Applications of the Logarithmic Conjugation Theorem

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Suppose Ω is an open subset of $\mathbf{R}^2 = \mathbf{C}$.

Definition: *Laplacian*

For $u\colon \Omega \to \mathbf{C}$, the *Laplacian* of u is denoted Δu and is the function $\Delta u\colon \Omega \to \mathbf{C}$ defined by

$$(\Delta u)(z) = \frac{\partial^2 u}{\partial x^2}(z) + \frac{\partial^2 u}{\partial y^2}(z)$$

for $z = (x, y) = x + iy \in \Omega$.

Definition: *harmonic*

u is called *harmonic* on Ω if

$$(\Delta u)(z) = 0$$

for all $z \in \Omega$.

Definition: ∂ and $\overline{\partial}$

For $f : \Omega \to \mathbf{C}$ define

$$\partial f = \frac{1}{2}\Big(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\Big),$$

$$\overline{\partial} f = \frac{1}{2}\Big(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\Big).$$

If f is analytic on Ω, then

$$(\partial f)(z) = f'(z).$$

Cauchy–Riemann equations

f is analytic on Ω if and only if $\overline{\partial} f = 0$.

d and d-bar

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d d-bar = d-bar d = $\frac{1}{4}$ Laplacian

$$\partial\overline{\partial} = \overline{\partial}\partial = \frac{1}{4}\Delta$$

Proof:

$$\partial(\overline{\partial} f) = \frac{1}{4}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big)\Big(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\Big)$$

$$= \frac{1}{4}\Big(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\Big)$$

$$= \frac{1}{4}\Delta f.$$

Similarly, $\overline{\partial}(\partial f) = \frac{1}{4}\Delta f$. ∎

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analytic functions are harmonic

If f is analytic on Ω, then $\mathrm{Re} f$, $\mathrm{Im} f$, and f are harmonic on Ω.

Proof. Suppose f is analytic on Ω.
Thus $\overline{\partial} f = 0$. Hence

$$\Delta f = 4\partial(\overline{\partial} f) = 0.$$

Thus f is harmonic, which implies that $\mathrm{Re} f$ and $\mathrm{Im} f$ are harmonic. ∎

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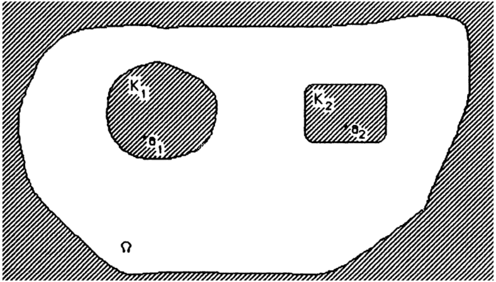
$\iff \overline{\partial}(\partial u) = 0$

$\iff \partial u$ is analytic. ∎

Logarithmic Conjugation Theorem



Definition: *finitely connected*

A domain $\Omega \subset \mathbf{C}$ is called *finitely connected* if $\mathbf{C} \setminus \Omega$ has only finitely many bounded connected components.

Logarithmic Conjugation Theorem

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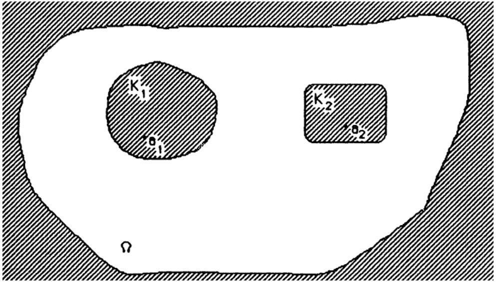
Suppose Ω is a finitely connected domain. Let $K_1, \ldots, K_n$ denote the bounded components of $\mathbf{C} \setminus \Omega$. Suppose $a_j \in K_j$ for $j = 1, \ldots, n$. If u is a real-valued harmonic function on Ω, then there exist an analytic function g on Ω and real numbers $c_1, \ldots, c_n$ such that

$$u(z) = \operatorname{Re} g(z) + c_1 \log|z - a_1| + \cdots + c_n \log|z - a_n|$$

for every $z \in \Omega$.



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isolated singularities of bounded harmonic functions

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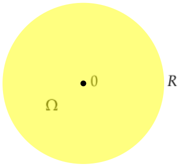
If a harmonic function is bounded near an isolated singularity, then that singularity is removable.

- Schwarz (1872)
- Picard (1923)
- Lebesgue (1923)



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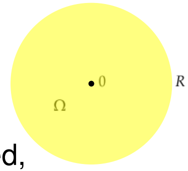

Proof: Suppose u is bounded, real-valued, and harmonic on

$$\Omega = \{z \in \mathbf{C} : 0 < |z| < R\}.$$

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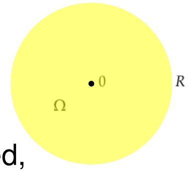

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$$\Omega = \{z \in \mathbf{C} : 0 < |z| < R\}.$$

By LCT, we can assume that there exist g analytic on Ω and $c \geq 0$ such that

$$(*) \qquad u(z) = \operatorname{Re} g(z) - c \log|z|.$$

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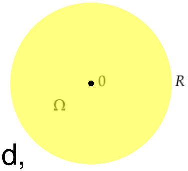

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$$\operatorname{Re} g(z) = u(z) + \log|z|^c.$$

Thus

$$|e^{g(z)}| = e^{\operatorname{Re} g(z)} = e^{u(z)}|z|^c.$$

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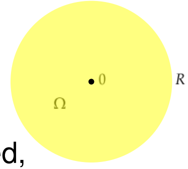

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Thus e^g is a bounded analytic function on Ω and hence has a removable singularity at 0. Hence g has a removable singularity at 0 (because if g has either an essential singularity or a pole at 0 then $\operatorname{Re} g$ is not bounded above near 0).

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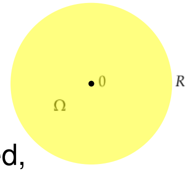

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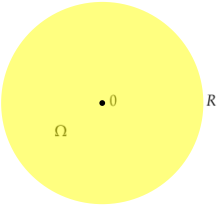
Now $(*)$ and boundedness of u imply that $c = 0$ and that u has a removable singularity at 0. ∎

Bôcher's Theorem

Suppose u is harmonic and positive on $\{z \in \mathbf{C} : 0 < |z| < R\}$. Then there exist g analytic on $\{z \in \mathbf{C} : |z| < R\}$ and $c \geq 0$ such that

$$u(z) = \operatorname{Re} g(z) - c \log|z|$$

for all $z \in \mathbf{C}$ with $0 < |z| < R$.



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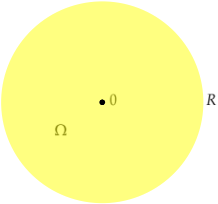

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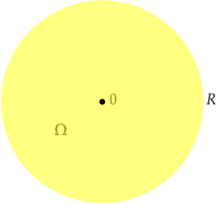

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Let m be a positive integer greater than c. Then

$$|z^m e^{-g(z)}| = |z^m|\, e^{-\operatorname{Re} g(z)} = e^{-u(z)} |z|^{m-c},$$

which is bounded for z near 0.

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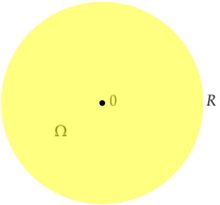

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Thus $z^m e^{-g}$ has a removable singularity at 0, which implies that e^{-g} has a pole or removable singularity at 0. This implies that g has a removable singularity at 0. This implies that $c \geq 0$. ∎

Liouville's Theorem for positive harmonic function on punctured plane

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If u is positive and harmonic on $\mathbf{R}^2 \setminus \{0\}$, then u is constant.

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is positive and harmonic on $\mathbf{C}$ and thus is constant. ∎

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The result above fails if $n > 2$ because

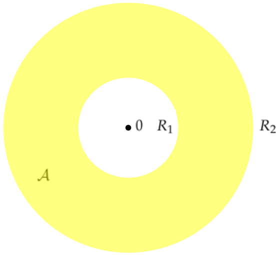
$$(x_1, \dots, x_n) \mapsto \frac{1}{\|(x_1, \dots, x_n)\|^{n-2}}$$

is positive and harmonic on $\mathbf{R}^n \setminus \{0\}$.

average of harmonic function on circles

Fix $0 \leq R_1 < R_2 \leq \infty$. Let

$$\mathcal{A} = \{z \in \mathbf{C} : R_1 < |z| < R_2\}.$$



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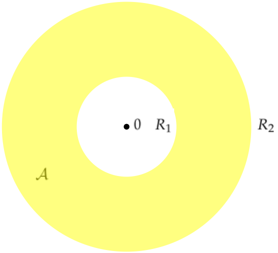

harmonic function on annulus

Suppose u is a real-valued harmonic function on $\mathcal{A}$. Then there exist $b, c \in \mathbf{R}$ such that

$$\frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\, d\theta = b + c \log r$$

for all $r \in (R_1, R_2)$.

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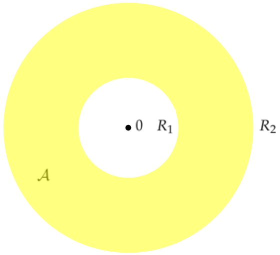

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Proof: By LCT, there exist g analytic on $\mathcal{A}$ and $c \in \mathbf{R}$ such that

$$u\big(re^{i\theta}\big) = \operatorname{Re} g\big(re^{i\theta}\big) + c\log r$$

for all $r \in (R_1, R_2)$ and all $\theta \in [0, 2\pi]$.

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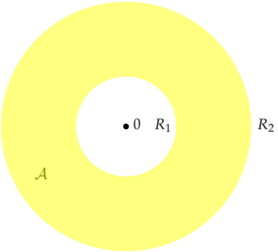

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$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} u\big(re^{i\theta}\big)\,d\theta &= \operatorname{Re}\frac{1}{2\pi}\int_0^{2\pi} g\big(re^{i\theta}\big)\,d\theta + c\log r\\ &= \operatorname{Re}\frac{1}{2\pi i}\int_{\{|z|=r\}}\frac{g(z)}{z}\,dz + c\log r\\ &= b + c\log r. \blacksquare\end{aligned}$$

series representation on annulus

Suppose u is a real-valued harmonic function on $\mathcal{A}$. Then there exist $c \in \mathbf{R}$ and $\{a_n\}_{n=-\infty}^{\infty} \subset \mathbf{C}$ such that

$$u(re^{i\theta}) = c \log r + \sum_{n=-\infty}^{\infty} \left(a_n r^n + \overline{a_{-n}} r^{-n}\right) e^{in\theta}$$

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$$g(z) = \sum_{n=-\infty}^{\infty} 2a_n z^n.$$

Proof: By LCT, there exist $c \in \mathbf{R}$ and g analytic on $\mathcal{A}$ such that

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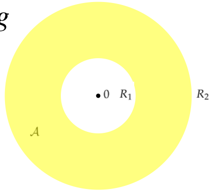
The analytic function g has a Laurent series:



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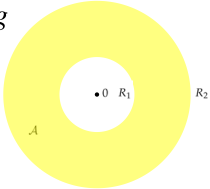

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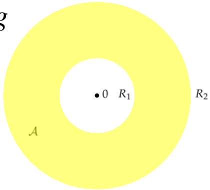

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$$u(re^{i\theta}) = c \log r + \sum_{n=-\infty}^{\infty} \left(a_n r^n e^{in\theta} + \overline{a_n} r^n e^{-in\theta}\right)$$

$$= c \log r + \sum_{n=-\infty}^{\infty} \left(a_n r^n + \overline{a_{-n}} r^{-n}\right) e^{in\theta}.$$

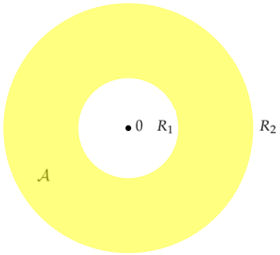
$\blacksquare$

Dirichlet problem for annulus

Now assume that $0 < R_1 < R_2 < \infty$.

Dirichlet problem on annulus

If $U \in C(\partial\mathcal{A})$, then $\exists$ unique $u \in C(\overline{\mathcal{A}})$ such that u is harmonic on $\mathcal{A}$ and $u|_{\partial\mathcal{A}} = U$.



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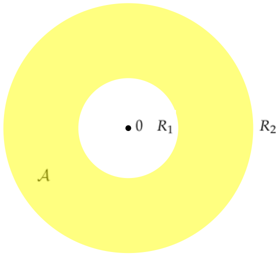

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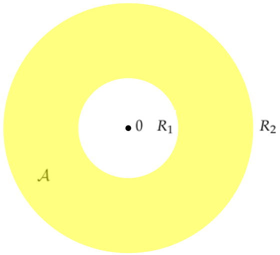
Dirichlet problem for annulus

Now assume that $0 < R_1 < R_2 < \infty$.



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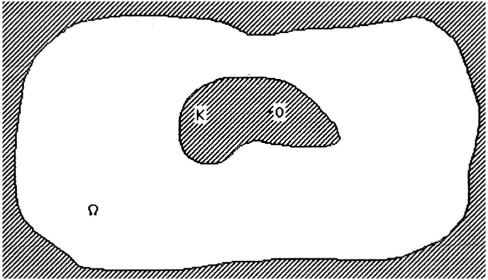
For arbitrary $u \in C(\partial\mathcal{A})$, use Stone-Weierstass theorem and maximum modulus theorem. ∎

$$u\left(re^{i\theta}\right) = c \log r + \sum_{n=-\infty}^{\infty} \left(a_n r^n + \overline{a_{-n}} r^{-n}\right) e^{in\theta}$$

Doubly Connected Mapping Theorem

Definition: *doubly connected*

An open connected set Ω is called *doubly connected* if $\mathbf{C} \setminus \Omega$ has exactly one bounded connected component.



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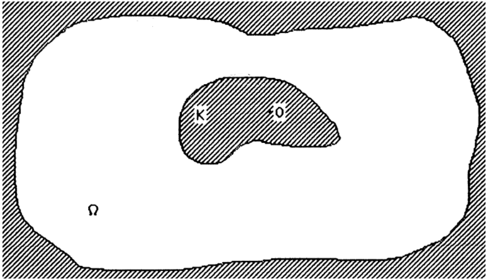

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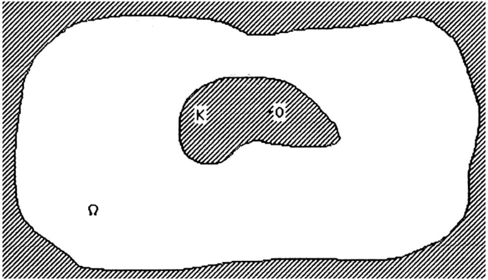
Every doubly connected domain is conformally equivalent to some annulus.

More precisely:

Suppose Ω is doubly connected. Then there exist an annulus $\mathcal{A}$ and an injective analytic function $f : \Omega \to \mathcal{A}$ from Ω onto $\mathcal{A}$.



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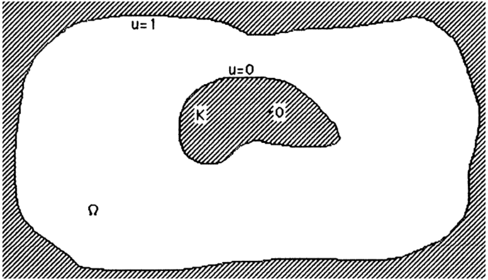

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Outline of proof: Let K denote the bounded connected component of $\mathbf{C} \setminus \Omega$. Assume $0 \in K$. Let u be the continuous real-valued function on $\overline{\Omega}$ such that

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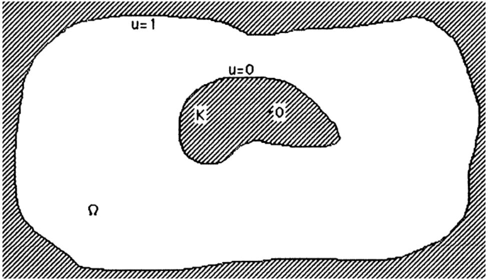

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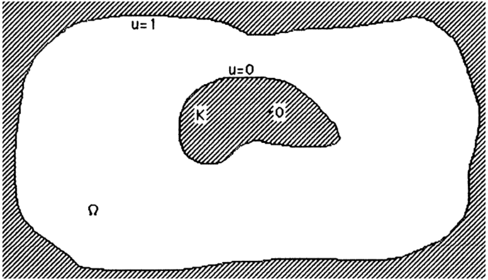
By the LCT, there exist g analytic on Ω and $c > 0$ such that

$$u(z) = c \log|z| + \operatorname{Re} g(z).$$



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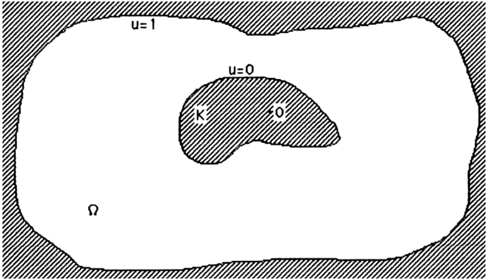

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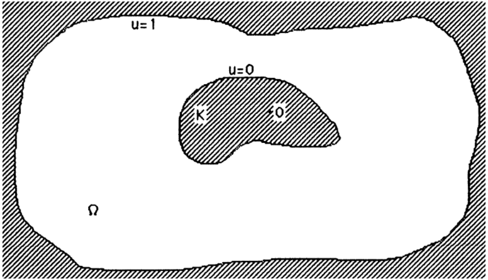

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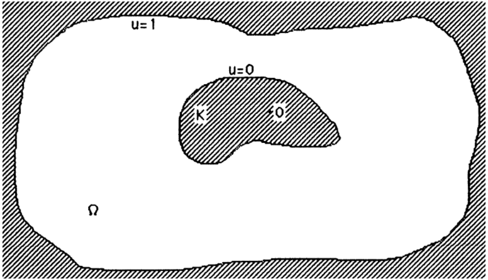


Let

$$\mathcal{A} = \left\{ z \in \mathbf{C} : 1 < |z| < e^{1/c} \right\}.$$

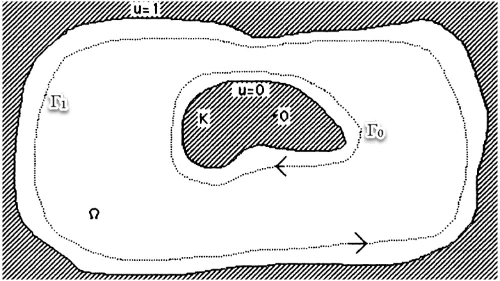
Thus

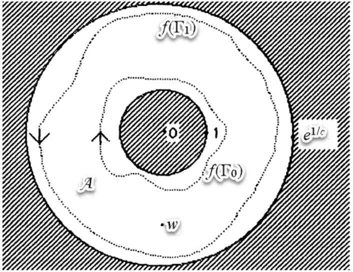
$$f : \Omega \to \mathcal{A}$$

is analytic (and it turns out that f is injective and maps Ω onto $\mathcal{A}$).

proof of Doubly Connected Mapping Theorem

Fix $w \in \mathcal{A}$. Let Γ_0 and Γ_1 be as in figures.





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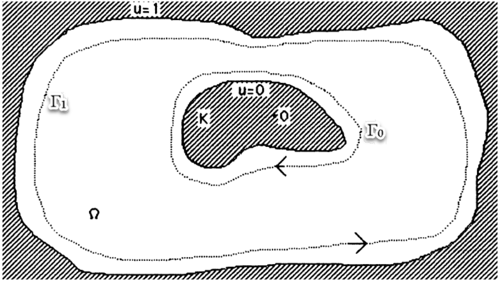
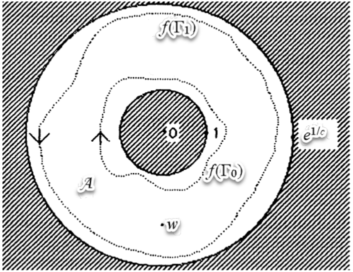
proof of Doubly Connected Mapping Theorem

Fix $w \in \mathcal{A}$. Let Γ_0 and Γ_1 be as in figures.

Argument principle: The number of times f takes on the value w in the region between Γ_0 and Γ_1 equals the winding number of $f(\Gamma_0) \cup f(\Gamma_1)$ about w.





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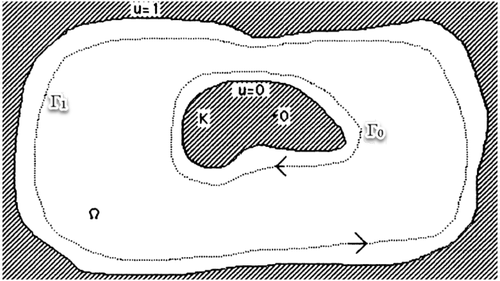

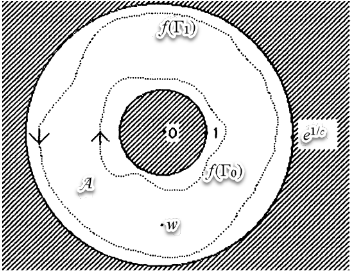

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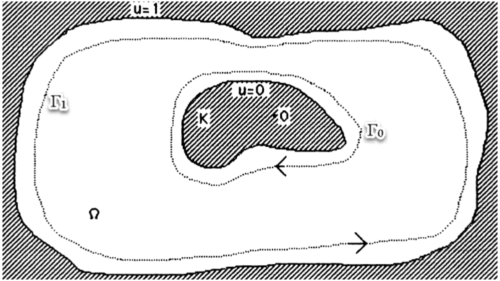
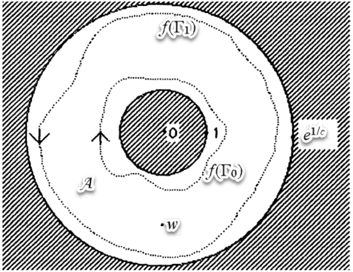
The winding number of $f(\Gamma_1)$ about w equals the winding number of $f(\Gamma_1)$ about 0, which equals the change in $\frac{1}{2\pi i} \log f(z)$ as z goes once around Γ_1.





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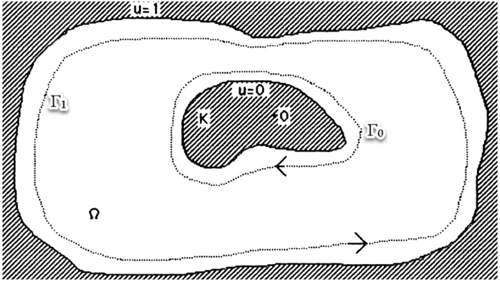
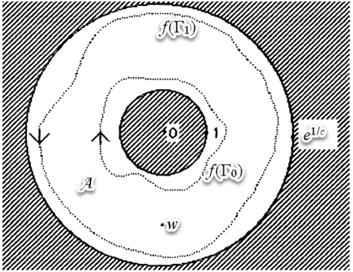
$$\log f(z) = \log z + \frac{g(z)}{c}.$$

Thus the change in $\frac{1}{2\pi i} \log f(z)$ around Γ_1 equals the change in $\frac{1}{2\pi i} \log z$ around Γ_1, which equals 1. ∎





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