Complex Dynamics
Part I: Entire Functions and Wandering Domains

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Complex dynamics (holomorphic dynamics): The study of the iteration of holomorphic functions on Riemann surfaces.

- Part I: Entire Functions and Wandering Domains
- Part II: Rational Functions on the Riemann Sphere
- Part III: Quasiconformal Mappings
- Part III: Sullivan's Proof of the No Wandering Domains Conjecture

Some references:

- John Milnor
  Dynamics in One Complex Variable
  arXiv 9201272[math.DS]
- Alan F. Beardon.
  Iteration of Rational Functions
- L. Carleson and T.W. Gamelin.
  Complex Dynamics
- I.N. Baker. Wandering domains in the iteration of entire functions.
  Proc. London Math. Soc. 1984
- Some notes of my own

Some Definitions

Definition

Given a domain $D \subseteq \mathbb{C}$, a function $f : D \to \mathbb{C}$ is *analytic* (or *holomorphic*) if f is differentiable at every point in D. A holomorphic function is *entire* if $D = \mathbb{C}$.

- A holomorphic function can be represented by a convergent power series at every point in its domain.
- Holomorphic functions are infinitely differentiable.

Given an entire function $f : \mathbb{C} \to \mathbb{C}$, consider $\{f^{\circ n}\}_{n=1}^{\infty}$, where

$$f^{\circ n} = \underbrace{f \circ f \circ \cdots \circ f}_{n}$$

Ex: $f(z) = z^2$

$f^{\circ 2}(z) = (z^2)^2 = z^4$

$f^{\circ 3}(z) = z^8$

$f^{\circ n}(z) = z^{2^n}$

For $f : \mathbb{C} \to \mathbb{C}$ and $S \subseteq C$, the set S is

- *forward invariant* if $f(S) \subseteq S$.
- *backward invariant* if $f^{-1}(S) \subseteq S$.
- *completely invariant* if it is both forward and backward invariant.

Two famous completely invariant sets:

- Fatou set $F(f)$ - the 'stable' set
- Julia set $J(f)$ - the 'chaotic' set

To formally define these sets, we need *normal families* of functions.

Let $\mathscr{H}(D, \mathbb{C})$ be the collection of holomorphic functions $f : D \to \mathbb{C}$.

Definition

A family of functions $\mathscr{F} \subseteq \mathscr{H}(D, \mathbb{C})$ is said to be *normal* if every infinite sequence $\{f_n\}_{n=1}^{\infty} \subseteq \mathscr{F}$ contains a subsequence which converges uniformly on compact sets.

A sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly on compact sets $\implies \exists\, g$ such that

$$\lim_{n\to\infty} \sup_{x\in K} |f_n(x) - g(x)| = 0$$

for any compact $K \subseteq D$.

Fatou and Julia sets

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function.

Definition

The *Fatou set* of f is given by

$$F(f) = \left\{ z \in \mathbb{C} : \{f^{\circ n}\}_{n=1}^{\infty} \text{ is normal in a nbhd. of } z \right\}.$$

The *Julia set* of f is the complement of the Fatou set,

$$J(f) = \mathbb{C} \setminus F(f).$$

How do I determine when or if $\{f^{\circ n}\}_{n=1}^{\infty}$ forms a normal family?

For today: Attracting fixed points.

Fixed point: $f(p) = p$.

A fixed point p is

- *Attracting* if $|f'(p)| < 1$.
- *Repelling* if $|f'(p)| > 1$.
- *Neutral* if $|f'(p)| = 1$.

Periodic point: $f^{\circ n}(p) = p$.

A periodic point p is

- *Attracting* if $|(f^{\circ n})'(p)| < 1$.
- *Repelling* if $|(f^{\circ n})'(p)| > 1$.
- *Neutral* if $|(f^{\circ n})'(p)| = 1$.

Basin of Attraction:

$$\{z \in \mathbb{C} : f^{\circ n}(z) \to p\}$$

Consider $f(z) = z^3$ and let $\mathbb{D} = \{z : |z| < 1\}$.

- $f(0) = 0$ and $f'(0) = 0$
- Basin of attraction $= \mathbb{D}$
- For any compact $K \subseteq \mathbb{D}$,

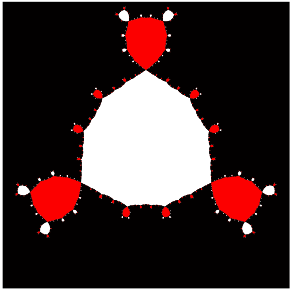
$$\sup_{z \in K} |f^{\circ n}(z) - 0| \to 0.$$

(note: $\{f^{\circ n}\}_{n=1}^{\infty}$ does not converge uniformly on $\mathbb{D}$)

Consider $g(z) = z^3 + i$

$g(0) = i$ and $g(i) = 0 \implies g^{\circ 2}(0) = 0$

$g^{\circ 2n}(z) \to 0$ in white, $g^{\circ 2n}(z) \to i$ in red



Lemma

Let p be an attracting fixed (or periodic) point of holomorphic f. Then $\{f^{\circ n}\}_{n=1}^{\infty}$ is a normal family on the basin of attraction.

'proof' Let B be the basin of attraction.
Fix λ such that $|f'(p)| < \lambda < 1$.
Let U be an bounded open nbhd of p such that

$$|f(z) - f(p)| \leq \lambda |z - p| \quad \forall\ z \in U.$$

Given compact $K \subseteq B$,

$$K \subseteq \bigcup_{k=1}^{\infty} f^{\circ -k}(U).$$

By compactness, find N such that,

$$K \subseteq \bigcup_{k=1}^{N} f^{\circ -k}(U).$$

Then for all $n \geq N$,

$$\begin{aligned}\sup_{z \in K} |f^{\circ n}(z) - p| &= \sup_{z \in K} |f^{\circ n}(z) - f^{\circ n}(p)| \\ &\leq \lambda^{n-N} \text{diam}(U).\end{aligned}$$

So $f^{\circ n} \to p$ uniformly on K.
If p is q-periodic, use $f = f^{\circ q}$. $\square$

Corollary

The basin of attraction is in $F(f)$ and its boundary is in $J(f)$.

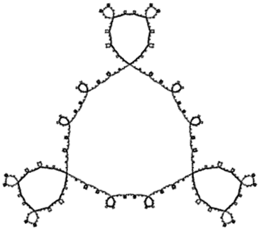
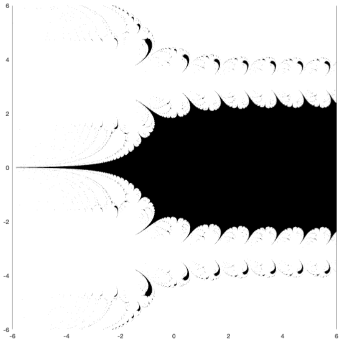


Figure: Julia set of $f(z) = z^3 + i$.

A transcendental example: $f(z) = z - 1 + e^{-z}$

f is Newton's method applied to $z \mapsto e^z - 1$

- $f(2\pi ki) = 2\pi ki$ for each $k \in \mathbb{Z}$
- $f'(2\pi ki) = 0$
- $f(z + 2\pi i) = f(z) + 2\pi i$
  $\implies f^{\circ n}(z + 2\pi i) = f^{\circ n}(z) + 2\pi i$



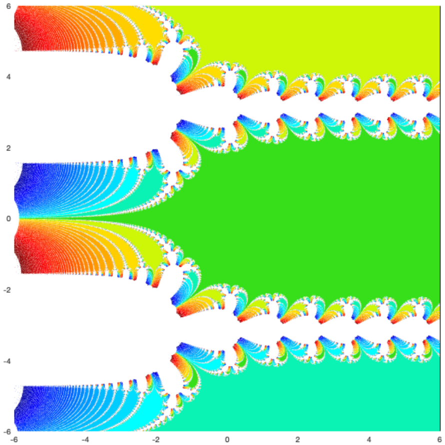


Figure: Basins of attraction of $f(z) = z - 1 + e^{-z}$.

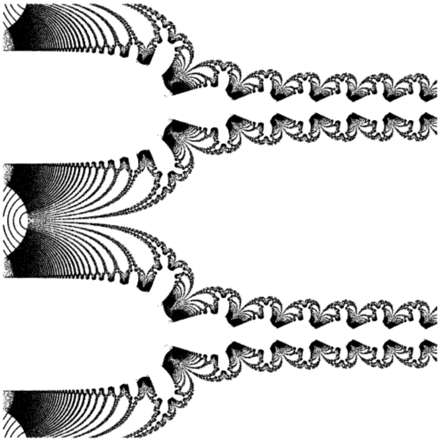


Figure: Part of $J(f)$ for $f(z) = z - 1 + e^{-z}$.

Consider $g(z) = z - 1 + e^{-z} + 2\pi i$

$$g(z) = f(z) + 2\pi i$$

Check that $f \circ g = g \circ f$

Lemma (I.N. Baker)

Suppose f and g are entire, $f \circ g = g \circ f$, and $g = f + c$ for some constant c. Then $J(f) = J(g)$.

proof: See Baker's *Wandering Domains in the Iteration of Entire Functions*

A *Fatou component* is a maximally connected subset of $F(f)$.

A Fatou component U is

- *Periodic* if $f^{\circ n}(U) = U$ for some n.
- *Eventually periodic* if $f^{\circ m}(U)$ is periodic for some m.
- *Wandering* if $f^{\circ m}(U) \cap f^{\circ n}(U) = \varnothing$ for all $n \neq m$.

The (known) components of $f(x) = z - 1 + e^{-z}$ are periodic or eventually periodic. What about $g(z)$?

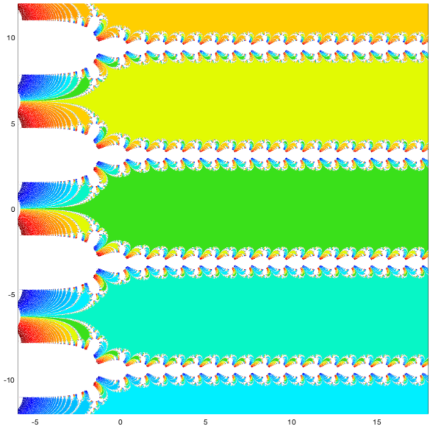


Figure: The large Fatou components of $g(z) = z - 1 + e^{-z} + 2\pi i$ wander upwards along the imaginary axis.

Why is this interesting?

I.N. Baker, 1976:

$$f(z) = \frac{z^2}{4e} \prod_{n=1}^{\infty} \left(1 + \frac{z}{\alpha_n}\right)$$

has (multiply-connected) wandering domains.

Theorem (D. Sullivan, 1982 (pub '85))

Rational functions (of deg ≥ 2) do not exhibit wandering domains.

Theorem (I.N. Baker, 1984)

For any ρ such that $1 \leq \rho \leq \infty$, $\exists$ entire $f : \mathbb{C} \to \mathbb{C}$ of order ρ, which has an infinity of different families of wandering domains.

Figure: Irvine Noel Baker (1932-2001).

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{f} & \mathbb{C} \\ \downarrow e^z & & \downarrow e^z \\ \mathbb{C}\setminus\{0\} & \xrightarrow{h} & \mathbb{C}\setminus\{0\} \end{array}$$

$$f(z) = z + \sin(z) \quad \text{and} \quad g(z) = z + \sin(z) + 2\pi$$

f has attracting fixed-points at $z_k = (2k+1)\pi$, $k \in \mathbb{Z}$

Since $f \circ g = g \circ f$ and $g = f + c$, $F(g) = F(f)$.

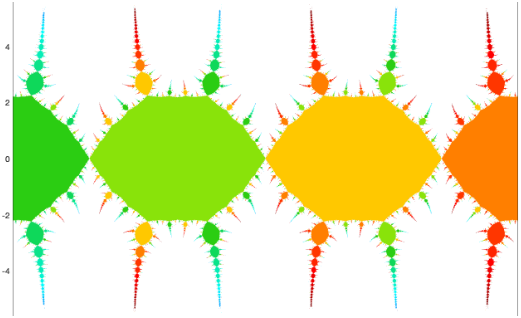


Figure: The large components of g wander along the real axis.