Definitions:

**Outer measure**

[3] Let $\Omega$ be a set. An outer measure, $\nu$, is an extended real-valued function that satisfies the following for any $A, B \in \mathcal{P}(\Omega)$.

(i) $\nu(A) \geq 0$. (nonnegative)

(ii) $\nu(\varnothing) = 0$.

(iii) If $A \subset B$ then $\nu(A) \leq \nu(B)$. (monotone)

(iv) $\nu\big(\bigcup_n A_n\big) \leq \sum_n \nu(A_n)$.

**Metric outer measure**

[3] Let $(\Omega, \rho)$ be a metric space. An outer measure, $\mu$, is a metric outer measure if

$$\mu(A \cup B) = \mu(A) + \mu(B)$$

for all $A, B \subset \Omega$ such that $\rho(A, B) > 0$.

**Diameter**

[4] Let $(\Omega, \rho)$ be a metric space and $S$ a nonempty subset of $\Omega$. The diameter of $S$ is given by

$$d(S) = \sup\{\rho(x, y) : x, y \in S\}.$$

By convention, $d(\varnothing) = 0$.

**Hausdorff measure**

[4] Let $E$ be a subset of a metric space $(\Omega, \rho)$. Let $r \geq 0$ and $\delta > 0$ be given. Define

$$\mathcal{H}^r_\delta(E) = \inf\left\{\sum_i d(A_i)^r : E \subset \bigcup_i A_i \text{ and } d(A_i) \leq \delta\right\}$$

The $r$-dimensional Hausdorff outer measure is given by

$$\mathcal{H}^r(E) = \lim_{\delta \downarrow 0} \mathcal{H}^r_\delta(E)$$

**Hausdorff dimension**

[3] A set $E$ is said to have Hausdorff dimension $r$, where

$$r = \sup\{s \in [0, \infty) : \mathcal{H}^s(E) > 0\}$$

with the condition that $\sup(\varnothing) = 0$.

## References

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[3] McDonald, John N. & Weiss, Neil A. (2013). *A Course in Real Analysis* (2nd ed.) Waltham, MA: Elsevier.

[4] Rogers, C. A. (1998). *Hausdorff Measures*. Cambridge University Press. (Original work published 1970)