The Embedding Theorems of Rådström and Hörmander

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1 Introduction

A topological space whose elements are subsets of a different topological space is known as a *hyperspace*. In general, hyperspaces can be difficult to work with; many seemingly natural topologies, especially on the collection of all nonempty subsets, fail to be Hausdorff or even T_1. On the other hand, if we restrict to well-chosen collections of subsets, a hyperspace can inherit many properties of its base space (see [5]). A broad introduction to hyperspaces is given in [1], which served as the primary source for these notes.

We will focus primarily on hyperspaces generated by convex sets of a normed space, equipped with the Hausdorff metric. In particular, we examine the embedding theorems of Rådström and Hörmander, which give isometric embeddings into a normed space of the compact convex and closed bounded convex sets, respectively.

In truth, Hörmander's original theorem [3] only requires that the base space is a locally convex space; the result for a normed space is given as a corollary. A similar result for locally convex spaces can also be found in [7]. For an application of the embedding theorems presented here, see [4].

2 Hyperspaces and the Hausdorff Metric

Let (X, d) be a metric space. Given a set $A \subseteq X$, the distance from a point $x \in X$ to the set A is given by

$$d(x, A) = \inf_{a \in A} d(x, a).$$

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Lemma 2.1. *If A is a nonempty subset of a metric space X, then $d(\cdot, A) : X \to \mathbb{R}$ is 1-Lipschitz continuous and $d(x, A) = d(x, \overline{A})$.*

Proof. Let $x, y \in X$ be given. For any $a \in A$, the triangle inequality gives us

$$d(x, a) \leq d(x, y) + d(y, a) \quad \text{and} \quad d(y, a) \leq d(x, y) + d(x, a)$$

so taking the infimum over all $a \in A$ gives

$$d(x, A) \leq d(x, y) + d(y, A) \quad \text{and} \quad d(y, A) \leq d(x, y) + d(x, A).$$

Therefore, $|d(x, A) - d(y, A)| \leq d(x, y)$, so $d(\cdot, A)$ is 1-Lipschitz continuous.

To verify that $d(x, A) = d(x, \overline{A})$, first note that the inclusion $A \subseteq \overline{A}$ implies $d(x, \overline{A}) \leq d(x, A)$, so it suffices to show the reverse inequality. Let $x \in X$ and $\varepsilon > 0$ be given, and fix $\bar{a} \in \overline{A}$ such that $d(x, \bar{a}) < d(x, A) + \varepsilon/2$. If $B_{\varepsilon/2}(\bar{a})$ is the open ball of radius $\varepsilon/2$ centered at $\bar{a}$, then $B_{\varepsilon/2}(\bar{a}) \cap A$ is nonempty. For any $a \in B_{\varepsilon/2}(\bar{a}) \cap A$,

$$d(x, A) \leq d(x, a) \leq d(x, \bar{a}) + d(\bar{a}, a) < d(x, \overline{A}) + \varepsilon.$$

Given that the above holds for any $\varepsilon > 0$, we have $d(x, A) \leq d(x, \overline{A})$. □

For any $\varepsilon > 0$, we define the *ε-neighborhood* of A as the set of points whose distance is at most ε from A;

$$N_\varepsilon(A) = \{x \in X : d(x, A) \leq \varepsilon\}.$$

Given two nonempty sets, say $A, B \subseteq X$, the *Hausdorff distance* between A and B is defined by

$$d_H(A, B) = \inf \big\{\varepsilon > 0 : A \subseteq N_\varepsilon(B) \text{ and } B \subseteq N_\varepsilon(A)\big\}.$$

Proposition 2.2. *For nonempty sets A and B in a metric space X, the following alternative definitions of the Hausdorff distance hold:*
(i) $d_H(A, B) = \max\big\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\big\}$*; and*
(ii) $d_H(A, B) = \sup_{x \in X} |d(x, A) - d(x, B)|$.

Proof. For (i): First, note that $x \in N_\varepsilon(A)$ if and only if $\varepsilon \geq d(x, A)$. Therefore,

$$\inf\{\varepsilon > 0 : B \subseteq N_\varepsilon(A)\} = \sup_{b \in B} d(b, A).$$

We can then extend the original definition of the Hausdorff distance by observing that

$$\begin{aligned} d_H(A, B) &= \inf\{\varepsilon > 0 : A \subseteq N_\varepsilon(B) \text{ and } B \subseteq N_\varepsilon(A)\} \\ &= \max\Big\{\inf\{\varepsilon > 0 : A \subseteq N_\varepsilon(B)\}, \inf\{\varepsilon > 0 : B \subseteq N_\varepsilon(A)\}\Big\} \\ &= \max\Big\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\Big\}. \end{aligned}$$

For (ii): For any $a \in A$ we have $d(a,B) = d(a,B) - d(a,A)$, so

$$\sup_{a\in A} d(a,B) \le \sup_{a\in A}\Big(d(a,B) - d(a,A)\Big) \le \sup_{x\in X}\Big(d(x,B) - d(x,A)\Big).$$

Conversely, let $x \in X$ and $\varepsilon > 0$ be given. Fixing $a_x \in A$ such that $d(x,a_x) < d(x,A) + \varepsilon/2$, and fix $b_a \in B$ such that $d(a_x,b_a) < d(a_x,B) + \varepsilon/2$ we get

$$\begin{aligned} d(x,B) &\le d(x,b_a) \\ &\le d(x,a_x) + d(a_x,b_a) \\ &< d(x,A) + d(a_x,B) + \varepsilon \\ &\le d(x,A) + \sup_{a\in A} d(a,B) + \varepsilon. \end{aligned}$$

Subtracting $d(x,A)$ from both sides and taking the supremum over $x \in X$ gives the reverse inequality. Therefore,

$$\sup_{a\in A} d(a,B) = \sup_{x\in X}\Big(d(x,B) - d(x,A)\Big).$$

Applying a symmetrical argument for $\sup_{b\in B} d(b,A)$ we get the desired result,

$$d_H(A,B) = \max\big\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\big\} = \sup_{x\in X}|d(x,A) - d(x,B)|. \qquad \square$$

Recalling that $d(x,A) = d(x,\overline{A})$, we see that $d_H(A,B) = d_H(\overline{A},\overline{B})$. Furthermore, it is entirely possible that the Hausdorff distance between two sets is infinite; for example, if A is bounded and B is unbounded. Therefore, if we want to form a metric space with the Hausdorff distance, then we need to restrict the class of subsets under examination. The largest such class is the set of nonempty closed and bounded subsets, which we denote $B(X)$.

Lemma 2.3. *Given a metric space (X,d), the Hausdorff distance d_H is a metric on $B(X)$, the collection of nonempty closed and bounded subsets of X.*

Proof. By restricting to bounded sets, we get $d_H(A,B) < \infty$ for all $A,B \in B(X)$. The fact that d_H is positive definite and symmetric are straightforward from the definition of d_H and Lemma 2.1. Letting $A,B,C \in B(X)$ be given, the triangle inequality is easily established using definition (ii) from Proposition 2.2;

$$\begin{aligned} d_H(A,B) &= \sup_{x\in X}|d(x,A) - d(x,B)| \\ &= \sup_{x\in X}|d(x,A) - d(x,C) + d(x,C) - d(x,B)| \\ &\le \sup_{x\in X}\Big(|d(x,A) - d(x,C)| + |d(x,C) - d(x,B)|\Big) \\ &\le \sup_{x\in X}|d(x,A) - d(x,C)| + \sup_{x\in X}|d(x,C) - d(x,B)| \\ &= d_H(A,C) + d_H(C,B). \end{aligned} \qquad \square$$

A topological space consisting of subsets of another topological space is commonly known as a *hyperspace*. In general, hyperspaces may not be metrizable (see [1] or [5] for examples), but we will only be considering hyperspaces equipped with the Hausdorff metric. In light of the previous lemma, one can also consider

$$K(X) = \{A \subseteq X : A \text{ is nonempty and compact}\}$$
$$F(X) = \{A \subseteq X : A \text{ is nonempty and finite}\}$$

as subspaces of $(B(X), d_H)$. Furthermore, if X is a normed space, then we may also consider the subspaces

$$CB(X) = \{A \subseteq X : A \text{ is nonempty, closed, bounded, and convex}\}$$
$$CK(X) = \{A \subseteq X : A \text{ is nonempty, compact, and convex}\}.$$

For our purposes, we will be primarily examining the metric spaces $(CK(X), d_H)$ and $(CB(X), d_H)$.

Remark. Suppose that X is a compact metric space and let $\mathscr{C}(X)$ be the Banach space of continuous real-valued functions on X. Using the results of this section; in particular, $d_H(A, B) = \|d(\cdot, A) - d(\cdot, B)\|_\infty$ by Proposition 2.2; we can see that the map $A \mapsto d(\cdot, A)$ is an isometric embedding of $B(X)$ into $\mathscr{C}(X)$. Although this is weaker than the embedding theorem we are aiming for, it serves as a good picture of how the main theorem will work.

3 Semi-Vector Spaces

Recall that a real vector space V is an abelian group together with distributive scalar multiplication over $\mathbb{R}$. This can be described as eight axioms, the first four of which apply to the group operation $+ : V \times V \to V$ and the last four apply to the scalar multiplication $\cdot : \mathbb{R} \times V \to V$: For any $u, v, w \in V$ and $\alpha, \beta \in \mathbb{R}$,

$$
\begin{aligned}
(i)&\ (u + v) + w = u + (v + w),\\
(ii)&\ u + v = v + u,\\
(iii)&\ v + 0 = v,\\
(iiii)&\ v + (-v) = 0,\\
(v)&\ 1v = v,\\
(vi)&\ \alpha(\beta v) = (\alpha\beta)v,\\
(vii)&\ \alpha(v + u) = \alpha v + \alpha u, \text{ and}\\
(viii)&\ (\alpha + \beta)v = \alpha v + \beta v.
\end{aligned}
$$

In general, a *semi-vector space* is abelian semigroup with multiplication by positive scalars; in other words, a semi-vector space satisfies (i) and (ii), as well as (v), (vi), and (vii) whenever $\alpha, \beta > 0$. We will explore two specific examples of semi-vector spaces, namely $CK(X)$ and $CB(X)$.

Let A and B be nonempty subsets of a vector space, and λ be a real number. We will use the Minkowski sum an scalar product, defined by

$$A + B = \{a + b : a \in A, b \in B\} \quad \text{and} \quad \lambda A = \{\lambda a : a \in A\}$$

respectively. Using the above operations, $CK(X)$ becomes a semi-vector space. In fact, one can check that $CK(X)$ also satisfies axioms (iii) with $0 = \{0\}$; (vi) and (vii) for all $\alpha, \beta \in \mathbb{R}$; and (viii) whenever $\alpha, \beta \geq 0$.

We can form a semi-vector space from $CB(X)$ in a similar manner, but one adjustment needs to be made. In general, the sum of two closed bounded convex sets may not be closed (in fact, if X is a Banach space, the sum is always closed if and only if X is reflexive). To alleviate this problem, we simply define the closed sum $\oplus$ by

$$A \oplus B = \overline{A + B}.$$

Using the closed sum in place of the standard Minkowski sum, one can verify that $CB(X)$ is a semi-vector space which satisfies precisely the same axioms as $CK(X)$. Moving forward, we will focus only on the collection $CB(X)$ and note that all of the results follow for $CK(X)$ since $CK(X) \subseteq CB(X)$ and $A \oplus B = A + B$ whenever $A, B \in CK(X)$.

Example 3.1. We give an example of two closed bounded convex sets whose sum is not closed. Consider the (non-reflexive) Banach space $\ell^1(\mathbb{N})$; that is, the space of sequences $a : \mathbb{N} \to R$ such that $\|a\|_1 = \sum_{n=1}^{\infty} |a(n)| < \infty$. Let $x^* : \ell^1(\mathbb{N}) \to \mathbb{R}$ be the continuous linear functional

$$x^*(a) = \sum_{n=1}^{\infty} \frac{n}{n+1} a(n).$$

Notice that $\|x^*\| = 1$, but $|x^*(a)| < 1$ whenever $\|a\|_1 \leq 1$. Let A and B be the sets

$$A = \{a : f(a) \geq 1, \|a\|_1 \leq 2\} \quad \text{and} \quad B = \{a : \|a\|_1 \leq 1\}.$$

We leave it to the reader to verify that $0 \in A \oplus B$, but $0 \notin A + B$.

Although there in general no additive inverses in $CB(X)$, we do get some nice cancellation properties. Notice that the first two results use the standard sum, rather than the closed sum.

Lemma 3.2. *For any $A, B \in CB(X)$, if $A \subseteq A + B$, then $0 \in B$.*

Proof. Let $a_0 \in A$ be given. By our hypothesis, we can find $a_1 \in A$ and $b_1 \in B$ such that $a_0 = a_1 + b_1$. Continuing in this manner, we can also write $a_1 = a_2 + b_2$, $a_2 = a_3 + b_3$, and so forth. This gives us sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ such that $a_0 - a_n = \sum_{k=1}^{n} b_k$. Using this relation and the fact that A is bounded, we see that

$$0 = \lim_{n\to\infty} \frac{1}{n}(a_0 - a_n) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} b_k.$$

Since B is closed and convex, and $\frac{1}{n}\sum_{k=1}^{n} b_k$ is a convex combination of elements of B which converges to 0, we have $0 \in B$. □

Corollary 3.3. *For any $A, B, C \in CB(X)$, if $A+C \subseteq B+C$, then $A \subseteq B$. Further, if $A + C = B + C$, then $A = B$.*

Proof. Fix any $a \in A$. By our hypothesis, $\{a\} + C \subseteq B + C$, so $C \subseteq B + \{-a\} + C$. It follows from the Lemma 3 that $0 \in B + \{-a\}$, so $a \in B$. Since $a \in A$ was arbitrary, we have $A \subseteq B$. The second statement follows from two applications of the first. □

Theorem 3.4. *If $A, B, C \in CB(X)$ and $\lambda \geq 0$, then*
(i) $d_H(A \oplus C, B \oplus C) = d_H(A, B)$ and
(ii) $d_H(\lambda A, \lambda B) = \lambda d_H(A, B)$.

Proof. Recalling that $d_H(A, B) = d_H(\overline{A}, \overline{B})$, we will prove (i) for the standard sum and the result for the closed sum will follow immediately. First, note that if $A \subseteq N_\varepsilon(B)$ for some $\varepsilon > 0$, then for any $a \in A$ and $c \in C$ we have

$$d(a + c, B + C) \leq d(a + c, B + c) = d(a, B) \leq \varepsilon$$

so $A + C \subseteq N_\varepsilon(B + C)$.

To show the reverse direction, we first show that $N_\varepsilon(B + C) = N_\varepsilon(B) + C$. Let x be an element of $N_\varepsilon(B+C)$. For any $\delta > 0$ we can find $b_x + c_x \in B+C$ such that

$$d(x, B + C) < \|x - (b_x + c_x)\| < \varepsilon + \delta.$$

The above implies that $x - c_x \in N_\varepsilon(B)$ since $d(x - c_x, B) \leq \|\|x - c_x - b_x\| < \varepsilon + \delta$, so we have $x \in N_\varepsilon(B) + C$ and therefore $N_\varepsilon(B + C) \subseteq N_\varepsilon(B) + C$. Conversely, if $y \in N_\varepsilon(B) + C$, then we can find $y' \in N_\varepsilon(B)$, $b_y \in B$, and $c_y \in C$ such that $\|y' - b_y\| < \varepsilon + \delta$ and $y = y' + c_y$. Furthermore,

$$d(y, B + C) = d(y' + c_y, B + C) \leq \|\|y' + c_y - (b_y + c_y)\| = \|y' - b_y\| < \varepsilon + \delta$$

so $y \in N_\varepsilon(B + C)$. Thus, $N_\varepsilon(B + C) = N_\varepsilon(B) + C$.

Now suppose that $A + C \subseteq N_\varepsilon(B + C)$. Then as we have just shown, $A + C \subseteq N_\varepsilon(B) + C$. By Corollary 3.3, $A \subseteq N_\varepsilon(B)$. We conclude that $A \subseteq N_\varepsilon(B)$ if and only if $A + C \subseteq N_\varepsilon(B + C)$. Thus, $d_H(A + C, B + C) = d_H(A, B)$.

Part (ii) follows from the fact that $\lambda \geq 0$ and the straightforward observation

$$d(\lambda a, \lambda B) = \inf_{\lambda b \in \lambda B} \|\lambda a - \lambda b\| = \inf_{b \in B} \lambda \|a - b\| = \lambda \inf_{b \in B} \|a - b\| = \lambda d(a, B).$$

Therefore,

$$\sup_{\lambda a \in \lambda A} d(\lambda a, \lambda B) = \sup_{a \in A} \lambda d(a, B) = \lambda \sup_{a \in A} d(a, B)$$

and so $d_H(\lambda A, \lambda B) = \lambda d_H(A, B)$. □

4 The Rådström Embedding Theorem

Rådstrm̈'s Embedding Theorem makes use of the semi-vector space structure of $CK(X)$ and $CB(X)$, as well as the fact that the Hausdorff metric on each is invariant under addition (see Theorem 3.4). The main thrust of the theorem relies on a technique for extending a semi-vector space to a real vector space. Similar to how the theorem is presented in [8], we leave verifying many details of the proof to the reader.

Theorem 4.1 (Rådström's Embedding Theorem)**.** *Let X be a normed space. There is a normed space Y such that $(CK(X), d_H)$ can be isometrically embedded as a convex cone in Y. Furthermore, we can construct Y to be minimal in the sense that for any normed space Z which contains $(CK(X), d_H)$, then Y is isomorphic to a subspace of Z containing $(CK(X), d_H)$.*

Proof. Let $\sim$ be an equivalence relation on $CK(X) \times CK(X)$ defined by

$$(A, B) \sim (C, D) \iff A + D = B + C$$

and let $[A, B]$ be the equivalence class containing (A, B). Let Y to be the set of equivalence classes $CK(X) \times CK(X)/\sim$. Define addition on Y by $[A, B] + [C, D] = [A + C, B + D]$ and scalar multiplication by $\lambda[A, B] = [\lambda A, \lambda B]$ for $\lambda \geq 0$ and $\lambda[A, B] = [|\lambda| B, |\lambda| A]$ for $\lambda < 0$. Finally define the norm on Y by

$$\big\|[A, B] - [C, D]\big\| = d_H\big(A + D, B + C\big).$$

We leave it to the reader to verify that the above operations are well-defined and that $(Y, \| \ \|)$ is in fact a normed space (some of these can be found in [8]). The desired embedding is then given by $A \mapsto [A, \{0\}]$ (note that this is equivalent to $A \mapsto [A + B, B]$ for any $B \in CK(X)$). Again, the reader may verify that this is map gives an isometric embedding and satisfies the homomorphism property for addition and nonnegative scalar multiplication.

Contrary to every other part of the proof, we will verify the most of the minimality of Y. Suppose there is a normed space Z such that Z contains an isometric and algebraic embedding of $CK(X)$. Let $a, b, c, d \in Z$ be such that $A \mapsto a$, $B \mapsto b$, etc. Map each element $[A, B] \in Y$ to $a - b$. Note that this embedding certainly contains the embedding of $CK(X)$ since $[A, \{0\}] \mapsto a$. The mapping injective since if $a-b = c-d$ then we have $a+d = b+c$ and by the algebraic property of the embedding, $A + D = B + C$, so by the definition of the equivalence class $[A, B] = [C, D]$. The isometry of this map can be seen since

$$\|[A, B] - [C, D]\| = d_H(A + D, B + C) = \|(a + d) - (b + c)\| = \|(a - b) - (c - d)\|.$$

Showing that addition and scalar multiplication are preserved also follows directly from the earlier definitions given. We conclude that Z does in fact contain an embedding of Y which itself contains the embedding of $CK(X)$. □

5 The Hörmander Embedding Theorem

An alternative to (or extension of) Rådström's Embedding Theorem is Hörmander's Embedding Theorem. To purse this result, we will make use of support functionals. Given a set $A \in X$, the *support functional* is a map on the dual space, $s_A : X^* \to \mathbb{R}$, defined by

$$s_A(x^*) = \sup_{a \in A} x^*(a).$$

In general, s_A is extended real-valued; however, for bounded sets (in particular for $A \in CB(X)$), $s_A(x^*)$ is finite. Furthermore, it is easy to see that s_A is convex since for any $x^*, y^* \in X^*$ and $t \in (0, 1)$,

$$\begin{aligned} s_A(tx^* + (1-t)y^*) &= \sup_{a \in A} \big(tx^*(a) + (1-t)y^*(a)\big) \\ &\leq t \sup_{a \in A} x^*(a) + (1-t) \sup_{a \in A} y^*(a) \\ &= t s_A(x^*) + (1-t) s_A(y^*). \end{aligned}$$

Lemma 5.1. *If $A, B \in CB(X)$ and $\lambda \geq 0$, then (i) $s_{A \oplus B} = s_A + s_B$ and (ii) $s_{\lambda A} = \lambda s_A$.*

Proof. Let $A, B \in CB(X)$ be given. By continuity of x^*, we have $s_A(x^*) = s_{\overline{A}}(x^*)$, so to prove (i) it is enough to show that $s_{A+B} = s_A + s_B$. First note that for any

$x^* \in X^*$,

$$\begin{aligned} s_{A+B}(x^*) &= \sup_{a+b\in A+B} x^*(a+b) \\ &= \sup_{a+b\in A+B} \big(x^*(a)+x^*(b)\big) \\ &\le \sup_{a\in A} x^*(a) + \sup_{b\in B} x^*(b). \end{aligned}$$

Conversely, let $x^* \in X^*$ and $\varepsilon > 0$ be given. Fix $a_0 \in A$ and $b_0 \in B$ be such that $s_A(x^*) < x^*(a_0) + \varepsilon/2$ and $s_B(x^*) < x^*(b_0) + \varepsilon/2$. Then,

$$s_A(x^*) + s_B(x^*) < x^*(a_0) + x_*(b_0) + \varepsilon = x^*(a_0+b_0) + \varepsilon \le s_{A+B}(x^*) + \varepsilon$$

so $s_A(x^*) + s_B(x^*) \le s_{A+B}(x^*)$. Thus, $s_{A+B} = s_A + s_B$.

The proof of (ii) comes from the simple observation that for $\lambda \ge 0$,

$$s_{\lambda A}(x^*) = \sup_{\lambda a \in \lambda A} x^*(\lambda a) = \sup_{a\in A} \lambda x^*(a) = \lambda \sup_{a\in A} x^*(a) = \lambda s_A(x^*). \qquad \square$$

Lemma 5.2. *Let X be a normed space and let $\mathcal{B}_1^*$ be the closed unit ball in the dual space X^*. For $A, B \in CB(X)$,*

$$d_H(A,B) = \sup_{x^*\in\mathcal{B}_1^*} |s_A(x^*) - s_B(x^*)|.$$

Proof. First recall that $\|x^*(a)\| \le \|x^*\|\|a\|$, so

$$\sup_{x^*\in\mathcal{B}_1^*} s_A(x^*) \le \sup_{a\in A} \|a\|.$$

Let $x^* \in \mathcal{B}_1^*$ and $\varepsilon > 0$ be given. Fix $b_0 \in B$ such that $s_B(x^*) < x^*(b_0) + \varepsilon$. For any $a \in A$, we have

$$s_B(x^*) < x^*(b_0) + \varepsilon = x^*(b_0 - a) + x^*(a) + \varepsilon \le \|b_0 - a\| + s_A(x^*) + \varepsilon.$$

Since the above holds for any $\varepsilon > 0$ and any $a \in A$ we have

$$s_B(x^*) - s_A(x^*) \le d(b_0, A) \le \sup_{b\in B} d(b, A)$$

and thus $\sup_{x^*\in\mathcal{B}_1^*} \big(s_B(x^*) - s_A(x^*)\big) \le \sup_{b\in B} d(b,A)$. For the reverse inequality, first note that if $\sup_{b\in B} d(b,A) = 0$ then the result is trivial, so suppose that $\sup_{b\in B} d(b,A) > 0$. Let λ be given such that $0 < \lambda < \sup_{b\in B} d(b,A)$ and fix $b_0 \in B$ such that $d(b_0, A) > \lambda$. Note that $d(b_0,A) = d(0, b_0 - A) > \lambda$. Letting G be the open ball of radius $d(0, b_0 - A)$,

$$G = \{x \in X : \|x\| < d(0, b_0 - A)\},$$

we see that $(b_0 - A) \cap G = \varnothing$. By the convex separation theorem (see Appendix), there is a nonzero continuous linear functional x_0^* such that

$$\inf\{x_0^*(u) : u \in b_0 - A\} \geq \sup\{x_0^*(v) : v \in G\}.$$

Without loss of generality, we may assume that $\|x_0^*\| = 1$ since $\frac{1}{\|x_0^*\|} x_0^*$ still satisfies the above inequality. Now we have

$$\begin{aligned}
\sup_{x^* \in \mathcal{B}_1^*} \big(s_B(x^*) - s_A(x^*)\big) &\geq s_B(x_0^*) - s_A(x_0^*) \\
&\geq x_0^*(b_0) - \sup_{a \in A} x_0^*(a) \\
&= \inf_{a \in A} x_0^*(b_0 - a) \\
&\geq \sup_{v \in G} x_0^*(v) \\
&= \sup_{v \in G} \|v\| \\
&> \lambda.
\end{aligned}$$

Thus, whenever $\sup_{a \in A} d(a, B) > \lambda$, we have $\sup_{x^* \in \mathcal{B}_1^*} \big(s_B(x^*) - s_A(x^*)\big) > \lambda$. We conclude that

$$\sup_{b \in B} d(b, A) = \sup_{x^* \in \mathcal{B}_1^*} \big(s_B(x^*) - s_A(x^*)\big)$$

Applying a symmetrical argument to $\sup_{b \in B} d(b, A)$ gives us

$$d_H(A, B) = \max\Big\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\Big\} = \sup_{x^* \in \mathcal{B}_1^*} |s_A(x^*) - s_B(x^*)|. \qquad \square$$

We are now able to prove our main theorem, but first we clarify some notation. Given a topological space X, we will use $\mathscr{C}_b(X)$ to represent the space of bounded continuous functions $f : X \to \mathbb{R}$. Recall that $\mathscr{C}_b(X)$ together with the uniform norm is always a Banach space.

Theorem 5.3 (Hörmander's Embedding Theorem)**.** *Let X be a real normed space and let $\mathcal{B}_1^*$ be the closed unit ball in X^*. The map $A \mapsto s_A$ is an algebraic and isometric embedding of $CB(X)$ as a convex cone in the Banach space $\mathscr{C}_b(\mathcal{B}_1^*)$.*

Proof. First, we need to verify that the $s_A \in \mathscr{C}(\mathcal{B}_1^*)$ whenever $A \in CB(X)$. Let $A \in CB(X)$ be given and let $\alpha = \sup_{a \in A} \|a\|$, which is finite as A is bounded. Given that $|x^*(a)| \leq \|x^*\|\|a\|$ and $x^* \in \mathcal{B}_1^*$, we see that

$$|s_A(x^*)| = |\sup_{a \in A} x^*(a)| \leq \sup_{a \in A} \|x^*\|\|a\| \leq \alpha$$

so s_A is bounded. Furthermore, we can see that s_A is α-Lipschitz continuous. For any $x^*, y^* \in X^*$,

$$\begin{aligned} |s_A(x^*) - s_A(y^*)| &= |\sup_{a\in A} x^*(a) - \sup_{a\in A} y^*(a)| \\ &\leq \sup_{a\in A} |x^*(a) - y^*(a)| \\ &\leq \sup_{a\in A} \|x^* - y^*\| \|a\| \\ &= \alpha \|x^* - y^*\|. \end{aligned}$$

By Lemma 5.2, this embedding is an isometry; and by Lemma 5.1, the embedding is a homomorphism with respect to the group operation and nonnegative scalar multiplication. □

A Convex Separation Theorems

First, recall that a *sublinear functional* $\rho : X \to \mathbb{R}$ is such that $\rho(x+y) \leq \rho(x)+\rho(y)$ and $\rho(\lambda x) = \lambda\rho(x)$ whenever $\lambda \geq 0$.

Theorem A.1 (The Hahn-Banach Theorem)**.** *Let X be a real vector space and let ρ be a sublinear functional on X. If M is a linear subspace of X and $f : M \to \mathbb{R}$ is such that $f(x) \leq \rho(x)$ for all $x \in M$, then there is a linear functional $F : X \to \mathbb{R}$ such that $F|_M = f$ and $F(x) \leq \rho(x)$ for all $x \in X$.*

Let X be a real topological vector space and let A and B be subsets of X. We say that A and B are *separated* if there is a continuous linear functional f and real number r such that

$$A \subseteq \{x \in X : f(x) \geq r\} \quad \text{and} \quad B \subseteq \{x \in X : f(x) \leq r\}.$$

If it further holds that

$$A \subseteq \{x \in X : f(x) > r\} \quad \text{and} \quad B \subseteq \{x \in X : f(x) < r\}$$

then A and B are said to be *strictly separated*.

Theorem A.2. *If X is a topological vector space and V is a nonempty open convex set not containing the origin, then there is a closed hyperplane H such that $V \cap H = \varnothing$.*

Proof. Let $v_0 \in V$ be fixed and let $U = v_0 - V$. Notice that U is a neighborhood of the origin and $v_0 \notin U$. Let$\rho : X \to \mathbb{R}$ be the sublinear functional defined by

$$\rho(x) = \inf\{t > 0 : x \in tU\}.$$

Notice that $\rho(u) < 1$ for any $u \in U$ and $\rho(v_0) \geq 1$. Let $M = \{tv_0 : t \in \mathbb{R}\}$ and define $f : M \to \mathbb{R}$ by

$$f(tv_0) = t\rho(v_0).$$

Then f is a nonzero continuous linear functional on M and $f(x) \leq \rho(x)$ for every $x \in M$. By the Hahn-Banach Theorem, there is a continuous linear functional $F : X \to \mathbb{R}$ such that $F|_M = f$ and $F(x) \leq \rho(x)$ for all $x \in X$. Since any $v \in V$ can be written as $v_0 - u$ for some $u \in U$,

$$F(v) = F(v_0 - u) = F(v_0) - F(u) = \rho(v_0) - F(u) \geq \rho(v_0) - \rho(u) > 0.$$

Thus, letting H be the closed hyperplane $\ker(F)$, we have $V \cap H = \varnothing$. □

Corollary A.3. *If X is a real topological vector space, $A, B \subseteq X$ are disjoint convex sets, and A is open, then A and B are separated.*

Proof. Let $G = A - B$. Since A is open and A and B are convex, so is G open and convex. Furthermore, given that $A \cap B = \varnothing$, we have $0 \notin G$. As G is an open convex set not containing the origin, by Theorem A.2, we can find a closed hyperplane H such that $G \cap H = \varnothing$. Let $f : X \to \mathbb{R}$ be a continuous linear functional such that $\ker(f) = H$. Without loss of generality, assume that $f(x) > 0$ for all $x \in G$. Then for any $a \in A$ and $b \in B$,

$$f(a) - f(b) = f(a - b) > 0$$

so $f(a) > f(b)$. Thus, we can fix $r \in \mathbb{R}$ such that

$$\sup_{b \in B} f(b) \leq r \leq \inf_{a \in A} f(a)$$

which tells us that

$$A \subseteq \{x \in X : f(x) \geq r\} \quad \text{and} \quad B \subseteq \{x \in X : f(x) \leq r\}.$$ □

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