Logarithmic Conjugation Theorem

Sheldon Axler

2 April 2021

Suppose Ω is an open subset of $\mathbf{R}^2 = \mathbf{C}$.

Definition: *Laplacian*

For $u \colon \Omega \to \mathbf{C}$, the *Laplacian* of u is denoted Δu and is the function $\Delta u \colon \Omega \to \mathbf{C}$ defined by

$$(\Delta u)(z) = \frac{\partial^2 u}{\partial x^2}(z) + \frac{\partial^2 u}{\partial y^2}(z)$$

for $z = (x, y) = x + iy \in \Omega$.

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Definition: *harmonic*

u is called *harmonic* on Ω if

$$(\Delta u)(z) = 0$$

for all $z \in \Omega$.

examples of harmonic functions

Example: If $b \in \mathbf{R}$ and

$$u(x,y) = x^7 + bx^5y^2 + 35x^3y^4 - 7xy^6,$$

then

$$(\Delta u)(x,y) = 2(x^5 + 10x^3y^4)(21 + b).$$

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Example: If $\zeta \in \mathbf{C}$ and

$$u(z) = \frac{1 - |z|^2}{|\zeta - z|^2},$$

then

$$(\Delta u)(z) = \frac{4(1 - |\zeta|^2)}{|\zeta - z|^4}.$$

Thus u is harmonic on $\mathbf{C} \setminus \{\zeta\}$ if and only if $|\zeta| = 1$.

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Example: If $t \in \mathbf{R}$ and

$$u(x, y) = \frac{y}{(x - t)^2 + y^2},$$

then

$$(\Delta u)(x, y) = 0.$$

Thus u is harmonic on $\mathbf{R}^2 \setminus \{(t, 0)\}$.

d and d-bar

Definition: ∂ *and* $\overline{\partial}$

For $f : \Omega \to \mathbf{C}$ define

$$(\partial f)(z) = \frac{1}{2}\left(\frac{\partial f}{\partial x}(z) - i\frac{\partial f}{\partial y}(z)\right)$$

$$(\overline{\partial} f)(z) = \frac{1}{2}\left(\frac{\partial f}{\partial x}(z) + i\frac{\partial f}{\partial y}(z)\right)$$

for $z = (x, y) = x + iy \in \Omega$.

If f is analytic on Ω, then

$$\begin{aligned}(\partial f)(z) &= \frac{1}{2}\Big(f'(z) + f'(z)\Big)\\ &= f'(z).\end{aligned}$$

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$$(\partial f)(z)=\frac{1}{2}\Big(f'(z)+f'(z)\Big)$$
$$=f'(z).$$

Cauchy–Riemann equations

f is analytic on Ω if and only if $\overline{\partial} f=0$.

Proof: Suppose $f=u+iv$. Then

$$\overline{\partial} f=\frac{1}{2}\Big(\Big[\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\Big]+i\Big[\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}\Big]\Big)$$

$$=\frac{1}{2}\Big(\Big[\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\Big]+i\Big[\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\Big]\Big).$$

Thus the condition $\overline{\partial} f=0$ is precisely the Cauchy–Riemann equations, which is equivalent to analyticity. ∎

d d-bar = d-bar d = $\frac{1}{4}$ Laplacian

$$\partial\overline{\partial} = \overline{\partial}\partial = \frac{1}{4}\Delta$$

Proof:

$$\begin{aligned}\partial(\overline{\partial}f) &= \frac{1}{4}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big)\Big(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\Big)\\ &= \frac{1}{4}\Big(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\Big)\\ &= \frac{1}{4}\Delta f.\end{aligned}$$

Similarly, $\overline{\partial}(\partial f) = \frac{1}{4}\Delta f$. ■

analytic $\Longrightarrow$ harmonic

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Similarly, $\overline{\partial}(\partial f) = \frac{1}{4}\Delta f$. ∎

analytic functions are harmonic

If f is analytic on Ω, then $\mathrm{Re}\, f$, $\mathrm{Im}\, f$, and f are harmonic on Ω.

Proof. Suppose f is analytic on Ω. Thus $\overline{\partial}f = 0$. Hence

$$\Delta f = 4\partial(\overline{\partial}f) = 0.$$

Thus f is harmonic, which implies that $\mathrm{Re}\, f$ and $\mathrm{Im}\, f$ are harmonic. ∎

u harmonic $\iff$ ∂u analytic

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u is harmonic on Ω if and only if
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Proof: Suppose $u\colon \Omega \to \mathbf{C}$. Then

$$\begin{aligned} u \text{ is harmonic} &\iff \Delta u = 0 \\ &\iff \overline{\partial}(\partial u) = 0 \\ &\iff \partial u \text{ is analytic. } \blacksquare \end{aligned}$$

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If $f\colon \Omega \to \mathbf{C}$, then

$$\bar\partial\,\bar f = \overline{\partial f}.$$

Thus

(a) f is analytic $\iff$ $\bar\partial f = 0$;

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(a) f is analytic $\iff$ $\overline{\partial} f = 0$;

(b) $\overline{f}$ is analytic $\iff$ $\partial f = 0$.

Definition: *harmonic conjugate*

If u is a real-valued harmonic function on Ω and f is an analytic function on Ω such that $u = \operatorname{Re} f$, then $\operatorname{Im} f$ is called a *harmonic conjugate* of u.

simply connected

Suppose Ω is simply connected and $u\colon \Omega \to \mathbf{R}$ is harmonic. Then u is the real part of some analytic function on Ω.

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Proof: ∂u is analytic on the simply connected set Ω. Thus there exists f analytic on Ω such that

$$\partial u = f'$$

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Proof: ∂u is analytic on the simply connected set Ω. Thus there exists f analytic on Ω such that

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Hence

$$\partial(u - f) = 0.$$

Thus $u - \bar{f}$ is analytic on Ω.

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$u - \bar{f}$ and f are analytic functions on Ω that have the same imaginary part. Thus

$$u - \bar{f} = f + c$$

for some real constant c. Hence

$$\begin{aligned} u &= f + \bar{f} + c \\ &= 2 \operatorname{Re} f + c \\ &= \operatorname{Re}(2f + c), \end{aligned}$$

showing that u is the real part of some analytic function on Ω. ∎

Given u harmonic, let's see how the proof above can be used to find an analytic function g such that $u = \operatorname{Re} g$.

What can go wrong?

Fix $a \in \mathbf{C}$. Then

$$
\begin{aligned}
\partial(\log|z-a|) &= \partial\left(\tfrac{1}{2}\log|z-a|^2\right) \\
&= \partial\left(\tfrac{1}{2}\log\big((z-a)(\bar{z}-\bar{a})\big)\right) \\
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However, there does not exist an analytic function f on $\mathbf{C} \setminus \{a\}$ such that

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Definition: *finitely connected*

A domain $\Omega \subset \mathbf{C}$ is called *finitely connected* if $\mathbf{C}\setminus\Omega$ has only finitely many bounded connected components.

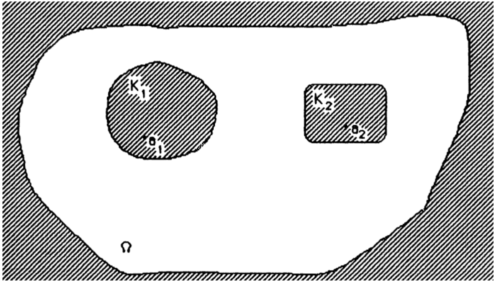
The next result states that on finitely connected domains, a real-valued harmonic function is the real part of an analytic function plus some log terms.

Logarithmic Conjugation Theorem

Suppose Ω is a finitely connected domain.
Let $K_1, \ldots, K_n$ denote the bounded components of $\mathbf{C} \setminus \Omega$. Suppose $a_j \in K_j$ for $j = 1, \ldots, n$.
If u is a real-valued harmonic function on Ω, then there exist an analytic function g on Ω and real numbers $c_1, \ldots, c_n$ such that

$$u(z) = \operatorname{Re} g(z) + c_1 \log|z - a_1| + \cdots + c_n \log|z - a_n|$$

for every $z \in \Omega$.



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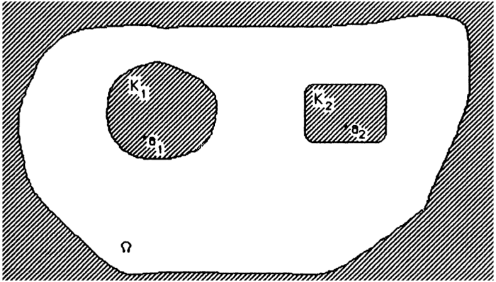

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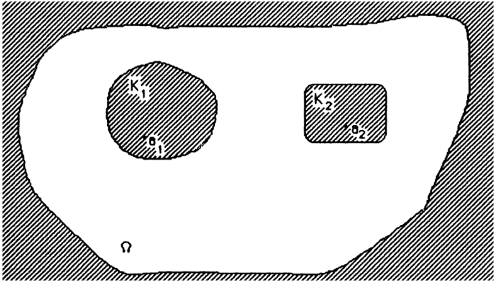

If Ω is an annulus centered at 0, then ∂u has a Laurent series

$$\sum_{n=-\infty}^{\infty} b_n z^n.$$

Only problem with finding an analytic anti-derivative is with the $\frac{1}{z}$ term.

references

- Sheldon Axler. Harmonic functions from a complex analysis viewpoint, *American Mathematical Monthly* 93 (1986), 246–258.
- J. L. Walsh, The approximation of harmonic functions by harmonic polynomials and by harmonic rational functions, *Bulletin of the American Mathematical Society* 35 (1929), 499–544.

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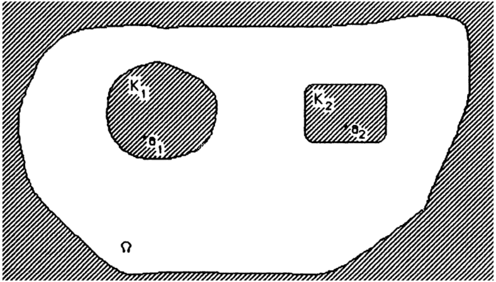

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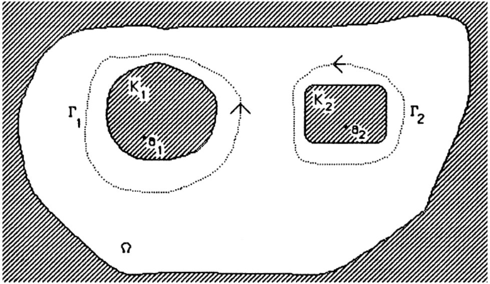
proof of Logarithmic Conjugation Theorem

Proof: For $j = 1, \ldots, n$, let

$$c_j = \frac{1}{\pi i} \int_{\Gamma_j} (\partial u)(w)\, dw.$$



Γ_j has winding number 1 around K_j and winding number 0 around the other components of $\mathbf{C} \setminus \Omega$.

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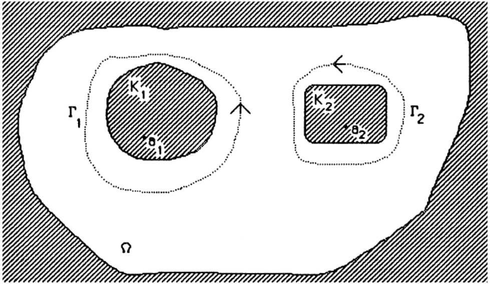
Let $w(t) = x(t) + iy(t)$ parameterize Γ_j. Then

$$c_j = \frac{1}{2\pi i} \int_0^1 \Big(\frac{\partial u}{\partial x}(w(t)) - i \frac{\partial u}{\partial y}(w(t)) \Big) (x'(t) + iy'(t))\, dt.$$



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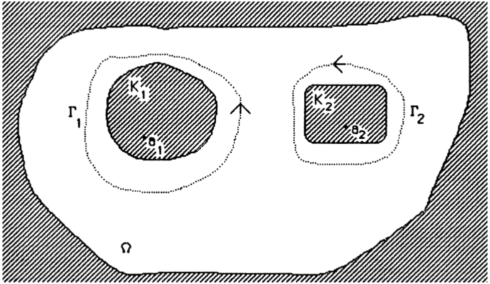

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Thus

$$\operatorname{Im} c_j = -\frac{1}{2\pi} \int_0^1 \left(\frac{\partial u}{\partial x}(x(t), y(t)) x'(t) + \frac{\partial u}{\partial y}(x(t), y(t)) y'(t) \right) dt$$

$$= -\frac{1}{2\pi} u\big(x(t), y(t)\big)\Big]_{t=0}^{t=1}$$

$$= 0.$$

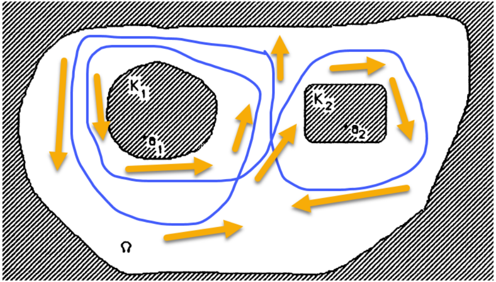
Thus $c_j \in \mathbf{R}$.

Γ_j has winding number 1 around K_j and winding number 0 around the other components of $\mathbf{C} \setminus \Omega$.

proof of Logarithmic Conjugation Theorem

Suppose γ is a closed curve in Ω. Let

$$m_j = \text{ winding \# of } \gamma \text{ around } K_j = \frac{1}{2\pi i}\int_\gamma \frac{1}{w - a_j}\, dw.$$



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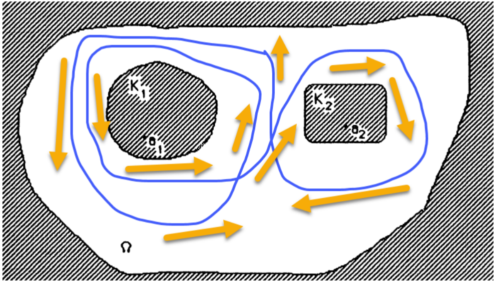
The Cauchy Integral Theorem implies

$$\int_\gamma (\partial u)(w)\,dw = m_1\int_{\Gamma_1}(\partial u)(w)\,dw + \cdots + m_n\int_{\Gamma_n}(\partial u)(w)\,dw$$

$$= \pi i(m_1c_1 + \cdots + m_nc_n)$$

$$= \int_\gamma \left(\frac{c_1}{2(w-a_1)} + \cdots + \frac{c_n}{2(w-a_n)}\right)dw.$$



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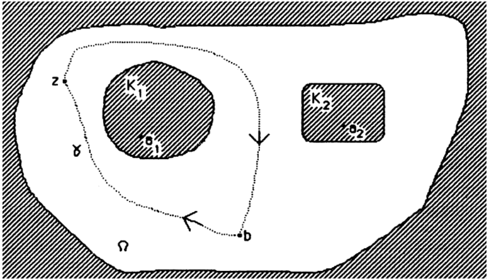


Fix $b \in \Omega$. Define $f : \Omega \to \mathbf{C}$ by

$$f(z) = \int_b^z \Big((\partial u)(w) - \frac{c_1}{2(w-a_1)} - \cdots - \frac{c_n}{2(w-a_n)}\Big)\,dw.$$

This makes sense because $\int_\gamma$ (above) $dw = 0$.

proof of Logarithmic Conjugation Theorem

Suppose γ is a closed curve in Ω. Let

$$m_j = \text{ winding \# of } \gamma \text{ around } K_j = \frac{1}{2\pi i}\int_\gamma \frac{1}{w-a_j}\,dw.$$

The Cauchy Integral Theorem implies

$$\int_\gamma (\partial u)(w)\,dw = m_1\int_{\Gamma_1}(\partial u)(w)\,dw + \cdots + m_n\int_{\Gamma_n}(\partial u)(w)\,dw$$

$$= \pi i(m_1c_1 + \cdots + m_nc_n)$$

$$= \int_\gamma \Big(\frac{c_1}{2(w-a_1)} + \cdots + \frac{c_n}{2(w-a_n)}\Big)\,dw.$$

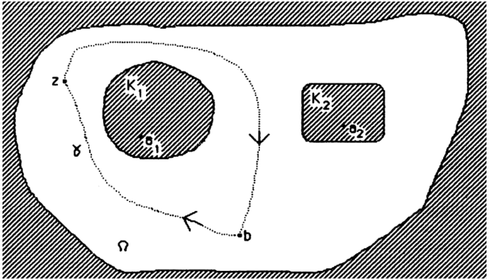
Fix $b \in \Omega$. Define $f : \Omega \to \mathbf{C}$ by

$$f(z) = \int_b^z \Big((\partial u)(w) - \frac{c_1}{2(w-a_1)} - \cdots - \frac{c_n}{2(w-a_n)}\Big)\,dw.$$

This makes sense because $\int_\gamma$ (above) $dw = 0$.

Thus f is analytic on Ω and

$$f'(z) = (\partial u)(z) - \frac{c_1}{2(z-a_1)} - \cdots - \frac{c_n}{2(z-a_n)}.$$



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We have an analytic function f on Ω such that

$$\begin{aligned}\partial f &= f' \\ &= \partial u - \frac{c_1}{2(z-a_1)} - \cdots - \frac{c_n}{2(z-a_n)} \\ &= \partial\big(u - c_1 \log|z-a_1| - \cdots - c_n \log|z-a_n|\big).\end{aligned}$$

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Thus

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Now $(*)$ and f are analytic functions on Ω that have the same imaginary part. Thus

$$u - \bar{f} - c_1 \log|z-a_1| - \cdots - c_n \log|z-a_n| = f + c$$

for some real constant c.

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Hence

$$\begin{aligned}u &= f + \bar{f} \\ &\quad + c_1 \log|z-a_1| + \cdots + c_n \log|z-a_n| + c \\ &= 2\,\mathrm{Re} f \\ &\quad + c_1 \log|z-a_1| + \cdots + c_n \log|z-a_n| + c \\ &= \mathrm{Re}(2f + c) \\ &\quad + c_1 \log|z-a_1| + \cdots + c_n \log|z-a_n|,\end{aligned}$$

as desired. ■

applications of $u = \operatorname{Re} g$ on simply connected domains

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  Proof: Suppose u is bounded harmonic on $\mathbf{C}$, with $u = \operatorname{Re} g$, where g is analytic on $\mathbf{C}$. Then

  $$|e^{g(z)}| = e^{u(z)}.$$

  Thus e^g is a bounded entire function, and hence is constant. Thus u is constant. ∎

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- Positive harmonic functions on $\mathbf{C}$ are constant.

  Proof: Suppose u is positive harmonic on $\mathbf{C}$, with $u = \operatorname{Re} g$, where g is analytic on $\mathbf{C}$. Then

  $$|e^{-g(z)}| = e^{-u(z)} \leq 1.$$

  Thus e^{-g} is a bounded entire function, and hence is constant. Thus u is constant. ∎