equiv} [a_1, a_2, \cdots, a_n] = \sum_{i=1}^{n} a_i \cdot 10^{-i} \ .$$

Example. $x = 0.5\left(\sqrt{5} - 1\right) = [0;\ 1,\ 1,\ 1,\ \cdots]$.
Convergents: $\left\{\frac{0}{1};\ \frac{1}{1},\ \frac{1}{2},\ \frac{2}{3},\ \frac{3}{5},\ \frac{5}{8},\ \frac{8}{13},\ \cdots\right\}$.

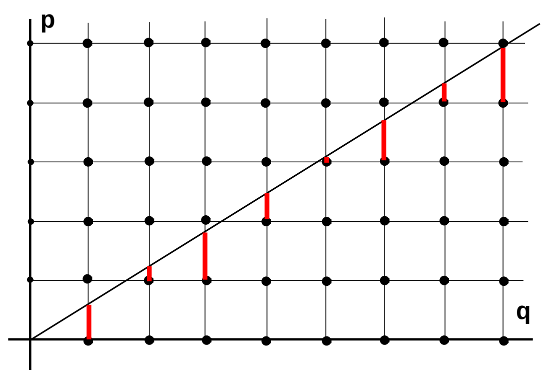
Example. $2\pi = [6;\ 3,\ 1,\ 1,\ 7,\ 2,\ 146,\ \cdots]$.
Convergents: $\left\{\frac{6}{1};\ \frac{19}{3},\ \frac{25}{4},\ \frac{44}{7},\ \frac{333}{53},\ \frac{710}{113},\ \frac{103993}{16551},\ \cdots\right\}$.

... and their Properties

Arnold's characterization of cont'd fr's. Golden mean.



So instead of $|x - \frac{p}{q}|$, consider rotations: $|qx - p|$.

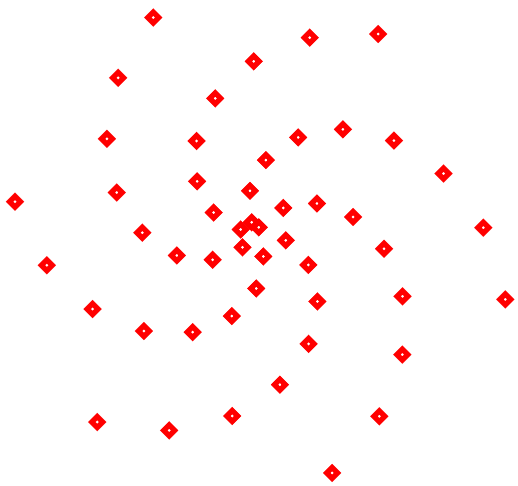


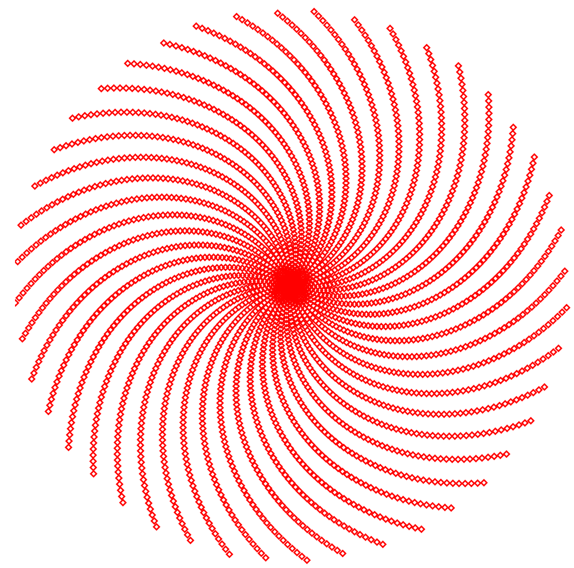
Main Theorem. If $\frac{p_n}{q_n}$ a cont'd fr'n convergent of x, then

$$|q_n x - p_n| < |q' x - p'|$$

for **all** $q' \in \{1, \cdots, q_{n+1} - 1\}$ and **all** p', **except** (p_n, q_n).
These are *optimal* approximations!

Polar Plots of Integers





$\boldsymbol{z(n) := ne^{in}}$ for $n \in \{1, \cdots, 50\}$ and $n \in \{1, \cdots, 3000\}$.

By the THM on previous page, the angular parts of $z(n)$ and $z(n_0)$ are closest if

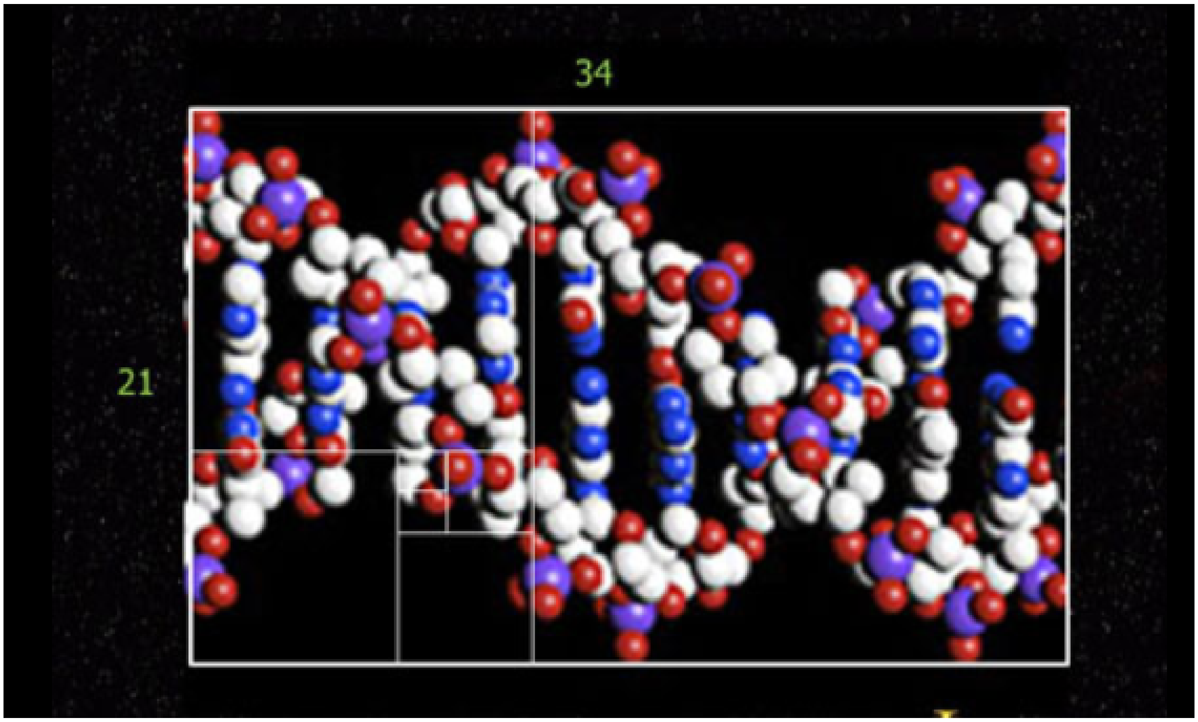
$$e^{i(n+n_0)} = e^{in} \cdot e^{in_0} = e^{i(n-2\pi m)}\, e^{in_0},$$

where $\dfrac{n}{m}$ is a cont'd fr'n convergent of 2π, or

$$\frac{n}{m} \quad \text{is one of} \quad \frac{6}{1}; \frac{19}{3}, \frac{25}{4}, \frac{44}{7}, \frac{333}{53} \cdots$$

Depending on number of pts plotted, you can see 6 (LEFT), 19, 25, 44 (RIGHT), 333, ... spiral arms.

Cont'd Fr's in Nature



The DNA molecule is based on the golden mean. It measures 34 angstroms long by 21 angstroms wide for each full cycle of its double helix spiral. 21/34 (approx. 1.6190476) is a continued fraction approximant of the golden mean (approx. 1.6180339) [15].

A word of caution: In 1994, Gardner [9] writes: "Since the Renaissance, an enormous literature has accumulated, most of it nonsense, about the applications of the golden ratio to architecture, painting, sculpture, nature, and even poetry and music."

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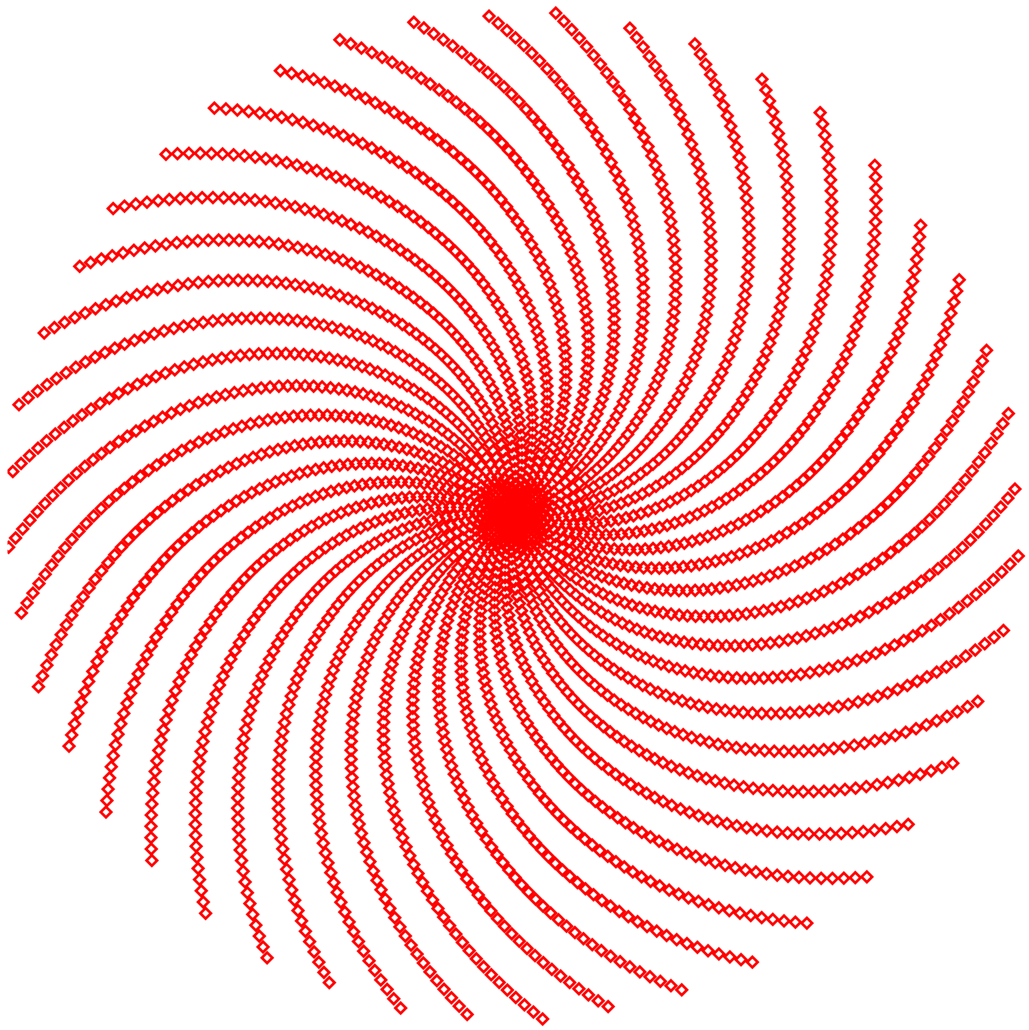
PRIME NUMBER THEOREM



A Question

Kronecker: "Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk"

Question. Compute $\operatorname{lcm}(1, 2, \cdots, n)$, then takes its nth root. Do this for all n. Do you get anything as n gets large??

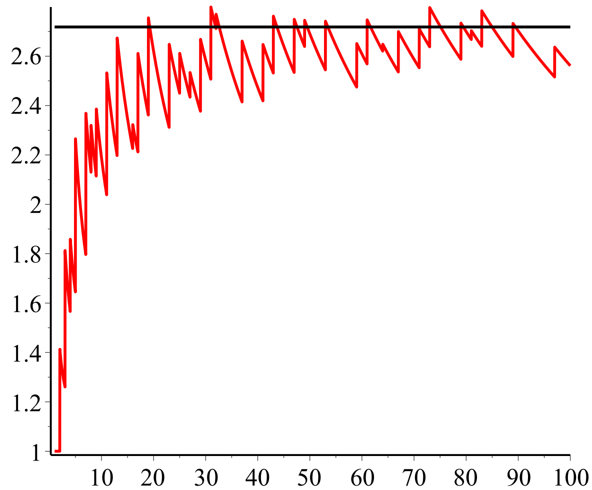
? ? ? ? ?

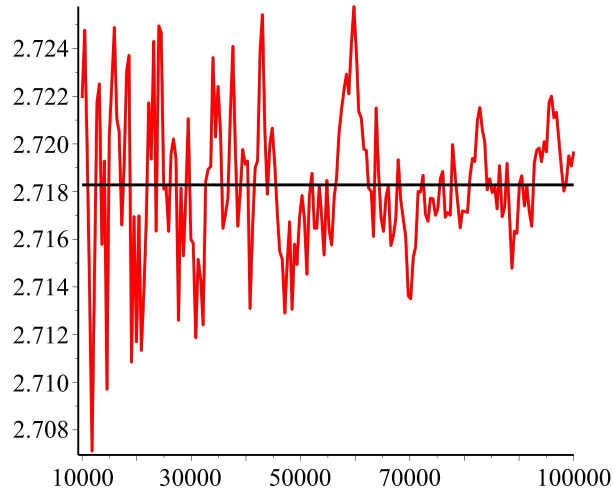
Let's try:

$n = 1:$ 1.
$n = 2:$ $2^{1/2} \approx 1.41.$
$n = 3:$ $(2 \cdot 3)^{1/3} \approx 1.82.$
$n = 4:$ $(2 \cdot 3 \cdot 2)^{1/4} \approx 1.86.$
$n = 5:$ $(2 \cdot 3 \cdot 2 \cdot 5)^{1/5} \approx 2.27.$
$n = 6:$ $(2 \cdot 3 \cdot 2 \cdot 5)^{1/6} \approx 1.98.$
ETC.

The Prime Number Theorem

Plots of $\left(\,\mathrm{lcm}\,(1,2,\cdots,n)\right)^{\frac{1}{n}}$





Prime Number Theorem I or PNT I.

$$\lim_{n\to\infty}\left(\,\mathrm{lcm}\,(1,2,\cdots,n)\right)^{\frac{1}{n}} = e\,.$$

So Kronecker's statement is false!

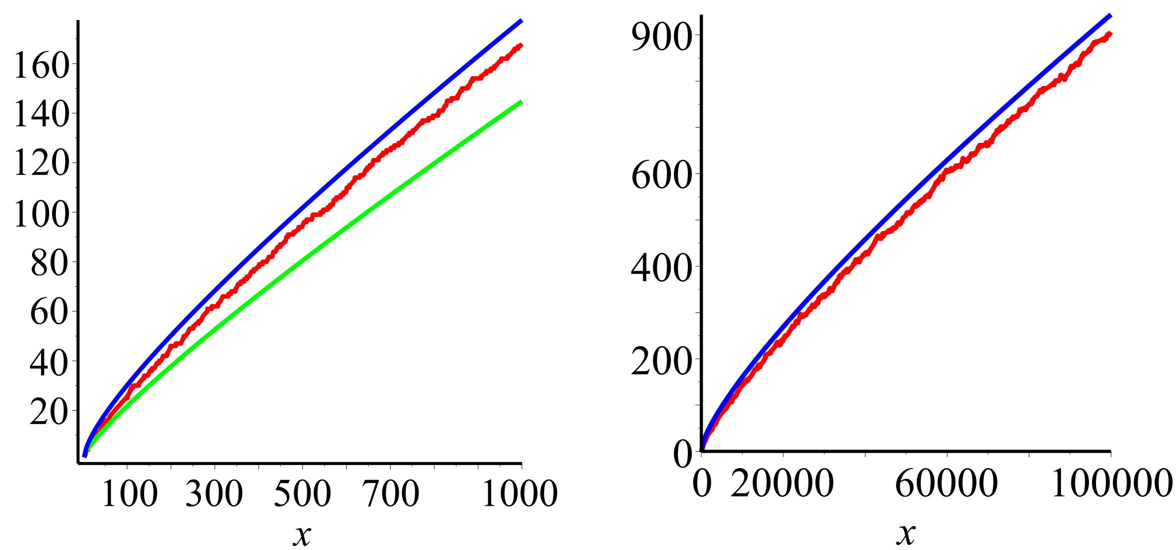
A More Familiar Version

Much NumThy is motivated by: **How many primes in $\mathbb{Z}$?**

PNT II. $\pi(x)$ is number of primes p with $2 \le p \le x$:

$$1. \quad \lim_{x\to\infty} \frac{\pi(x)}{(x/\ln x)} = 1 \qquad \text{and} \qquad 2. \quad \lim_{x\to\infty} \frac{\pi(x)}{\int_2^x \ln t\, dt} = 1$$

(ln is natural logarithm.)



Left, $x \le 10^3$: $\int_2^x \ln t\, dt$, $\pi(x)$, $x/\ln x$.
Right, $x \le 10^5$: $\int_2^x \ln t\, dt - x/\ln x$, $\pi(x) - x/\ln x$.

Skewes number: least x with $\pi(x) \ge \int_2^x \ln t\, dt$. Less than 10^{317}.

Chebyshev's Complaint

We take n to be even.

Cheb. realized that $\binom{n}{n/2}$ encodes ***marvelous*** properties:

1. It is an integer (by combinatorial defn).

2. Via the binomial thm: $\frac{2^n}{n+1} < \binom{n}{n/2} \le 2^n$.

3. $\binom{n}{n/2}$ contains primes in $(n/2, n]$: $\prod_{\frac{n}{2}<p\le n} p \le \binom{n}{n/2}$.

4. Unique factorization plus trick gives: $\binom{n}{n/2} \le n^{\pi(n)}$.

From **2**, **3**, and **4**, he derives thm below. *Very* close (but no cigar)! Result superseded by PNT proved in 1896 [10] and [8]!

Recall **definition**: $\pi(x)$ **is # primes** p **with** $2 \le p \le x$**.**

Chebyshev's Thm. [4] in 1850

$$\forall x \ge K \ : \ \frac{\pi(x)}{x/\ln x} \in [0.89, 1.11] \text{ (approximately)}.$$

If a limit exists, it must be 1.

(9 years of effort failed to produce better results.)

"Fast and Loose Proof" of PNT, 1

Via **2** and **3** of Cheb., you get that the product of primes between $x/2$ and x is less than 2^x. Take ln and divide by x:

1. $$\frac{\theta(x)}{x} := \frac{1}{x}\sum_{p\le x} \ln p \qquad \text{is bounded}\,.$$

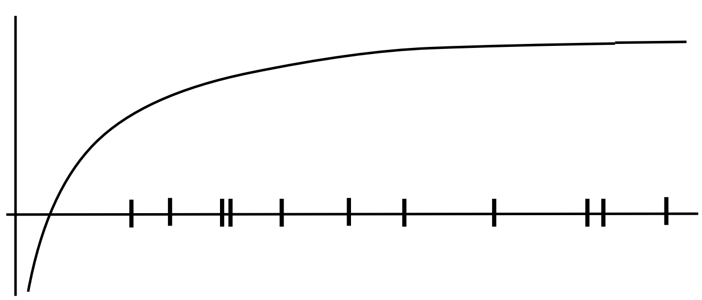
Definition. For $z \in \mathbb{C}$ and if well-defined, let

$$g(z) := \int_1^\infty \left(\frac{\theta(x)}{x} - 1\right) x^{-z-1}\,dx\,.$$

2. **If** $g(0)$ exists (is finite), then $\lim_{x\to\infty} \frac{\theta(x)}{x} = 1.$
(This looks quite plausible and is in fact easy, though clever.)

3. $\lim_{x\to\infty} \frac{\theta(x)}{x} = 1$ if and only if $\lim_{x\to\infty} \frac{\pi(x)\ln x}{x} = 1.$ **BINGO !**

(ln p increases so *agonizingly* slowly that almost all p's have



roughly same logarithm as that of the greatest prime).

The problem is the If, or: does $g(0)$ exist???

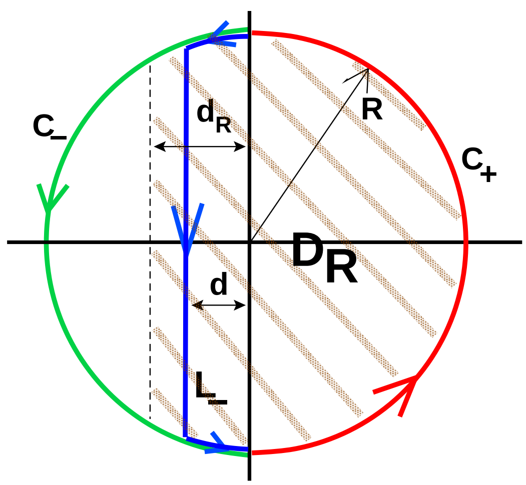
“Fast and Loose Proof” of PNT, 2

Riemann: 8 pages on numthy and compl. anal. [13] (1859).
Revolution: investigate analyticity of *complex* function $g(z)$.
Changed course of number theory and mathematics in general.
Proof took another 40 years [10] and [8] (1896).

4. After considerable tricks, it turns out that $\boldsymbol{g(z)}$ can be written as expression with $\boldsymbol{\zeta(1+z)}$ **in the denominator**.

Recall RH. Zeroes of $\zeta(1+z)$ lie on $\mathrm{Re}(z) \le -1/2$.

Zeroes lie on $\mathrm{Re}(z) < 0$ is provable variant of RH. So $g(z)$ anal on $\mathrm{Re}(z) \ge 0$.



Intricate argument w. contour integr. in $\mathbb{C}$ proves:
5. “Tauberian” Thm. Recall $g(z) := \int_1^\infty f(x)x^{-z}\,dx$.
If g has anal. cont'n to $\mathrm{Re}(z) \ge 0$, then $g(0) = \int_1^\infty f(x)\,dx$ exists.

6. Bingo! $\int_1^\infty f(x)\,dx$ exists. So **1, 2, 3** above prove PNT.

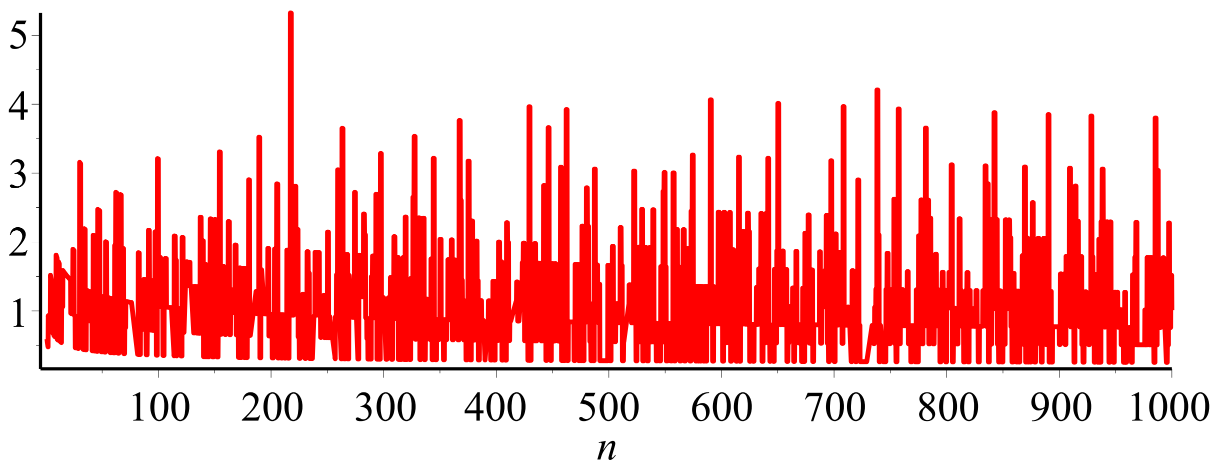
Prime Gaps

PNT III. p_n is the nth prime in $\mathbb{N}$. $\displaystyle\lim_{n\to\infty} \frac{p_n}{n\ln n} = 1.$

Another "somewhat unjustified" heuristic gives:

$$\begin{aligned} p_n - p_{n-1} &\sim n\ln n - (n-1)\ln(n-1) \\ &= \ln n + (n-1)\ln\left(\frac{n}{n-1}\right) \\ &\approx \ln n + \ln e\,. \end{aligned}$$

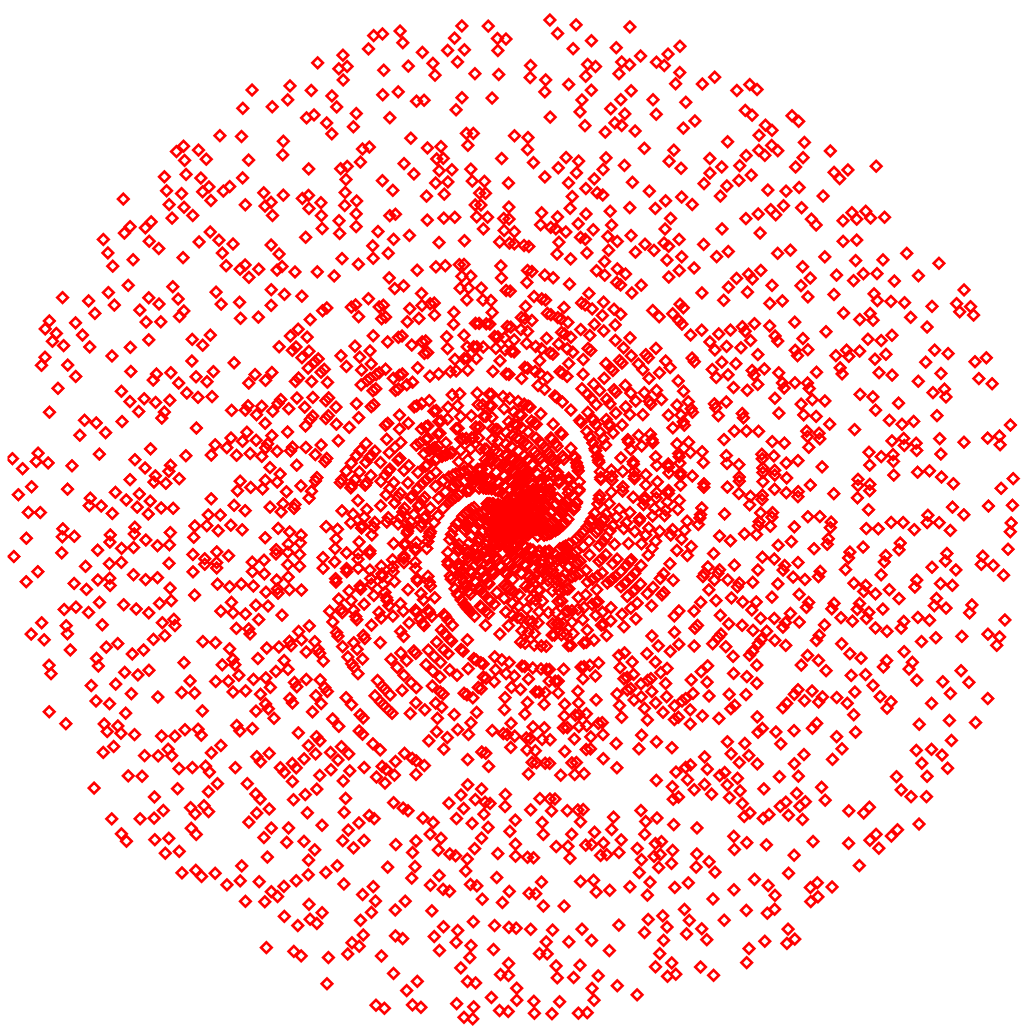
Intuitively: the gap between p_n and p_{n-1} is about $1 + \ln n$.



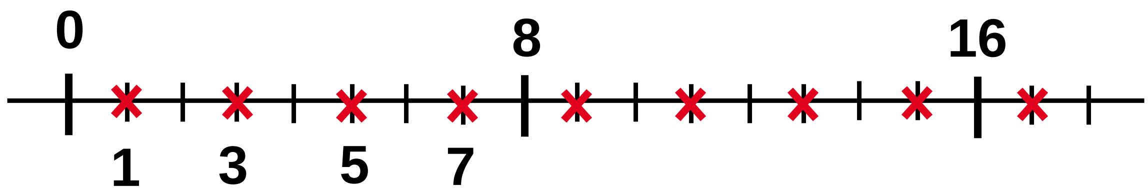
Prime gaps $p_n - p_{n-1}$ divided by $1 + \ln n$ for n in $\{1, \cdots, 1000\}$.

Prime gaps are an active area of research. See, for instance, the wikipedia entry on "prime gaps".

PRIMES IN ARITHMETIC PROGRESSIONS



The Setting



If $\gcd(x, 8) = 2$ (say), then $x = 2$ or x has non-triv divisor 2.
So with finite exceptions: $\boldsymbol{p}$ **prime, then** $\boldsymbol{\gcd(p, 8) = 1}$.
Or: (almost) all primes p are rel prime to 8.

Conclusion. An **arithmetic progression** AP with spacing q

$$\boldsymbol{AP = \{a, a+q, a+2q, a+3q, \cdots\}}$$

can contain > 1 primes **only** if a is rel prime to q.
Note: $p \in AP$ means $p =_q a$.

Defn. $\mathbb{Z}_{\boldsymbol{q}}^{\times}$: integers modulo q <u>and</u> rel prime to q.
$\boldsymbol{\phi}(q)$: cardinality of $\mathbb{Z}_q^{\times}$ (Euler totient fn).

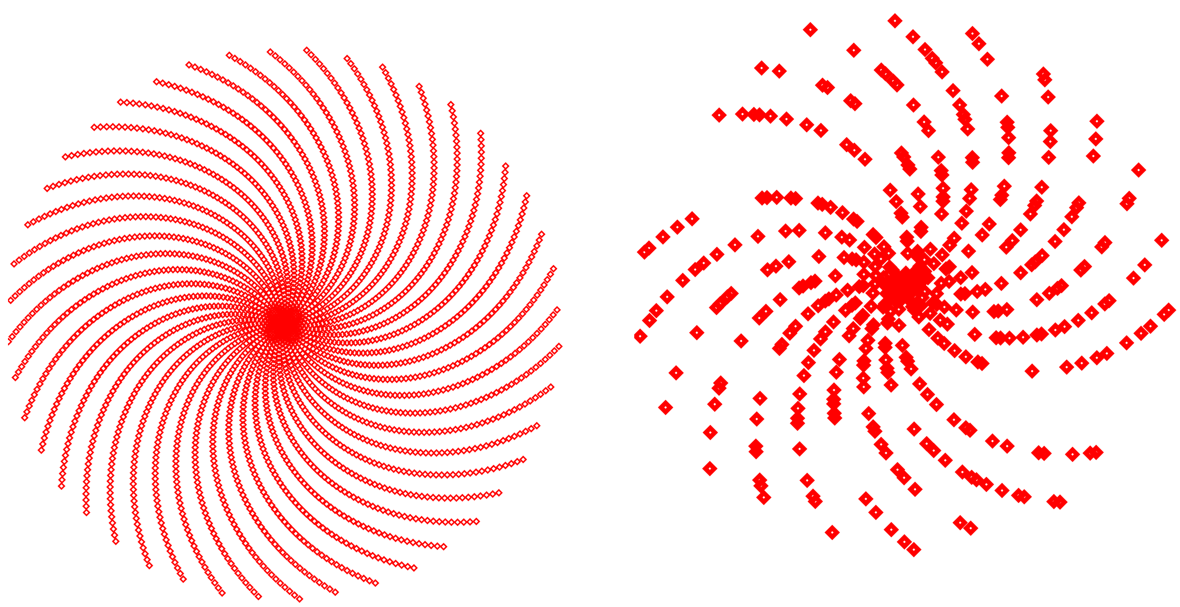
Example. $\mathbb{Z}_8^{\times}$ has elements 1, 3, 5, 7; q and $\phi(8) = 4$.
Multiplication table: $1^2 =_8 3^2 =_8 5^2 =_8 7^2 =_8 1$.
Elmts have mult. inverses $\Longrightarrow$ mult. **Abelian group**.
$\mathbb{Z}_q^{\times}$ are precisely the units of the ring $\mathbb{Z}$ mod q.

Question. $\phi(q)$ ways of choosing $a \in \mathbb{Z}_q^{\times}$. What is the density of primes in each of these arithmetic progressions???

PNT for Arithmetic Progressions

PNT Arithm Progr: $\Pi_{q,a}(x)$ is number of primes p with $p \leq x$ and $p =_q a$ with a rel prime to q. Then :

1. $\displaystyle \lim_{x\to\infty} \frac{\Pi_{q,a}(x)}{(x/\ln x)} = \frac{1}{\phi(q)}$ and 2. $\displaystyle \lim_{x\to\infty} \frac{\Pi_{q,a}(x)}{\int_2^x \ln t\, dt} = \frac{1}{\phi(q)}$



Recall (left): our picture of $z(n) = ne^{in}$, $n \leq 3000$.
Now (right): new picture of $z(n) = p_n e^{ip_n}$, $n \leq 430$.

These are all primes ≤ 3000. Note: $3000/\ln(3000) \approx 375$.

Observe that: 20 of 44 branches are populated.
These correspond exactly to rel primes mod 44: $\phi(44) = 20$.
Primes are roughly equally distributed over these.

Dirichlet's Characters

Definition. A **character** of a finite Abelian group $\mathbb{Z}_q^\times$ is a homomorphism $\chi : \mathbb{Z}_q^\times \to \mathbb{C}^\times$, i.e. $\chi(a)\chi(b) = \chi(ab)$.

In **examples below,** $\chi(a)$ are 4th rts of 1. $\phi(5) = \phi(8) = 4$.

$\mathbb{Z}_5^\times$	χ_0	χ_1	χ_2	χ_3
1	1	1	1	1
2	1	i	-1	-i
3	1	-i	-1	i
4	1	-1	1	-1

$\mathbb{Z}_8^\times$	$\chi_{(0,0)}$	$\chi_{(0,1)}$	$\chi_{(1,0)}$	$\chi_{(1,1)}$
1	1	1	1	1
3	1	1	-1	-1
5	1	-1	1	-1
7	1	-1	-1	1

Euler's Thm. Each element a of $\mathbb{Z}_q^\times$ satisfies $a^{\phi(q)} =_q 1$.
Thus $\chi(a)^{\phi(q)} = 1$.
Thus $\chi(a) = e^{2\pi i k/\phi(q)}$.

Characters form basis of a Discr Fourier Transf[6]
with added cond'n: respects group operation (multiplication).

Theorem. i) The rescaled characters of $\mathbb{Z}_q^\times$ form an orthonormal basis of the vector space $\mathbb{C}^{\phi(q)}$.
ii) The row and column sums of the χ table are 0, except:
. a) column corresponding to 'identity' char. χ_1 and
. b) row corresponding to '1'.
. Summing (a) or (b) yields $\phi(q)$.

Defn. The set of characters χ of $\mathbb{Z}_q^\times$ are denoted by X_q.

[6]Simplified a bit here.

Massage Problem to Look like PNT

Replace zeta function by $Z_{q,a}$:

$$Z_{q,a}(z) := \exp\left(\sum_{\chi\in X_q} \overline{\chi(a)}\,\ln\left(\sum_{n=1}^{\infty}\frac{\chi(n)}{n^z}\right)\right).$$

We do this because ...

1. χ_1 is the identity character and behaves like $\zeta(z)$. The other χ's have average zero and are analytic in $\operatorname{Re} z > 0$.

$$\ln Z_{q,a}(z) = \left(\sum_{n=1}^{\infty}\frac{\chi_1(n)}{n^z}\right)\cdot\exp\left(\sum_{\chi\in X_q,\chi\neq\chi_1} \overline{\chi(a)}\,\ln\left(\sum_{n=1}^{\infty}\frac{\chi(n)}{n^z}\right)\right)$$

2. Using properties of characters:

$$\ln Z_{q,a}(1+z) = \phi(q)\sum_{p\nmid q}\sum_{\substack{n=1\\ p^n=_q a}}^{\infty}\frac{1}{np^{n(1+z)}}.$$

$$\text{Compare: } \ln\zeta(1+z) = \sum_{p}\sum_{n=1}^{\infty}\frac{1}{np^{n(1+z)}}.$$

The pairs (n,p) such that $p =_q a$ dominate, because all other contributions $n \geq 2$ are bounded for $\operatorname{Re} z > 0$. So $\ln Z$ almost counts primes with residue a mod q.

The Massage Works, ...

because all these computations mean that ...

1. Z behaves like ζ (same singularities).
2. Z 'almost' counts primes with residue a mod q.

Handy Definition. For any $\chi \in X_q$, analytic cont'n of

$$L(\chi, z) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^z}\,.$$

is called a **Dirichlet *L*-function** in the literature.

After massage, Follow Steps of PNT

1. $$\frac{\Theta_{q,a}(x)}{x} := \frac{\phi(q)}{x} \prod_{\substack{p \le x \\ p =_q a}} \ln p$$ is bounded by Cheb.

Definition. For $z \in \mathbb{C}$ and if well-defined, let

$$G(z) := \int_1^\infty \left(\frac{\Theta_{q,a}(x)}{x} - 1 \right) x^{-z-1}\, dx\,.$$

2. If $g(0)$ exists (is finite), then $\lim_{x\to\infty} \frac{\Theta_{q,a}(x)}{x} = 1$. Easy.

3. $\lim_{x\to\infty} \frac{\Theta_{q,a}(x)}{x} = 1$ iff $\frac{\Pi_{q,a}(x)\ln x}{x} = 1$. Same proof as PNT.

4. $\boldsymbol{G(z)}$ can be written as expression with $\boldsymbol{Z_{q,a}(1+z)}$ **in the denominator**.

Zeroes of $Z_{q,a}(1+z)$ lie on $\mathrm{Im}(z) < 0$. So:

5. Same **"Tauberian" Thm.** applies. $G(z) := \int_1^\infty F(x) x^{-z}\, dx$. If G has anal. cont'n to $\mathrm{Re}(z) \ge 0$, then $G(0) = \int_1^\infty F(x)\, dx$ exists.

6. Bingo again! So $\int_1^\infty f(x)\, dx$ exists.

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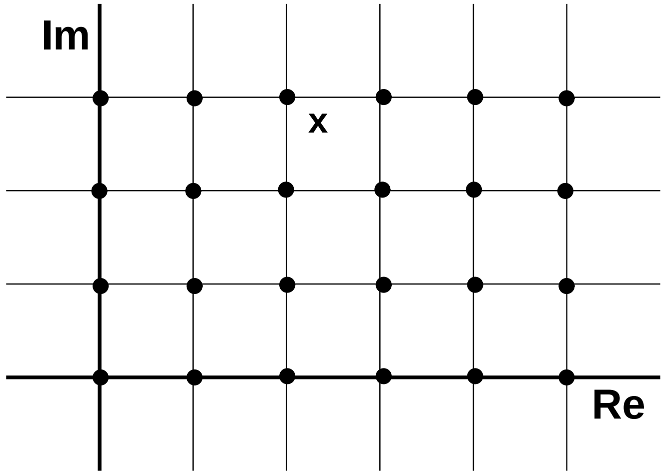
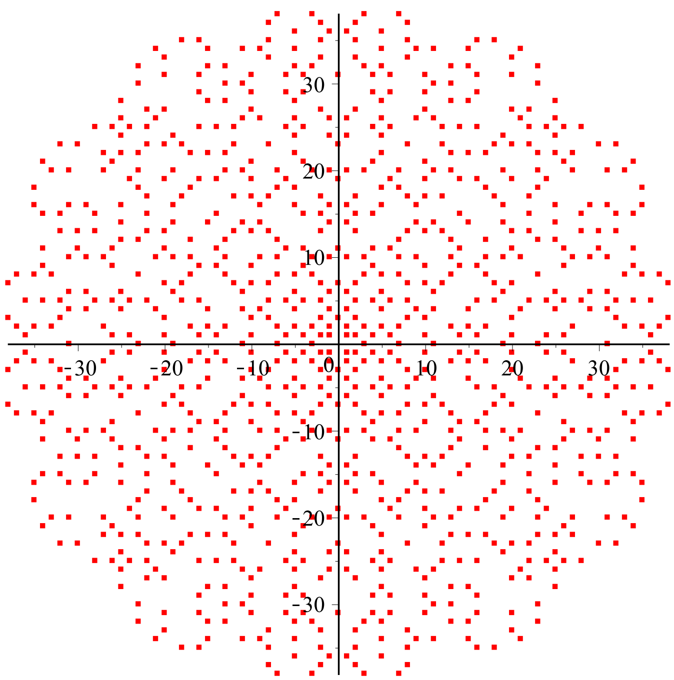
REMARKS IF TIME PERMITS

Gaussian Primes



The Gaussian integers form a lattice in the complex plane.



There are approx. 950 Gaussian primes within a radius 40.

A Gaussian integer π is prime if and only if:

1) π lies on a coord. axis and $|\pi|$ prime in $\mathbb{Z}$ with $|\pi| =_4 3$, or
2) π not on coord. axis and $|\pi|^2$ is prime with $|\pi|^2 =_4 1$.

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