The Embedding Theorems of Rådström and Hörmander: A connection between hyperspaces of convex sets and bounded continuous functions

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Figure: (Top) Lars Hörmander (1931-2012). (Bottom) Lars Hörmander with John Milnor in 1962.

Let X be a normed space.

Goal: Find a normed space Y so that the compact convex sets (resp. closed bounded convex sets) of X are 'nicely' embedded as points in Y.

Rådström's Embedding Theorem:
Using the techniques of extending a semi-vector space to a vector space, we can construct a norm space Y for which there is a 'nice' embedding of the nonempty compact convex sets of X into Y.

Hörmander's Embedding Theorem:
Using support functionals, we can 'nicely' embed the nonempty closed bounded convex sets of X into the Banach space of bounded continuous functions on the closed unit ball of X^*.

Outline

- The Hausdorff Metric
- The Semi-Vector Space of Compact Convex Sets
- Rådström's Embedding Theorem
- The Support Functional $s_A : X^* \to \mathbb{R}$
- Hörmander's Embedding Theorem

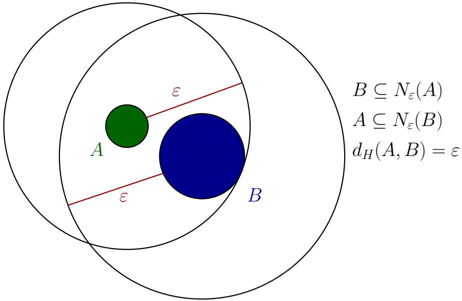
The Hausdorff Metric

Some Notation:

- $(X, \| \ \|)$ is a norm space (with scalar field $\mathbb{R}$)
- For $p \in X$ and $A \subseteq X$
  - $d(p, A) = \inf\{\|p - a\| : a \in A\}$
  - $N_\varepsilon(A) = \{x \in X : d(x, A) \leq \varepsilon\}$

The Hausdorff Distance:

$$d_H(A, B) = \inf \left\{ \varepsilon > 0 : A \subseteq N_\varepsilon(B), B \subseteq N_\varepsilon(A) \right\}$$



Hyperspaces with the Hausdorff Metric

Given a metric space (X, d), define

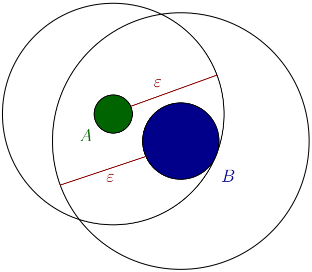
- $B(X)$, the nonempty closed bounded subsets of X
- $K(X)$, the nonempty compact subsets of X
- $F(X)$, the nonempty finite subsets of X

Given a normed space $(X, \| \ \|)$, define

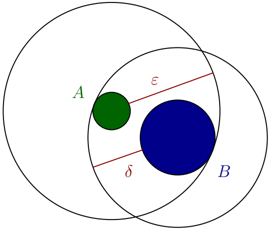
- $CB(X)$, the nonempty closed bounded convex subsets of X
- $CK(X)$, the nonempty compact convex subsets of X

The Hausdorff distance is a metric on each of these collections.

(i) $d_H(A,B) = \inf\left\{\varepsilon > 0 : A \subseteq N_\varepsilon(B), B \subseteq N_\varepsilon(A)\right\}$
(ii) $d_H(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\}$
(iii) $d_H(A,B) = \sup_{x\in X}\left|d(x,A) - d(x,B)\right|$



$B \subseteq N_\varepsilon(A)$
$A \subseteq N_\varepsilon(B)$
$d_H(A,B) = \varepsilon$



$\sup_{b\in B} d(b,A) = \varepsilon$
$\sup_{a\in A} d(a,B) = \delta$
$d_H(A,B) = \varepsilon$

'proof'

(i) = (ii): $\inf\{\varepsilon > 0 : A \subseteq N_\varepsilon(B)\} = \sup_{a\in A} d(a,B)$.

(ii) = (iii): $\sup_{a\in A} d(a,B) = \sup_{x\in X} d(x,B) - d(x,A)$.

Observation: (iii) is equivalent to $\|d(\cdot,A) - d(\cdot,B)\|_\infty$.

Semi-Vector Structure of $CK(X)$ and $CB(X)$

Minkowski sum: $A + B = \{a + b : a \in A, b \in B\}$

Scalar product: $\lambda A = \{\lambda a : a \in A\}$

- If A and B are convex, so is $A + B$.
- If A and B are convex and compact, so is $A + B$
- However, if $A, B \in CB(X)$, then $A + B$ may not be closed
    - For a Banach space X, $A + B$ is always closed if and only if X is reflexive.

Define the closed sum $A \oplus B = \overline{A + B}$.

- $(CK(X), +)$ is an abelian semi-group (or monoid, since $A + \{0\} = A$).
- $(CB(X), \oplus)$ is an abelian semi-group.
- Using the defined scalar product makes each a semi-vector space.

Semi-Vector Spaces

A vector space is an abelian group with scalar multiplication:

$+ : V \times V \to V$

- $(u+v)+w = u+(v+w)$
- $u+v = v+u$
- $v+0 = v$
- $v+(-v) = 0$

$\cdot : \mathbb{R} \times V \to V$

- $1v = v$
- $\alpha(\beta v) = (\alpha\beta)v$
- $\alpha(u+v) = \alpha u + \alpha v$
- $(\alpha+\beta)v = \alpha v + \beta v$

The semi-vector space $CK(X)$ is an abelian semigroup which satisfies:

$+ : CK(X) \times CK(X) \to CK(X)$

- $(A+B)+C = A+(B+C)$
- $A+B = B+A$
- $A+\{0\} = A$
-

$\cdot : [0,\infty) \times CK(X) \to CK(X)$

- $1A = A$
- $\alpha(\beta A) = (\alpha\beta)A$
- $\alpha(A+B) = \alpha A + \alpha B$
- $(\alpha+\beta)A = \alpha A + \beta A$

Theorem (Rådström's Embedding Theorem)

Let X be a normed space. There is a normed space Y such that $(CK(X), d_H)$ can be isometrically embedded as a convex cone in Y.

'proof'

Verify for $A, B, C \in CK(X)$:

- If $A + C \subseteq B + C$, then $A \subseteq B$.
  Moreover, if $A + C = B + C$, then $A = B$.
- $d_H(A + C, B + C) = d_H(A, B)$.
- $d_H(\lambda A, \lambda B) = \lambda d_H(A, B)$ for $\lambda \geq 0$.

Define an equivalence relation on $CK(X) \times CK(X)$ by

$$(A, B) \sim (C, D) \iff A + D = B + C$$

Define Y as the quotient of $CK(X) \times CK(X)$ with this equivalence relation.

Theorem (Rådström's Embedding Theorem)

Let X be a normed space. There is a normed space Y such that $(CK(X), d_H)$ can be isometrically embedded as a convex cone in Y.

'proof' cont.

$$(A,B) \sim (C,D) \iff A+D = B+C, \qquad Y = (CK(X) \times CK(X))/\sim$$

Let $[A,B]$ be the equivalence class containing (A,B).

- $[A,B] + [C,D] = [A+C, B+D]$
- $\lambda[A,B] = \begin{cases} [\lambda A, \lambda B] & \text{if } \lambda \geq 0 \\ [|\lambda| B, |\lambda| A] & \text{if } \lambda < 0 \end{cases}$
- $\big\|[A,B] - [C,D]\big\| = d_H\big(A+D, B+C\big)$

Finally, the embedding is given by $A \mapsto [A, \{0\}]$.

The Support Functional

Given $A \in CB(X)$, define $s_A : X^* \to \mathbb{R}$ by

$$s_A(x^*) = \sup_{a \in A} x^*(a)$$

For $A, B \in CB(X)$ and $\lambda \geq 0$,

$$s_{A \oplus B} = s_{A+B} = s_A + s_B \quad \text{and} \quad s_{\lambda A} = \lambda s_A.$$

Lemma

Let $\mathcal{B}_1^*$ be the closed unit ball in the dual space X^*. For $A, B \in CB(X)$,

$$d_H(A, B) = \sup_{x^* \in \mathcal{B}_1^*} |s_A(x^*) - s_B(x^*)|.$$

Lemma

$d_H(A,B) = \sup_{x^* \in \mathcal{B}_1^*} |s_A(x^*) - s_B(x^*)|$ *for* $A, B \in CB(X)$.

$s_A(x^*) = \sup_{a \in A} x^*(a)$

'proof'

Show that

$$\sup_{b \in B} d(b, A) = \sup_{x^* \in \mathcal{B}_1^*} \big(s_B(x^*) - s_A(x^*)\big).$$

Then by symmetry

$$d_H(A,B) = \max\Big\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\Big\} = \sup_{x^* \in \mathcal{B}_1^*} |s_B(x^*) - s_A(x^*)|.$$

'proof' cont.

For $x^* \in \mathcal{B}_1^*$ and $\varepsilon > 0$ given, fix $b_0 \in B$ so that $s_B(x^*) < x^*(b_0) + \varepsilon$.

$$\begin{aligned} s_B(x^*) &< x^*(b_0) + \varepsilon \\ &= x^*(b_0 - a) + x^*(a) + \varepsilon \\ &\leq \|b_0 - a\| + s_A(x^*) + \varepsilon. \end{aligned}$$

Given that the above holds for any $\varepsilon > 0$ and $a \in A$,

$$\begin{aligned} s_B(x^*) - s_A(x^*) &\leq d(b_0, A) \\ &\leq \sup_{b \in B} d(b, A) \end{aligned}$$

And so $\sup_{x^* \in \mathcal{B}_1^*} \left(s_B(x^*) - s_A(x^*)\right) \leq \sup_{b \in B} d(b, A)$. ✓

$\sup_{x^* \in \mathcal{B}_1^*} \big(s_B(x^*) - s_A(x^*)\big) \leq \sup_{b \in B} d(b, A)$ ✓

'proof' cont. (the reverse inequality)

Suppose $\sup_{b \in B} d(b, A) > 0$.

Let λ be given such that $0 < \lambda < \sup_{b \in B} d(b, A)$. Fix $b_0 \in B$ such that

$$0 < \lambda < d(b_0, A) \leq \sup_{b \in B} d(b, A).$$

Note that $d(b_0, A) = d(0, b_0 - A)$ and let G be the open ball

$$G = \{x \in X : \|x\| < d(0, b_0 - A)\}.$$

$(b_0 - A) \cap G = \varnothing$. By the convex separation theorem $\exists x_0^*$ such that

$$\inf\{x_0^*(u) : u \in b_0 - A\} \geq \sup\{x_0^*(v) : v \in G\}$$

$\lambda < d(b_0, A)$ and $G = \{x \in X : \|x\| < d(0, b_0 - A)\}$
$\inf\{x_0^*(u) : u \in b_0 - A\} \geq \sup\{x_0^*(v) : v \in G\}$.
'proof' cont.
WLOG $\|x_0^*\| = 1$ since $\frac{1}{\|x_0^*\|} x_0^*$ still satisfies the above.

$$\begin{aligned}
\sup_{x^* \in \mathcal{B}_1^*} \big(s_B(x^*) - s_A(x^*)\big) &\geq s_B(x_0^*) - s_A(x_0^*) \\
&\geq x_0^*(b_0) - \sup_{a \in A} x_0^*(a) \\
&= \inf_{a \in A} x_0^*(b_0 - a) \\
&\geq \sup_{v \in G} x_0^*(v) \\
&= \sup_{v \in G} \|v\| \\
&> \lambda.
\end{aligned}$$

$\sup_{x^* \in \mathcal{B}_1^*} \big(s_B(x^*) - s_A(x^*)\big) \geq \sup_{b \in B} d(b, A) \quad \checkmark$

$\mathcal{B}_1^*$ is the closed unit ball in X^*

$\mathscr{C}_b(\mathcal{B}_1^*)$ is the set of bounded continuous functions $f : \mathcal{B}_1^* \to \mathbb{R}$

$\|f\|_\infty = \sup\{|f(x^*)| : x^* \in \mathcal{B}_1^*\}$

Theorem (Hörmander's Embedding Theorem)

The mapping $A \mapsto s_A$ is an algebraic and isometric embedding of $(CB(X), d_H)$ as a cone in the Banach space $(\mathscr{C}_b(\mathcal{B}_1^*), \|\cdot\|_\infty)$.

Verify that $s_A \in \mathscr{C}_b(\mathcal{B}_1^*)$. Let $\alpha = \sup_{a \in A} \|a\|$,

- $|s_A(x^*)| \le \alpha \|x^*\|$
- $|s_A(x^*) - s_A(y^*)| \le \alpha \|x^* - y^*\|$

Lemma

For $A, B \in CB(X)$, $d_H(A, B) = \sup_{x^* \in \mathcal{B}_1^*} |s_A(x^*) - s_B(x^*)|$.

For $A, B \in CB(X)$ and $\lambda \ge 0$,

$$s_{A \oplus B} = s_A + s_B \quad \text{and} \quad s_{\lambda A} = \lambda s_A.$$

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