Looking for the Space of Bounded Deformations

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About the Speaker

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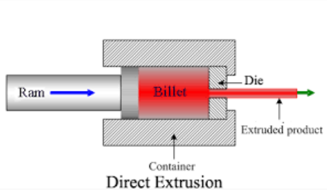
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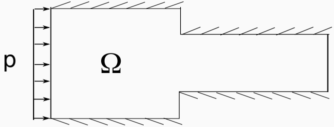
Billet
Die
Extruded product
Container
Direct Extrusion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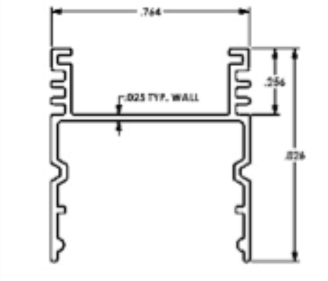
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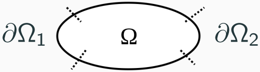
How Widely Used is Extrusion in Manufacturing?

- Tesla uses a skateboard design for the extruded aluminum frame of its battery pack to improve the rigidity of its vehicles for example,
- Tesla uses aluminum extruded enclosure design for the battery enclosure for example,
- The average North American passenger vehicle contained about 28 lbs of aluminum extrusion and by 2025 the estimate is that it will contain nearly 50 lbs/vehicle,
- In the UK alone, the forecast by 2035 is for 2.2 million vehicles to have battery enclosures – 50% which are EV's and 50% which are PHEV's,
- The Norwegian Company, Sapa, opened a $3 Million lab in Troy MI to develop extrusion processes, products, et al. in 2016.

Variational Problem for Limit Analysis (Mura and Lee) - 1962



- Traction BC's, T_0, on $\partial\Omega_1$,
- Displacement BC's, u_0, on $\partial\Omega_2$.

Suppose $u(x) \in \mathbb{R}^n$, $n = 2,\ 3$. Find $(u, \sigma, m) \in \mathcal{U} \times \mathcal{P} \times \mathbb{R}$ that is an extremal point of the following functional:

$$L(u, \sigma, m) = \int_\Omega \left(k\sqrt{2\, e(u)^T e(u)} + \sigma\, \nabla \cdot u\right) - m \int_{\partial\Omega_1} \left(T_0 \cdot u - 1\right)$$

- u denotes the displacement, $e(u) = \nabla \odot u$ denotes the strain tensor, and σ denotes the hydrostatic pressure (Gurtin),
- k is the yield stress in simple shear of a von Mises material and m denotes the limit load (Lubliner).

Minimization of Functional for Stokes Problem

Let $n = 2, 3$, $H = \{u \in W^{1,2}(\Omega)^n : \nabla \cdot u = 0$ and $u = 0$ on $\partial\Omega\}$, and $f \in L^2(\Omega)^n$.

Stokes Problem:

Find $u \in H$ that minimizes $J(u) = \int_\Omega (\nabla \otimes u)^T (\nabla \otimes u) - f \cdot u$.

Theorem 1 (Vainberg - 1973).

Let H be a Hilbert Space and $J : H \to \mathbb{R}$ such that J is:

- *differentiable, i.e.* $\lim_{\alpha \to 0} \left[\frac{J(u+\alpha v) - J(v)}{\alpha} \right]$ *exists for* $u, v \in H$
- *strictly convex, i.e. for* $0 < \theta < 1$
  $J(\theta u + (1-\theta)v) < \theta J(u) + (1-\theta)J(v)\ \forall u, v \neq 0,\ u \neq v \in H$
- *and coercive, i.e.* $\lim_{\| u \|_H \to \infty} \left[\frac{J(u)}{\| u \|_H} \right] = \infty$,

then, there exists a unique $u \in H$ such that $J(u) \leq J(v)\ \forall\ v \in H$.

Regularity of Stokes Problem

- $u \in H$; thus, by the Trace Theorem (Lions -1958), $u|_{\partial\Omega} \in L^q(\partial\Omega)^n$ for $2 \leq q \leq 4$, when $n = 3$, and $2 \leq q < \infty$, when $n = 2$,
- For the Stokes Problem: $\int_\Omega (\nabla \otimes u)^T (\nabla \otimes u) = \| \nabla \otimes u \|_0^2$,
- But, by Poincaré (1894): $\| \nabla \otimes u \|_0^2 \geq c(\Omega) \| u \|_0^2$,
  thus:

$$\frac{J(u)}{\| u \|_H} \geq c \| u \|_H - \| f \|_0 \quad \text{which implies coercivity in the limit.}$$

Questions:

1. Korn's Inequality[1] (1909): Does the norm, $\| \nabla \odot u \|_0^2$, dominate as in Poincaré's case?
2. What about the case $u \in W^{1,1}(\Omega)^n$?

[1] Poincare meets Korn via Maxwell: Extending Korns First Inequality to Incompatible Tensor Fields by Patrizio Neff, Dirk Pauly, and Karl-Josef Witsch, Journal of Diff. Eqs., 2015

Non-existence[2] of u in $W^{1,1}(\mathbb{B})^2$ for a First-Order PDE

Lemma 2 (Bourdaud and Wojciechowski, Bourgain and Brezis, ...).

Let $\mathbb{B}$ be the open unit ball in $\mathbb{R}^2$. There does not exists a $u \in W^{1,1}(\mathbb{B})^2$ such that:

$$\nabla \cdot u = f \quad \forall f \in L^1(\mathbb{B}).$$

Proof.

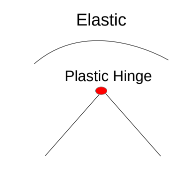
Let $u_0,\ u_1,\ u_2 \in W^{1,1}(\mathbb{B})$ with $f = u_0 + \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}$. By the Sobolev Embedding Theorem, $W^{1,1}(\mathbb{B}) \hookrightarrow L^2(\mathbb{B})$ so that:

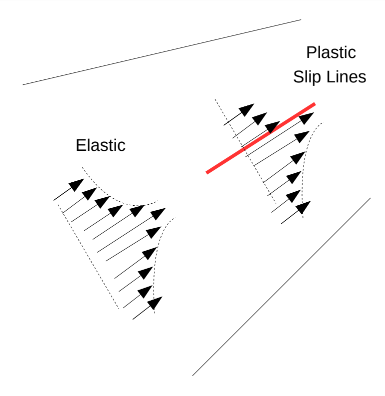
$$L^1(\mathbb{B}) \subset \{v_0, v_1, v_2 \in L^2(\mathbb{B}) : v_0 + \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}\} = (W^{1,2}(\mathbb{B}))^*.$$

But, $W^{1,2}(\mathbb{B}) \not\subseteq (L^1(\mathbb{B}))^* = L^\infty(\mathbb{B})$. □

[2] Kristensen, From Ornstein's non-inequalities to rank-one convexity

$W^{1,1}(\Omega)^n$ is not large enough – How about $BV(\Omega)^n$ or ...





- Definition of Space of Solutions for $\mathcal{U}$ and $\mathcal{P}$?
- Where are the Boundary Values of u – Trace Theorem?
- Existence/Uniqueness of Minimizers - Will regularization/relaxation be needed?
- Fine Properties – Behaviour and regularity of u?

Candidate Spaces

- $W^{1,1}(\Omega)^n = \{u \in L^1(\Omega)^n : D_j u_i \in L^1(\Omega),\ 1 \leq i, j \leq n\}$,
- $\mathcal{M}(\Omega) = \{\text{Bounded Radon measures on } \Omega\}$,
- $BV(\Omega)^n = \{u \in L^1(\Omega)^n : D_j u_i \in \mathcal{M}(\Omega),\ 1 \leq i, j \leq n\}$,
- $BD(\Omega) = \{u \in L^1(\Omega)^n : E(u)_{ij} \in \mathcal{M}(\Omega),\ 1 \leq i, j \leq n\}$,
- $BD(\Omega) \backslash BV(\Omega)^n$ – Construction[3] of an element that is in $BD(\Omega)$ but not in $BV(\Omega)^n$.

[3] Fine Properties of Functions of Bounded Deformation by Ambrosio, Garcia, dal Maso in the Archive for Rational Mechanics, 1997

Trace Result:

There exists a trace operator, $\gamma_0 : BD(\Omega) \to L^1(\Gamma)$ such that,

$$\gamma_0(u) = u\,|_{\Gamma} \quad \forall u \in BD(\Omega) \cap C^0(\bar{\Omega})^n$$

This operator is onto $L^1(\Gamma)$

Closure – Point of Departure

- 1978 - Suquet, Un espace fonctionnel pour les équations de la plasticité
- 1980 - Temam and Strang, Functions of Bounded Deformation
- 1989 - Glowinski and Le Tallec, Augmented Lagrangian and Operator-Splitting Methods in non-linear Mechanics
- 1997 - Ambrosio, Coscia, dal Maso, Fine Properties of Functions of Bounded Deformation
- 2000 - Fuchs and Seregin, Variational Methods for Problems from Plasticity Theory and Generalized Newtonian Fluids
- 2000 - Barroso, Fonseca, and Toader, A relaxation theorem in the space of functions of bounded deformation
- 2008 - Vicente da Silva and Antão, Upper Bound Limit Analysis with a Parallel Mixed Finite Element Formulation
- 2012 - Kristensen and Rindler, Piecewise affine approximations for functions of bounded variation
- 2015 - Babadjian, Traces of Functions of Bounded Deformation
- 2017 - Conti, Focardi, and Iurlano, Which special functions of bounded deformation have bounded variation?

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