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Analysis Seminar 10/12/2018

An Introduction to the Hausdorff Measure

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Abstract

This talk will cover the definition and basic properties of the Hausdorff measure and show the uniqueness of the Hausdorff dimension of a set. Afterwards, the Hausdorff dimension and measure of the Cantor set will be explored as an example.

Outer measure

[3] Let Ω be a set. An outer measure, ν, is an extended real-valued function that satisfies the following for any $A, B \in \mathcal{P}(\Omega)$.

(i) $\nu(A) \geq 0$. (nonnegative)

(ii) $\nu(\varnothing) = 0$.

(iii) If $A \subset B$ then $\nu(A) \leq \nu(B)$. (monotone)

(iv) $\nu\big(\bigcup_n A_n\big) \leq \sum_n \nu(A_n)$.

Diameter

[4] Let (Ω, ρ) be a metric space and S a nonempty subset of Ω. The diameter of S is given by

$$d(S) = \sup\{\rho(x, y) : x, y \in S\}.$$

By convention, $d(\varnothing) = 0$.

Hausdorff measure

[4] Let E be a subset of a metric space (Ω, ρ). Let $r \geq 0$ and $\delta > 0$ be given. Define

$$\mathcal{H}^r_\delta(E) = \inf\Big\{\sum_i d(A_i)^r : E \subset \bigcup_i A_i \text{ and } d(A_i) \leq \delta\Big\}$$

The r-dimensional Hausdorff outer measure is given by

$$\mathcal{H}^r(E) = \lim_{\delta \downarrow 0} \mathcal{H}^r_\delta(E)$$

Proof $\mathcal{H}^r_\delta(E)$ is nonincreasing as a function of δ

Claim: Let E be a subset of a metric space (Ω, ρ). Then for any $\varepsilon > \delta > 0$,

$$\mathcal{H}^r_\varepsilon(E) \leq \mathcal{H}^r_\delta(E)$$

Pf: Let ε and δ be given such that $\varepsilon > \delta > 0$. Then for any δ-cover of E, say $\{D_i\}_i$,

$$0 < d(D_i) \leq \delta < \varepsilon$$

for each i. As such, $\{D_i\}_i$ is also a ε-cover of E. Therefore,

$$\left\{\sum_i d(A_i)^r : E \subset \bigcup_i A_i \text{ and } d(A_i) \leq \delta\right\} \subset \left\{\sum_i d(A_i)^r : E \subset \bigcup_i A_i \text{ and } d(A_i) \leq \varepsilon\right\}$$

Thus,

$$\inf\left\{\sum_i d(A_i)^r : E \subset \bigcup_i A_i \text{ and } d(A_i) \leq \delta\right\} \geq \inf\left\{\sum_i d(A_i)^r : E \subset \bigcup_i A_i \text{ and } d(A_i) \leq \varepsilon\right\}$$

so $\mathcal{H}^r_\varepsilon(E) \leq \mathcal{H}^r_\delta(E)$ as desired. □

From here, it follows that

$$\mathcal{H}^r(E) = \lim_{\delta \downarrow 0} \mathcal{H}^r_\delta(E) = \sup_{\delta > 0} \mathcal{H}^r_\delta(E)$$

Proof $\mathcal{H}^r(E)$ is nonincreasing as a function of r

Claim: Let (Ω, ρ) be a metric space and $E \subset \Omega$. $\mathcal{H}^r(E)$ is a nonincreasing function of r.

Pf: Let r, s, and δ be given such that $0 \leq r < s$ and $0 < \delta < 1$. Let $\{U_i\}_i$ be a δ-cover of E. Then for each i, $d(U_i)^s \leq d(U_i)^r$, so $\sum_i d(U_i)^s \leq \sum_i d(U_i)^r$. Therefore, $\mathcal{H}^s_\delta(E) \leq \mathcal{H}^r_\delta(E)$.

Since this holds for any δ close to 0,

$$\mathcal{H}^s(E) = \lim_{\delta \downarrow 0} \mathcal{H}^s_\delta(E) \leq \lim_{\delta \downarrow 0} \mathcal{H}^r_\delta(E) = \mathcal{H}^r(E)$$

Uniqueness of, and motivation for, Hausdorff Dimension

Claim: Let (Ω, ρ) be a metric space. For any set $E \subset \Omega$, if $\mathcal{H}^r(E)$ is a positive real number for some $r \in [0, \infty)$, then $\mathcal{H}^t(E) = \infty$ whenever $0 \leq t < r$ and $\mathcal{H}^s(E) = 0$ whenever $r < s < \infty$.

Pf: Suppose $\mathcal{H}^r(E)$ is a positive real number and let s be given such that $r < s$. For any δ-cover of E, say $\{U_i\}_i$, since $d(U_i) \leq \delta$,

$$d(U_i)^s \leq \delta^{s-r} d(U_i)^r$$

for each i. Therefore,

$$\sum_i d(U_i)^s \leq \delta^{s-r} \sum_i d(U_i)^r$$

As such, $\mathcal{H}^s_\delta(E) \leq \delta^{s-r}\mathcal{H}^r_\delta(E)$. Taking the limit as δ approaches 0 reveals that if $\mathcal{H}^r(E)$ is finite, then $\mathcal{H}^s(E)$ must be 0.

Similarly, if $t < r$ then $\mathcal{H}^r_\delta(E) \leq \delta^{r-t}\mathcal{H}^t_\delta(E)$. Equivalently,

$$\delta^{t-r}\mathcal{H}^r_\delta(E) \leq \mathcal{H}^t_\delta(E)$$

Since $t - r < 0$, taking the limit on both sides reveals that if $\mathcal{H}^r(E)$ is nonzero, then $\mathcal{H}^t(E)$ must be infinite. □

Hausdorff dimension

[3] A set E is said to have Hausdorff dimension r, where

$$r = \sup\{s \in [0, \infty) : \mathcal{H}^s(E) > 0\}$$

with the condition that $\sup(\varnothing) = 0$.

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[1] [3] **Measuring the Cantor set**

Let C be the Cantor set and let $\delta_n = 1/3^n$. For each n construct a δ-cover of C, say $\{D_i\}_i$, with $d(D_i) = \delta_n$ for each i. Then,

$$H^s(C) = \lim_{n\to\infty} H^s_{1/3^n}(C) \leq \lim_{n\to\infty} \sum_{k=1}^{2^n} \left(\frac{1}{3^n}\right)^s = \lim_{n\to\infty} \frac{2^n}{3^{ns}}$$

In order to find the dimension, we want to find an s such that the limit is nonzero and finite. We evaluate the limit using the logarithm technique. Then

$$\lim_{n\to\infty} \log\left(\frac{2^n}{3^{ns}}\right) = \lim_{n\to\infty} n\big(\log(2) - s\log(3)\big)$$

It should be clear that the limit is $+\infty$ or $-\infty$ unless

$$s = \frac{\log(2)}{\log(3)} \approx 0.6309$$

in which case,

$$\lim_{n\to\infty} n\big(\log(2) - s\log(3)\big) = 0$$

If follows that $H^s(C) \leq 1$. We also get the opposite inequality since for any δ-cover of C, say $\{U_i\}_i$, there is an $n \in \mathbb{N}$ such that the $\bigcup_i D_i \subset \bigcup_i U_i$. As such, $1 \leq \sum_i d(D_i)^s \leq \sum_i d(U_i)^s$. Thus, $H^s(C) = 1$. □

References

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