Computing with Harmonic Functions

Sheldon Axler

16 October 2020

Fix a positive integer n.
Suppose Ω is an open subset of $\mathbf{R}^n$.

Definition: *Laplacian*

For $u\colon \Omega \to \mathbf{R}$ in C^2, the *Laplacian* of u is denoted Δu and is the function $\Delta u\colon \Omega \to \mathbf{R}$ defined by

$$\Delta u = {D_1}^2 u + \cdots + {D_n}^2 u.$$

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Definition: *harmonic*

u is called *harmonic* on Ω if

$$(\Delta u)(x) = 0$$

for all $x \in \Omega$.

Example: If $\Omega \subset \mathbf{R}$ is an open interval and $u\colon \Omega \to \mathbf{R}$, then u is harmonic on Ω if and only if there exist $m, b \in \mathbf{R}$ such that

$$u(x) = mx + b$$

for all $x \in \Omega$.

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Conversely, if Ω is an open simply connected subset of $\mathbf{R}^2$ and u is harmonic on Ω, then $u = \mathrm{Re} f$ for some analytic function f on Ω.

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Example: The function $u\colon \mathbf{R}^3 \to \mathbf{R}$ defined by

$$u(x_1, x_2, x_3) = 6{x_1}^2 x_2 x_3 - {x_2}^3 x_3 - x_2 {x_3}^3$$

is harmonic on $\mathbf{R}^3$.

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Example: The function $u\colon \mathbf{R}^3 \to \mathbf{R}$ defined by $u(x_1, x_2, x_3) =$

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$$\|x\| = \sqrt{{x_1}^2 + \cdots + {x_n}^2}.$$

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Example: If $\zeta \in \mathbf{R}^n$ and $\|\zeta\| = 1$, then

$$x \mapsto \frac{1 - \|x\|^2}{\|x - \zeta\|^n}$$

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mean value property

Let B denote the open unit ball in $\mathbf{R}^n$:

$$B = \{x \in \mathbf{R}^n : \|x\| < 1\}.$$

Thus ∂B is the unit sphere in $\mathbf{R}^n$:

$$\partial B = \{x \in \mathbf{R}^n : \|x\| = 1\}.$$

Let σ denote surface area measure on ∂B, normalized so that $\sigma(\partial B) = 1$.

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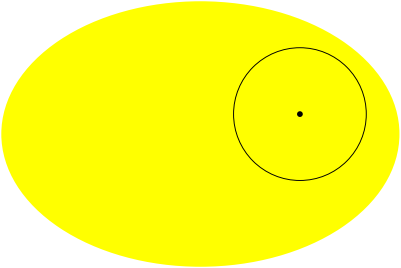

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Suppose $\Omega \subset \mathbf{R}^n$ and $u \colon \Omega \to \mathbf{R}$ is continuous. We say that u has the *mean value property* if

$$u(x) = \int_{\partial B} u(x + r\zeta)\, d\sigma(\zeta)$$

for all $x \in \Omega$ and all $r \in (0, \infty)$ such that $x + r\overline{B} \subset \Omega$.

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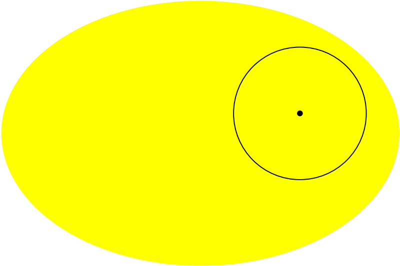

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A continuous function on an open subset of $\mathbf{R}^n$ is harmonic if and only if it has the mean value property.

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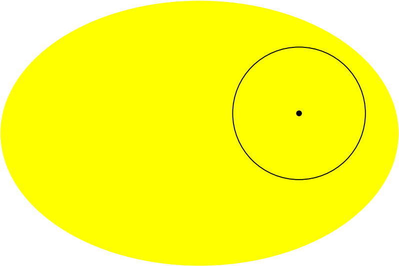

max and min attained on $\partial\Omega$

Suppose $\Omega \subset \mathbf{R}^n$ is open and bounded. If $u\colon \overline{\Omega} \to \mathbf{R}$ is continuous on $\overline{\Omega}$ and harmonic on Ω, then u attains its maximum and minimum values over $\overline{\Omega}$ on $\partial\Omega$.

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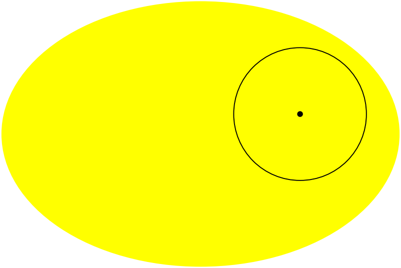

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u *determined by* $u|_{\partial\Omega}$

Suppose $\Omega \subset \mathbf{R}^n$ is open and bounded. Suppose $u\colon \overline{\Omega} \to \mathbf{R}$ is continuous on $\overline{\Omega}$, harmonic on Ω. If $u|_{\partial\Omega} \equiv 0$, then $u|_{\Omega} \equiv 0$.

Dirichlet problem

Dirichlet problem: Suppose Ω is an open subset of $\mathbf{R}^n$. Given $f \in C(\partial\Omega)$, find $u \in C(\overline{\Omega})$ such that u is harmonic on Ω and $u|_{\partial\Omega} = f$.

Johann Dirichlet (*1805–1859*)

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If f gives the temperature at each point on $\partial\Omega$, then u gives the temperature at each point in Ω at equilibruium.

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solution to Dirichlet problem on B

Suppose $f \in C(\partial B)$. Define $u\colon \overline{B} \to \mathbf{R}$ by

$$u(x) = \begin{cases} \displaystyle\int_{\partial B} \frac{1 - \|x\|^2}{\|x - \zeta\|^n} f(\zeta)\, d\sigma(\zeta) & \text{if } x \in B, \\ f(x) & \text{if } x \in \partial B. \end{cases}$$

Then $u \in C(\overline{B})$ and u is harmonic on B.

Thus the function u above is the solution to the Dirichlet problem for f.

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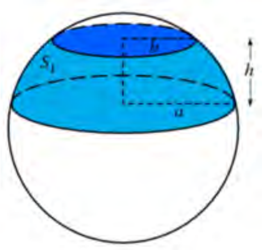
attempt to evaluate via iterated slice integration

iterated slice integration

If g is a function on $\partial B = \partial B_n$, then

$$\int_{\partial B_n} g \, d\sigma_n = \frac{\text{s.a. of } \partial B_{n-1}}{\text{s.a. of } \partial B_n} \int_{-1}^{1} (1 - t^2)^{\frac{n-?}{2}} \int_{\partial B_{n-1}} g\left(\sqrt{1 - t^2}\zeta, t\right) d\sigma_{n-1}(\zeta) \, dt.$$



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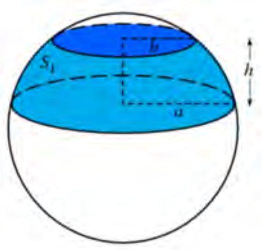
This formula gives a result due to Archimedes:

The surface area of a slice of the sphere between two parallel planes depends only upon the distance h between the planes and not on the position of the planes.



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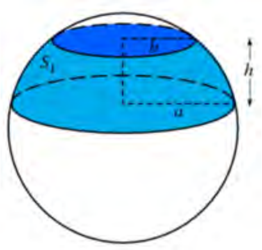

The result of Archimedes holds only for $n = 3$.

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For $f \in C(\partial B)$, the function u on B defined by

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Example: Suppose $n = 3$ and

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Then

$$u(x) = \tfrac{1}{7}\left(6{x_1}^2 x_2 x_3 + x_2 x_3 - {x_2}^3 x_3 - x_2 {x_3}^3\right).$$

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Poisson integral of a polynomial

Suppose f is a polynomial of n variables. Then the Poisson integral u of f is also a polynomial of n variables and

$$\deg u \le \deg f.$$

Poisson integral of a polynomial is a polynomial

Definition: $\mathcal{P}_m$

For m a positive integer, let

$$\mathcal{P}_m = \{\text{polynomials on } \mathbf{R}^n \text{ with degree} \leq m\}.$$

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Proof Let u denote the Poisson integral of f. We guess that

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for some $g \in \mathcal{P}_{m-2}$.

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is harmonic and vanishes on ∂B.

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Thus $g = 0$. Hence L is injective.

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Thus $g = 0$. Hence L is injective. Thus L is surjective.

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Thus L is surjective.
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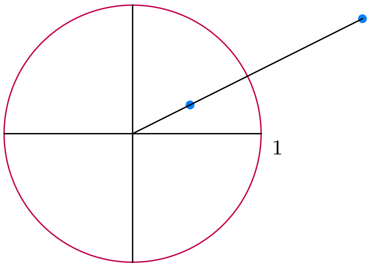
∎

reflecting in unit sphere

The function $x \mapsto \dfrac{x}{\|x\|^2}$ maps the unit ball in $\mathbf{R}^n$ to the exterior of the unit ball.



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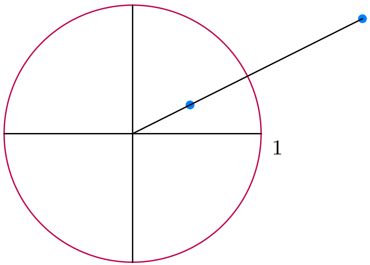

Suppose $n = 2$ and u is harmonic on $\mathbf{R}^2 = \mathbf{C}$. Then

$$u\Big(\frac{z}{\|z\|^2}\Big) = u\Big(\frac{z}{z\bar{z}}\Big) = u\Big(\frac{1}{\bar{z}}\Big),$$

which is a harmonic function of z.

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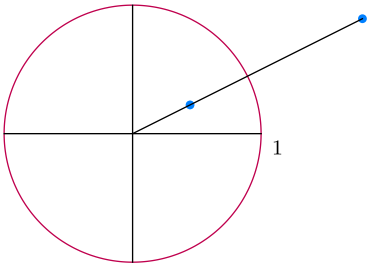

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which is a harmonic function of z.

If $n > 2$ and u is harmonic on $\mathbf{R}^n$, then $x \mapsto u\left(\dfrac{x}{\|x\|^2}\right)$ is rarely harmonic.

Definition: *reflection in the sphere*

For $x \in \mathbf{R}^n \cup \{\infty\}$, define

$$x^* = \frac{x}{\|x\|^2}.$$

Note that $(x^*)^* = x$ for all $x \in \mathbf{R}^n \cup \{\infty\}$.

Kelvin transform

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Suppose u is a function defined on $\Omega \subset \mathbf{R}^n$. The *Kelvin transform* $K[u]$ is the function on Ω^* defined by

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$$K[1] = \|x\|^{2-n}$$

$$K[\|x\|^{2-n}] = 1$$

$$K\big[K[u]\big] = u$$

Kelvin transform and harmonic functions

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- A function is harmonic if and only if its Kelvin transform is harmonic.
- More precisely, suppose $u\colon \Omega \to \mathbf{R}$ is defined on an open set $\Omega \subset \mathbf{R}^n$. Then u is harmonic on Ω if and only if $K[u]$ is harmonic on Ω^*.

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Laplacian of Kelvin transform

If u is a C^2 function on an open subset of $\mathbf{R}^n \setminus \{0\}$, then

$$\Delta\big(K[u]\big) = K\big[\|x\|^4 \Delta u\big].$$

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instructive example

Suppose $f(x_1, x_2, x_3) = {x_1}^3 {x_2}^2 x_3$. Then

$$D_f \frac{1}{\|x\|} = \frac{\partial^3}{\partial {x_1}^3} \frac{\partial^2}{\partial {x_2}^2} \frac{\partial}{\partial x_3} \left(\frac{1}{\|x\|} \right)$$

$$= \frac{315 x_1 (\|x\|^4 x_3 - 3\|x\|^2 {x_1}^2 x_3 - 9\|x\|^2 {x_2}^2 x_3 + 33 {x_1}^2 {x_2}^2 x_3)}{\|x\|^{13}},$$

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which is a harmonic function.

Take Kelvin transform of both sides of this equation, getting

$$\mathcal{K}\left[D_f \frac{1}{\|x\|} \right] = 315(\|x\|^4 x_1 x_3 - 3\|x\|^2 {x_1}^3 x_3 - 9\|x\|^2 x_1 {x_2}^2 x_3 + 33 {x_1}^3 {x_2}^2 x_3).$$

The function on the right is a harmonic polynomial.

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$$f = \tfrac{1}{315\cdot 33}\mathcal{K}\Big[D_f\frac{1}{\|x\|}\Big] + \|x\|^2(\text{polynomial of degree 4}).$$

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This always happens, with 4 replaced by $m - 2$ and $315 \cdot 33$ replaced by an appropriate constant.

Let $c_m = \prod_{k=0}^{m-1} (2 - n - 2k)$.

using Kelvin transform to compute Poisson integral

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algorithm for solving Dirichlet problem on B for polynomials

Suppose f is a polynomial on $\mathbf{R}^n$ homogeneous of degree m. Then

- $K[D_f\|x\|^{2-n}]$ is a harmonic polynomial on $\mathbf{R}^n$ homogeneous of degree m.
- Also,

$$f = \frac{1}{c_m}K[D_f\|x\|^{2-n}] + \|x\|^2 g$$

for some polynomial g on $\mathbf{R}^n$ homogeneous of degree $m-2$.

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The result above leads to a very fast algorithm for computing the Poisson integral of a polynomial.

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Dirichlet problem for ellipsoids

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Suppose Ω is an ellipsoid in $\mathbf{R}^n$ and $f \in \mathcal{P}_m$. Then there exists a harmonic polynomial $u \in \mathcal{P}_m$ such that $u|_{\partial\Omega} = f$.

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Example: The harmonic polynomial that equals ${x_1}^{10}$ on

$$\{x \in \mathbf{R}^3 : 2{x_1}^2 + 3{x_2}^2 + 4{x_3}^2 = 1\}$$

has value

$$\frac{50094521382345255444054646238540058 4789}{3972633695067359598012898420409222152 51461}$$

at the origin.

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More examples:

Neumann problem

Suppose $q\colon \mathbf{R}^n \to \mathbf{R}$ and

$$\Omega = \{x \in \mathbf{R}^n : q(x) < 1\}$$

and

$$\partial\Omega = \{x \in \mathbf{R}^n : q(x) = 1\}.$$

For $x \in \partial\Omega$, the vector $(\nabla q)(x)$ is normal (perpendicular) to $\partial\Omega$.

For $v\colon \overline{\Omega} \to \mathbf{R}$ and $x \in \partial\Omega$, the *directional derivative* of v in the normal direction is $(\nabla v)(x) \cdot (\nabla q)(x)$.

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Example: Suppose $q(x) = \|x\|^2$. Then $\Omega = B$ and

$$(\nabla q)(x) = 2x$$

for each $x \in \partial B$.

solution of Neumann problem on ball

Neumann problem: Given $f \in C(\partial B)$, find $v \in C(\overline{B})$ such that v is harmonic on Ω and

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Solution: Suppose f is a polynomial on $\mathbf{R}^n$ of degree m. Let u be the solution of the Dirichlet problem corresponding to f. Then

$$u = \sum_{k=0}^{m} u_k,$$

where each u_k is harmonic and homogeneous of degree k.

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for each k and each $x \in \partial B$. Thus setting

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Note that $u_0 = \int_{\partial B} f\, d\sigma$. There is no solution unless

$$\int_{\partial B} f\, d\sigma = 0.$$

Neumann problem on ellipsoids

Sheldon Axler and Peter Shin, The Neumann Problem on Ellipsoids, *Journal of Applied Mathematics and Computing* 57 (2018), 261–278.

Peter Shin's institutional affiliation is Department of Radiology and Biomedical Imaging, UC San Francisco.

Neumann problem on ellipsoids

Suppose $\beta_1, \ldots, \beta_n$ are positive numbers.
Let

$$q(x) = q(x_1, \ldots, x_n) = \beta_1 {x_1}^2 + \cdots + \beta_n {x_n}^2,$$

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Neumann problem

Suppose $f \in \mathcal{P}_m$. Then the following are equivalent:

(a) $\displaystyle\int_{\partial\Omega} \frac{f}{\|\nabla q\|}\, dA = 0.$

(b) There exists a harmonic polynomial $v \in \mathcal{P}_m$ such that

$$\nabla v \cdot \nabla q = f \text{ on } \partial\Omega.$$

(c) There exists a harmonic function v on $\overline{\Omega}$ such that

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Let V be volume measure on $\mathbf{R}^n$.

Let $D_{\mathbf{n}} v = \nabla v \cdot \dfrac{\nabla q}{\|\nabla q\|}$

Green's Second Identity

If v and w are smooth functions on $\overline{\Omega}$, then

$$\int_{\Omega} (w\Delta v - v\Delta w)\, dV = \int_{\partial\Omega} (w\, D_{\mathbf{n}} v - v\, D_{\mathbf{n}} w)\, dA.$$

Suppose v is harmonic on $\overline{\Omega}$. Take $w \equiv 1$. Then

$$0 = \int_{\partial\Omega} D_{\mathbf{n}} v\, dA$$

Thus (c) implies (a).

integrals on ellipsoids

Recall that A is surface area measure on $\partial\Omega$.

formula for surface area integral on ellipsoid

Suppose $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an n-tuple of nonnegative even integers. Then

$$\int_{\partial\Omega} \frac{x^\alpha}{\|\nabla q(x)\|}\, dA(x) = \frac{\pi^{n/2}}{2\Gamma(\frac{n}{2}+1)\sqrt{\prod_{j=1}^n \beta_j^{\alpha_j+1}}} \cdot \frac{(\alpha_1 - 1)!! \cdots (\alpha_n - 1)!!}{(n+2)(n+4)\cdots(n+|\alpha|-2)}.$$

If $\Omega = B$, this was proved by Hermann Weyl in 1939.

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Examples:

references

- Sheldon Axler and Peter J. Shin. The Neumann problem on ellipsoids, *Journal of Applied Mathematics and Computing* 57 (2018), 261–278.
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