Complex Dynamics
Part IV: No Wandering Domains

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- Part I: Entire Functions and Wandering Domains
- Part II: Rational Functions on the Riemann Sphere
- Part III: Quasiconformal Mappings
- **Part IV: Sullivan's Proof of the No Wandering Domains Conjecture**

Some references:

- Dennis Sullivan. Quasiconformal homeomorphisms and dynamics I. Solution of the Fatou-Julia problem on wandering domains. *Annals of Mathematics*, 122(2):401-418, 1985.
- Saeed Zakeri. Sullivan's proof of Fatou's no wandering domain conjecture. Find it here.
- Alan F. Beardon. *Iteration of Rational Functions*
- John Milnor, *Dynamics in One Complex Variable*. arXiv 9201272[math.DS]
- Slides from previous talks

Notation and Definitions

The simply connected Riemann surfaces:

- $\mathbb{C}$ is the complex plane
- $\mathbb{D}$ is the unit disk, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$
- $\widehat{\mathbb{C}}$ is the **Riemann sphere**, i.e. the extended complex plane $\mathbb{C} \cup \{\infty\}$

For a **holomorphic** function f, consider

$$f^{\circ n} = \underbrace{f \circ f \circ \cdots \circ f}_{n}$$

A sequence of functions is **normal** if it contains a subsequence that converges locally uniformly.

The **Fatou set**

$$F(f) = \{z : \{f^{\circ n}\}_n \text{ is normal on a nbhd of } z\}$$

The **Julia set**

$$J(f) = \text{ the complement of } F(f)$$

$F(f)$ and $J(f)$ are **completely invariant**:

$$z \in F(f) \implies f(z) \in F(f)$$

and

$$z \in F(f) \implies f^{-1}(z) \subseteq F(f)$$

Any component $U \subseteq F(f)$ is either

- periodic: $f^{\circ n}(U) \subseteq U$
- eventually periodic: $f^{\circ n}(U) \subseteq f^{\circ m}(U)$
- wandering: $f^{\circ n}(U) \cap f^{\circ m}(U) = \varnothing$

Holomorphic dynamics on $\mathbb{D}$:

Theorem

For every holomorphic map $f : \mathbb{D} \to \mathbb{D}$, the Julia set $J(f)$ is empty.

Vacuously, every Fatou component of $f : \mathbb{D} \to \mathbb{D}$ is periodic.

Holomorphic dynamics on $\mathbb{C}$:

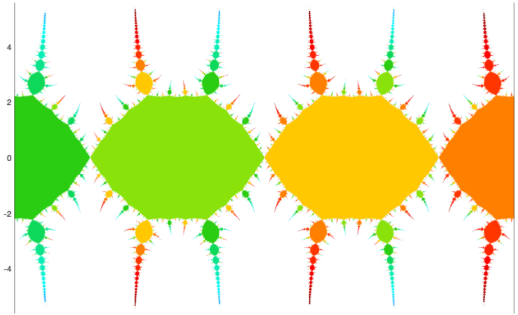

Figure: $f(z) = z + \sin(z) + 2\pi$ has wandering domains.

Lets talk about $\widehat{\mathbb{C}}$.
Every holomorphic function $R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a **rational function**:

$$R(z) = \frac{P(z)}{Q(z)}, \quad \deg(R) = \max\{\deg(P), \deg(Q)\}$$

Theorem (Sullivan, 1985)

Let $R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational function. Every component U of the Fatou set $F(R)$ is eventually periodic.

I.e. there are no wandering domains for holomorphic functions $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$.

Available proofs of theorem:

- D. Sullivan, *Annals of Mathematics*, 1985
- L. Bers, *American J. Math.*, 1987
- A. Beardon, *Iteration of Rational Functions*, 1991
- S. Zakeri, Online Notes, 2002

Outline:

- Basic properties of $J(R)$ and $F(R)$
- Narrowing possible wandering domains
- Quasiconformal maps and R
- Constructing a contradiction

We will restrict to R with $\deg(R) \geq 2$.

$\deg(R) = 0 \implies R$ is constant.

$\deg(R) = 1 \implies R(z) = \frac{az+b}{cz+d}, \quad ac - bd \neq 0$

The degree 1 rational functions are the **Möbius transformations**; the conformal isomorphisms of $\widehat{\mathbb{C}}$.

Theorem

Let R be a rational function with $\deg(R) \geq 2$. Then $J(R)$ is nonempty and infinite.

Further, $J(R)$ is **perfect**: closed and without isolated points.

It is possible that $J(R) = \widehat{\mathbb{C}}$ and $F(R) = \varnothing$. This is true for

$$R(z) = \frac{(z-2)^2}{z^2}.$$

Moving forward, we now suppose that R has a wandering domain W and let $W_n = R^{\circ n}(W)$.

- Narrowing possible wandering domains

Lemma

For all sufficiently large n, W_n is simply connected and $R : W_n \to W_{n+1}$ is injective.

Important: we may assume that R acts conformally on W.

Recall: f is **quasiconformal** if it satisfies the **Beltrami equation**

$$\bar{\partial} f = \mu \partial f$$

where μ is the **complex dilatation** with $\|\mu\|_\infty < 1$.

Given μ on W, we use the pull-back and push-forward of μ by R,

$$\mu\big(R(z)\big) = \big(R'(z)/\overline{R'(z)}\big)\mu(z)$$

On the grand orbit of W, $\bigcup_{n\in\mathbb{Z}} R^{\circ n}(W)$, define μ as above.
On points not in the grand orbit, set $\mu(z) = 0$.

- Quasiconformal maps and R

Suppose that ϕ is quasiconformal with complex dilatation μ (as we constructed).

Lemma

$\phi \circ R \circ \phi^{-1}$ is a rational function with $\deg(\phi \circ R \circ \phi^{-1}) = \deg(R)$.

'proof'

$$\begin{array}{ccc} \widehat{\mathbb{C}}[\mu] & \xrightarrow{R} & \widehat{\mathbb{C}}[\mu] \\ \downarrow \phi & & \downarrow \phi \\ \widehat{\mathbb{C}} & \xrightarrow{\phi \circ R \circ \phi^{-1}} & \widehat{\mathbb{C}} \end{array}$$

- Constructing a contradiction

Let $d = \deg(R)$.
The space of rational functions of degree d has $4d+2$ real degrees of freedom (think $\mathbb{CP}^{2d+1}$).

On the other hand, the set of functions $\mu : \mathbb{D} \to \mathbb{D}$, is infinite dimensional.

Let $h : \mathbb{D} \to W$ be a conformal equivalence.

1. Given μ on $\mathbb{D}$, define η on W by

$$\eta(h(z)) = \left(h'(z)/\overline{h'(z)}\right)\mu(z).$$

2. Use R to define η on the grand orbit of W and set $\eta(z) = 0$ elsewhere.

3. Let ϕ be the unique solution to $\bar{\partial}\phi = \eta\partial\phi$ which fixes 0, 1, and ∞.

4. $\phi \circ R \circ \phi^{-1}$ is a rational map with the same degree as R.

We now have a sequence of mappings

$$\mu \mapsto \eta \mapsto \phi \mapsto \phi \circ R \circ \phi^{-1}$$

Exercise:
Show that this produces a space of rational maps strictly larger than the space of rational functions of $\deg(R)$.