Geodesic Bicombings and Convex Sets

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Overview

Goal: To introduce geodesic bicombings and examine some properties of convex sets in linear and nonlinear spaces.

This talk is based on:

Geodesic bicombings on some hyperspaces. arXiv:2106.12681 [math.MG]

Dominic Descombes. *Spaces with convex geodesic bicombings*. Ph.D. Thesis, ETH Zürich, 2015.

- Geodesic bicombings
- Convex sets
- Hausdorff distance
- Hörmander's embedding theorem
- Bicombings on the space of closed bounded convex sets

Geodesics

A **path** is a continuous function $\gamma : [0, T] \to X$, $[0, T] \subseteq \mathbb{R}$.

A **geodesic** from x to y is a an arc length parametrized path γ, whose *length* is the distance between x and y.

A geodesic is a path $\gamma : [0, d(x, y)] \to X$ such that

$$d\big(\gamma(s), \gamma(t)\big) = |s - t|$$

for all $s, t \in [0, d(x, y)]$.

A **geodesic space** is a metric space such that every pair of points is connected by a geodesic.

For any x and y in normed space,

$$\gamma(t) = x + \frac{t(y - x)}{\|x - y\|}$$

is a geodesic connecting x and y.

For convenience, we will mostly use **linearly reparametrized** geodesics, $\gamma : [0, 1] \to X$

$$d\big(\gamma(s), \gamma(t)\big) = \lambda|s - t|, \quad \lambda = d(\gamma(0), \gamma(1))$$

For $x, y \in (X, \| \ \|)$, we have the linearly reparametrized geodesic, $\gamma(t) = (1 - t)x + ty$.

Hierarchy of Geodesic Spaces (of Global Nonpositive Curvature)

Inner product spaces	$\subset$	Strictly convex normed spaces	$\subset$	Normed spaces
$\cap$		$\cap$		$\cap$
$\mathrm{CAT}(0)$ spaces	$\subset$	Busemann spaces	$\subset$	Spaces with convex bicombing

A complete $\mathrm{CAT}(0)$ space is a **Hadamard space**.

Geodesic Bicombings

A **geodesic bicombing** on a metric space X is a function $\sigma : X \times X \times [0,1] \to X$ such that $\sigma(x,y,\cdot)$ is a linearly reparameterized geodesic from x to y.

Setting $\sigma(x,y,t) = \sigma_{xy}(t)$,

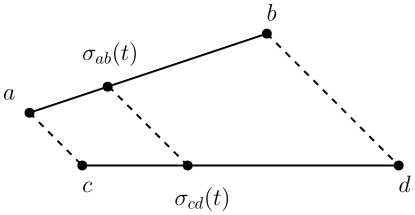
- $\sigma_{xy}(0) = x$
- $\sigma_{xy}(1) = y$
- $d\big(\sigma_{xy}(s), \sigma_{xy}(t)\big) = d(x,y)|s-t|$ for all $s,t \in [0,1]$.

Normed space bicombing

$$\sigma(x,y,t) = (1-t)x + ty$$

A bicombing σ is **conical** if for any $a,b,c,d \in X$,

$$d\big(\sigma_{ab}(t), \sigma_{cd}(t)\big) \le (1-t)d(a,c) + td(b,d)$$



Geodesic Bicombings Contd.

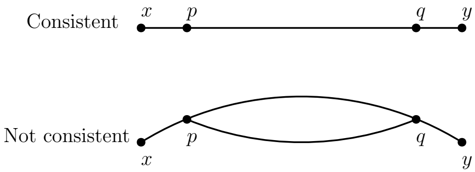
A bicombing is **convex** if for any $a, b, c, d \in X$,

$$t \mapsto d\big(\sigma_{ab}(t), \sigma_{cd}(t)\big)$$

is convex.

A bicombing is **consistent** if for $p = \sigma_{xy}(s)$ and $q = \sigma_{xy}(t)$,

$$\sigma_{pq}(\lambda) = \sigma_{xy}((1-\lambda)s + \lambda t).$$



- convex $\implies$ conical
- consistent and conical $\implies$ convex

Normed space bicombing

$$\sigma(x, y, t) = (1-t)x + ty$$

This bicombing is conical:

$$\begin{aligned} d\big(\sigma_{ab}(t), \sigma_{cd}(t)\big) &= \|\sigma_{ab}(t) - \sigma_{cd}(t)\| \\ &= \|(1-t)a + tb - ((1-t)c + td)\| \\ &\le (1-t)\|a - c\| + t\|b - d\| \\ &= (1-t)d(a,c) + td(b,d) \end{aligned}$$

Exercise: It is also consistent (and so convex).

Convex Sets and the Hausdorff Distance

A set $C \subseteq X$ is **convex** if for any $x, y \in C$ and $t \in [0,1]$, we have $\sigma(x,y,t) \in C$.

The **closed convex hull** of C, $\overline{\text{co}}(C)$, is the intersection of all closed convex sets containing C.

Moving forward, we will examine

$$B(X) = \{A \subseteq X : A \text{ is nonempty, closed, and bounded}\}$$

$$CB(X) = \{A \in B(X) : A \text{ is convex}\}$$

Given $A \subseteq X$, the closed ε-neighborhood of A is

$$N_\varepsilon(A) = \{x \in X : d(x,A) \leq \varepsilon\}$$

Hausdorff Distance

Given two sets $A, B \in B(X)$, the **Hausdorff distance** between them is

$$\begin{aligned} d_H(A,B) &= \inf\{\varepsilon > 0 : A \subseteq N_\varepsilon(B), B \subseteq N_\varepsilon(A)\} \\ &= \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\} \\ &= \sup_{x\in X} |d(x,A) - d(x,B)| \end{aligned}$$

Lemma

If X admits a conical bicombing σ, then for any $A, B \subseteq X$

$$d_H(\overline{\text{co}}(A), \overline{\text{co}}(B)) \leq d_H(A, B).$$

'Proof'

1. σ is conical $\implies N_\varepsilon(\overline{\text{co}}(B))$ is convex.
2. $B \subseteq \overline{\text{co}}(B)$, so
$$A \subseteq N_\varepsilon(B) \implies A \subseteq N_\varepsilon(\overline{\text{co}}(B)).$$
3. $N_\varepsilon(\overline{\text{co}}(B))$ is a closed convex set containing $A \implies \overline{\text{co}}(A) \subseteq N_\varepsilon(\overline{\text{co}}(B))$.
4. By symmetry,
$$B \subseteq N_\varepsilon(A) \implies \overline{\text{co}}(B) \subseteq N_\varepsilon(\overline{\text{co}}(A)).$$

Corollary

If X admits a conical bicombing σ, then

$$\overline{\text{co}} : B(X) \to CB(X)$$

is 1-Lipschitz.

Question

If X admits a geodesic bicombing σ and

$$\overline{\text{co}} : B(X) \to CB(X)$$

is 1-Lipschitz, does that imply σ is conical?

Hörmander's Embedding Theorem

Let X be a real normed space.

Support functional

For $A \subseteq X$ and $x^* \in X^*$

$$s_A(x^*) = \sup_{a \in A} x^*(a)$$

Let U^* be the closed unit ball in X^*.

Theorem (Hörmander's embedding)

The map $A \mapsto s_A$ is an algebraic and isometric embedding of $CB(X)$ as a convex cone in the Banach space $C_b(U^*)$.

1. $d_H(A, B) = \sup_{u^* \in U^*} |s_A(u^*) - s_B(u^*)|$
2. $ts_A = s_{tA}$ for $t \geq 0$
3. $s_A + s_B = s_{\overline{A+B}}$

Corollary

If X is a real normed space, the map

$$\Sigma : CB(X) \times CB(X) \times [0, 1] \to CB(X)$$

defined by

$$\Sigma(A, B, t) = \overline{(1-t)A + tB}$$

is a consistent convex bicombing on $CB(X)$.

Bicombing $CB(X)$

Recall our normed space bicombing

$$\sigma(x, y, t) = (1-t)x + ty.$$

Let's generalize the notation of our bicombing:

$$\begin{aligned}\Sigma(A, B, t) &= \overline{(1-t)A + tB} \\ &= \overline{\bigcup_{a \in A} \bigcup_{b \in B} (1-t)a + tb} \\ &= \overline{\bigcup_{a \in A} \bigcup_{b \in B} \sigma(a, b, t)}\end{aligned}$$

Would this bicombing work in $CB(X)$ if X is nonlinear?

Theorem

If X admits a consistent convex bicombing σ, then

$$\Sigma(A, B, t) = \overline{\text{co}}\left(\bigcup_{a \in A} \bigcup_{b \in B} \sigma(a, b, t)\right)$$

is a conical bicombing on $CB(X)$.

Questions

1. Is this bicombing consistent?
2. Is this bicombing convex?