Convexity in Metric Spaces of Nonpositive Curvature

Logan S. Fox

Portland State University

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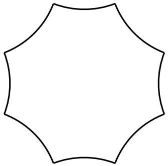
Overview

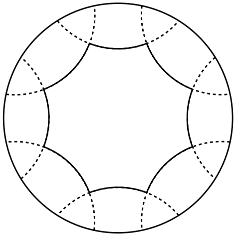
Goal: To motivate **convexity** as a **geometric** property, by examining convexity in (possibly) nonlinear spaces.

Trick Question: Is this set convex?



Answer: It depends.

It is convex in the Poincaré disk.



- Geodesic metric spaces
- Examples of nonpositively curved spaces
- Hadamard Spaces
- Convexity in Hadamard Spaces
- Busemann Spaces

Geodesic Spaces

A **path** is a continuous function $\gamma : [0, T] \to X$, where $[0, T] \subseteq \mathbb{R}$.

A **geodesic** is a path $\gamma : [0, T] \to X$ such that

$$d\big(\gamma(s), \gamma(t)\big) = |s - t|$$

for all $s, t \in [0, T]$.

In particular, $t = d(\gamma(0), \gamma(t))$ for all $t \in [0, T]$.

A **geodesic space** is a metric space such that every pair of points is connected by a geodesic.

Every normed space is a geodesic space:

For $x, y \in (X, \| \, \|)$, setting $T = \|x - y\|$,

$$\gamma(t) = x + \frac{t(y - x)}{\|x - y\|}$$

is a geodesic connecting x and y.

For convenience, we also consider **linearly reparametrized** geodesics, $\gamma : [0, 1] \to X$

$$d\big(\gamma(s), \gamma(t)\big) = \lambda |s - t|$$

where $\lambda = d(\gamma(0), \gamma(1))$.

Now for $x, y \in (X, \| \, \|)$, we can use the lin. reparam. geodesic, $\gamma(t) = (1 - t)x + ty$.

Spaces of (Global) Nonpositive Curvature

Hadamard Spaces $\subsetneq$ Busemann Spaces $\subsetneq$ Spaces with Convex Bicombings

Examples of Hadamard spaces (also $\mathrm{CAT}(0)$ spaces):

- Euclidean space, $\mathbb{E}^n$
- Hyperbolic space, $\mathbb{H}^n$
- $\mathscr{L}^2$ space
- Hilbert spaces
- Tree graphs

Examples of Busemann spaces:

- Hadamard spaces
- $\mathscr{L}^p$ spaces, $1 < p < \infty$
- Any strictly convex Banach space

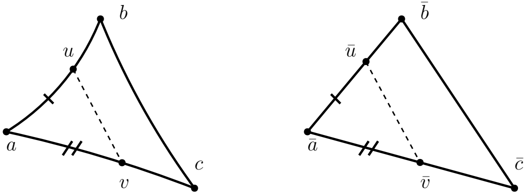
Spaces with convex bicombings:

- Busemann spaces
- $\mathscr{L}^1$ space
- $\mathscr{L}^\infty$ space
- $(CB(X), d_H)$, see Hörmander's embedding theorem

Hadamard Spaces

A Hadamard space is a complete $\mathrm{CAT}(0)$ space.

A $\mathrm{CAT}(0)$ space is a geodesic space with ‘thinner than Euclidean’ triangles:



$\Delta(a, b, c) \subseteq X$ $\qquad$ $\Delta(\bar{a}, \bar{b}, \bar{c}) \subseteq \mathbb{R}^2$

Edge lengths of both triangles are equal.
E.g. $d(a, b) = \|\bar{a} - \bar{b}\|$

CAT(0) condition: for all $u, v \in \Delta(a, b, c)$ and the edge-isometric points $\bar{u}, \bar{v} \in \Delta(\bar{a}, \bar{b}, \bar{c})$,

$$d(u, v) \leq \|\bar{u} - \bar{v}\|.$$

Equivalently, the **Hadamard condition**: for all $x, y, z \in X$, if m is the midpoint of x and y, then

$$2d(z, m)^2 + \frac{1}{2}d(x, y)^2 \leq d(z, x)^2 + d(z, y)^2.$$

Exercise: If X is a Hilbert space, $m = \frac{x+y}{2}$ and we get equality above.

Convexity in Hadamard Spaces

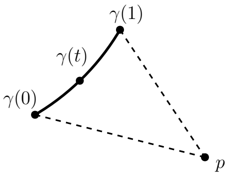
$f : X \to \mathbb{R}$ is **convex** if $f \circ \gamma$ is convex for every geodesic γ:

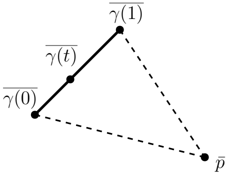
$$f \circ \gamma(tx + (1-t)y) \leq t f \circ \gamma(x) + (1-t) f \circ \gamma(y)$$

Lemma

In a Hadamard space, distance to a point is a convex function.

Proof: Let $p \in X$, $\gamma : [0,1] \to X$, $t \in (0,1)$,





Domain $[0,1]$ for convenience:

- $t = t1 + (1-t)0$
- $\overline{\gamma(t)} = t\overline{\gamma(1)} + (1-t)\overline{\gamma(0)}$

$$\begin{aligned} d(\gamma(t), p) &\leq \|\overline{\gamma(t)} - \bar{p}\| \\ &\leq t\|\overline{\gamma(1)} - p\| + (1-t)\|\overline{\gamma(0)} - \bar{p}\| \\ &= t d(\gamma(1), p) + (1-t) d(\gamma(0), p) \end{aligned}$$

This same idea yields general convexity,

$$\begin{aligned} &d\big(\gamma(tx + (1-t)y), p\big) \\ &\qquad \leq t d\big(\gamma(x), p\big) + (1-t) d\big(\gamma(y), p\big) \end{aligned}$$

for all $x, y \in \text{dom}(\gamma)$ and $t \in (0,1)$. $\square$

Lemma

Hadamard spaces are uniquely geodesic.

Proof: Suppose $\gamma : [0,1] \to X$ and $\eta : [0,1] \to X$ are geodesics with

$$\gamma(0) = \eta(0) \quad \text{and} \quad \gamma(1) = \eta(1)$$

Draw $\Delta\left(\overline{\gamma(0)}, \overline{\eta(t)}, \overline{\gamma(1)}\right) \subseteq \mathbb{R}^2$ and notice

$$d(\gamma(t), \eta(t)) \leq \|\overline{\gamma(t)} - \overline{\eta(t)}\| = 0$$

so $\gamma = \eta$. □

In a Hilbert space, the unique geodesic is the (reparametrized) affine segment:

$$\gamma(t) = (1-t)x + ty, \quad t \in [0,1]$$

A subset C of a Hadamard space is **convex** if for every $x, y \in C$, the geodesic connecting x and y is contained in C.

We will examine one particular question of convex optimization: the **metric projection**.

Define $d(x, C) = \inf\{d(x,c) : c \in C\}$.

Can we find $p \in C$ such that $d(x,p) = d(x,C)$?

Theorem

Let X be a Hadamard space. If $C \subseteq X$ is nonempty closed and convex, then for any $x \in X$, $\exists !\ p \in C$ such that $d(x,p) = d(p,C)$.

Metric Projection on Convex Sets

Theorem

Let X be a Hadamard space. If $C \subseteq X$ is nonempty closed and convex, then for any $x \in X$, $\exists!\ p \in C$ such that $d(x,p) = d(p,C)$.

Proof: Let $\{c_n\}_n \subseteq C$ be such that

$$d(x, c_n) \to d(x, C).$$

Let $c_{m,n}$ be the midpoint of c_n and c_m.

For $\varepsilon > 0$ fix N large so that

$$n \geq N \implies d(x, c_n) < d(x, C) + \varepsilon.$$

$$4d(x, c_{m,n})^2 + d(c_m, c_n)^2 \leq 2d(x, c_m)^2 + 2d(x, c_n)^2$$

$$\begin{aligned} d(c_m, c_n)^2 &\leq 2d(x, c_m)^2 + 2d(x, c_n)^2 - 4d(x, c_{m,n})^2 \\ &\leq 2d(x, c_m)^2 + 2d(x, c_n)^2 - 4d(x, C)^2 \\ &< 4(d(x, C) + \varepsilon)^2 - 4d(x, C)^2 \\ &= \varepsilon(8d(x, C) + 4\varepsilon) \end{aligned}$$

$\implies \{c_n\}_n$ is Cauchy

$\implies c_n \to p \in C$ and $d(x,p) = d(x,C)$.

Uniqueness of limits $\implies$ uniqueness of p. □

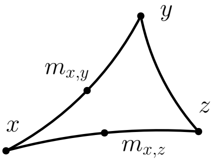
Corollary

Let X be a Hilbert space. If $C \subseteq X$ is nonempty closed and convex, then for any $x \in X$, $\exists!\ p \in C$ such that $d(x,p) = d(x,C)$.

Busemann Spaces

Busemann spaces generalize Hadamard spaces by *only* requiring convexity of distance.

Busemann condition (triangles):



$$d(m_{x,y}, m_{x,z}) \leq \frac{1}{2} d(y, z)$$

where $m_{x,y}$ is the midpoint of x and y.

Equivalently, for any two (linearly reparametrized) geodesics γ and η, the map

$$D_{\gamma,\eta}(s,t) = d\big(\gamma(s), \eta(t)\big)$$

is convex.

Busemann spaces retain:

- Convexity of distance
- Unique geodesics
- Definition for convex sets / functions

However, not all convexity properties are retained. E.g. our metric projection theorem does not hold in complete Busemann spaces.

Strictly Convex Normed Spaces are Busemann

A normed space is **strictly convex** if closed balls are strictly convex sets:
If x and y are distinct points with $\|x\| = \|y\| = R$, then

$$\|tx + (1-t)y\| < R \quad \text{for } t \in (0,1).$$

Lemma

Every strictly convex normed space is Busemann.

Proof: Suppose $\gamma, \eta : [0,1] \to X$ are geodesics with

$$\gamma(0) = \eta(0) \quad \text{and} \quad \gamma(1) = \eta(1).$$

If for some t, $\gamma(t) \neq \eta(t)$, then setting $p = \frac{1}{2}\gamma(t) + \frac{1}{2}\eta(t)$, we get

$$\|\gamma(0) - p\| = \|\gamma(0) - (\tfrac{1}{2}\gamma(t) + \tfrac{1}{2}\eta(t))\| < \|\gamma(0) - \gamma(t)\|$$

$$\text{and } \|\gamma(1) - p\| < \|\gamma(1) - \gamma(t)\|$$

$$\begin{aligned} \text{so } \|\gamma(0) - \gamma(1)\| &= \|\gamma(0) - \gamma(1) - p + p\| \\ &\leq \|\gamma(0) - p\| + \|\gamma(1) - p\| \\ &< \|\gamma(0) - \gamma(t)\| + \|\gamma(1) - \gamma(t)\| \\ &= \|\gamma(0) - \gamma(1)\| \text{ (a contradiction)} \end{aligned}$$

Thus, the only geodesics are of the form $\gamma(t) = (1-t)x + ty$. (Exercise: check Busemann condition)